THE

SCIENTIFIC PAPERS

OF

JOHN COUCH ADAMS.

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JOHN [illegible] ADAMS

[illegible], LL.D., F.R.S.
LATE LOWNDEAN [illegible] ASTRONOMY
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[illegible]TED BY

WILL[illegible] ADAMS, Sc.D., F.R.S.

[illegible] MEMOIR BY

J. W. [illegible]LAISHER, Sc.D., F.R.S.

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JOHN COUCH ADAMS,

M.A., Sc.D., D.C.L., LL.D., F.R.S.,
LATE LOWNDEAN PROFESSOR OF ASTRONOMY AND GEOMETRY
IN THE UNIVERSITY OF CAMBRIDGE.

VOL. I.

EDITED BY

WILLIAM GRYLLS ADAMS, Sc.D., F.R.S.

WITH A MEMOIR BY

J. W. L. GLAISHER, Sc.D., F.R.S.

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PREFACE TO VOLUME I.

THE present volume of the Collected Works of the late Professor JOHN COUCH ADAMS contains all the original papers which were published by him during his lifetime, extending from 1844 (when he was 25 years of age) to 1890. They consist of about 50 Astronomical Papers which were for the most part printed in the Memoirs or Monthly Notices of the Royal Astronomical Society and 11 Papers on Pure Mathematics. Besides these there are many papers on various branches of Astronomy which were left in an incomplete state among Professor Adams' manuscripts. These are being prepared for publication by Professor Sampson.

There is also a great quantity of unpublished work in an incomplete state on Legendre's and Laplace's Coefficients and on Terrestrial Magnetism which was taken up from time to time extending over a period of 40 years, but no part of which has been published except a short paper (No. 60) on Legendre's Coefficients. It is hoped that a considerable portion of this unpublished work may shortly be brought into shape for publication, and that it will form the continuation of these Collected Works.

Since the Appendix to Paper 19 (p. 124 of this volume) was printed, more exact expressions of the coefficients for Jupiter's Satellites II, III and IV have been found among Professor Adams' unpublished papers. Thus in forming the Tables for Satellite II, in addition to the terms $-2^s{\cdot}5 \sin(\Pi - \Lambda_{II}) - 1^s{\cdot}5 \sin(\Pi - \Lambda_{III})$ given on p. 118 of this volume, another term $+0^s{\cdot}127 \sin(\Pi - \Lambda_{IV})$ was employed in the calculation for the period 1890—1900. In place of the expressions given on p. 124 for this period, 1890—1900, the more exact values of the coefficients are

$$\begin{array}{lll} \text{For Satellite II} & +0^s{\cdot}756 \sin(5\bar{u} - 2u_0 - 17^\circ{\cdot}7), \\ \text{Satellite III} & +\ 2{\cdot}233 \sin(5\bar{u} - 2u_0 - 17^\circ{\cdot}7), \\ \text{Satellite IV} & +12{\cdot}33\ \sin(5\bar{u} - 2u_0 - 17^\circ{\cdot}7). \end{array}$$

The full paper on the attraction of an indefinitely thin ellipsoidal shell on an external point, which was given before the Cambridge Philosophical Society, has been reproduced (see p. 414 of this volume) by the aid of the notes taken by Professor Greenhill at Professor Adams' lectures on the Figure of the Earth.

In 1876 a translation of the paper on the discovery of the planet Neptune was published in Liouville's *Journal de Mathématiques* with the addition of an Appendix by Professor Adams which forms the seventh paper of this Volume. In March 1867, a paper "Sur les étoiles filantes de Novembre" was published in the Paris *Acad. Sci. Compt. Rend.*, LXIV. which was also communicated to, but not published by, the Cambridge Philosophical Society. A paper on the lunar inequalities due to the ellipticity of the Earth was overlooked when the papers on Astronomy were being printed: these papers are printed at the end of this Volume.

The biographical notice prefixed to this volume has been written by Dr J. W. L. Glaisher.

My thanks are due to Mr W. H. Wesley, the Assistant Secretary of the Royal Astronomical Society, for kind help which he has given me.

W. GRYLLS ADAMS.

KING'S COLLEGE,
LONDON.
Oct. 8th, 1896.

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BIOGRAPHICAL NOTICE.

JOHN COUCH ADAMS was born on June 5, 1819, at the farmhouse of Lidcot, seven miles from Launceston in Cornwall. His father, Thomas Adams, was a tenant farmer, and his ancestors for at least four generations had been tenant farmers in or near Laneast. His mother, whose maiden name was Tabitha Knill Grylls, possessed a small estate which was bequeathed to her by her aunt, Grace Couch. She had also inherited her uncle's library, and these books, which included some on astronomy, were Adams's early companions. He was the eldest of seven children. His brother Thomas, born April 28, 1821, was a missionary in Tonga and completed the translation of the Bible into the Tongan language: he died in 1885. His brother George, born November 5, 1823, assisted his father at Lidcot and became a farmer. His youngest brother, William Grylls Adams, born February 16, 1836, is the editor of this volume. He had three sisters who all died before him. From his mother, who belonged to a musical family, he inherited a correct ear and a love of music. At a village school in Laneast he made rapid progress, and with the schoolmaster, Mr R. C. Sleep, as his fellow student he was learning algebra before he was ten years old. At the age of twelve he went to a private school at Devonport, kept by the Rev. John Couch Grylls, a first cousin of his mother.

He remained under Mr Grylls's tuition for several years, first at Devonport and afterwards at Saltash and Landulph, and received the usual school training in classics and mathematics. Astronomy had been his passion from very early boyhood, and at fourteen years of age he made copious notes and drew tiny maps of the constellations. He read with avidity all the astronomical books to which he could obtain access, and in particular he studied the astronomical articles in Rees's *Cyclopædia*, which he met with in the library of the Devonport Mechanics' Institute, where he used to spend his spare time in reading astronomy and mathematics. In the same library he came across a copy of Vince's *Fluxions*, which was his first introduction to the higher mathematics.

The intense interest which as a boy he felt in all astronomical questions is shown by the number of carefully written out manuscripts, belonging to this period, which exist among his papers, as well as by his letters to his parents and brothers. Some of the manuscripts are copies from books, others contain calculations of his own. On October 17, 1835, he wrote from Landulph to his parents telling them that he had watched for the comet three weeks before without success, and that at last he had seen it: "you may conceive with what pleasure I viewed this, the first comet I had ever had a sight of, which at its visit 380 years ago threw all Europe into consternation, but which now affords the highest pleasure to astronomers by proving the accuracy of their calculations and predictions." The annular eclipse of the sun of May 15, 1836, interested him greatly and on May 13 he wrote from Stoke a long letter to his brother Thomas at Lidcot in order to give him "a brief description of the large eclipse of the sun which will take place next Sunday." He proceeds "As the almanacs only give the time &c. to this eclipse for London and some other remarkable places, I have taken some pains to calculate it, and I herewith send you, what I believe has not been done for some time, a calculation of this eclipse for the meridian and latitude of Litcott." He finds that it will begin at 1 h. 28 m. p.m., that the greatest eclipse will be at 3 h. 0 m. and that it will end at 4 h. 22 m., the digits eclipsed being 10. He also gives a diagram showing the eclipse as it will appear from Lidcot. At the conclusion of the letter, he adds "There will also happen next Thursday evening between 6 and 7 o'clock a remarkable conjunction of the Moon and the planets Jupiter and Venus, which I wish you would observe. These planets are now approaching each other and will then be very near, as also will the moon." This early calculation of an eclipse (the manuscript of which still exists) is especially interesting in connexion with the remarkable theoretical calculations which he was to undertake and carry out so successfully only a few years later. On April 24, 1837, he wrote from Stoke "I observed the eclipse last Thursday with a small spy-glass which I borrowed: the moon looked most delightful after the end of the eclipse. At the request of Mr Bate, a young man of my acquaintance, who reports for the *Telegraph*, I wrote next morning a few lines on the eclipse, which were inserted in the paper the following day....Mr Richards, the editor of the *Telegraph*, tells me that my article on the eclipse has been copied into several of the London papers."

He was also interested in practical astronomy, and there was long preserved in the home at Lidcot a simple instrument constructed by him, when very young, in order to determine the elevation of the sun. It consisted of a vertical circular card with graduated edge, from the centre of which a plumb bob was suspended. Two small square pieces of card, with a pin-hole in each, projected from the circular disc at right angles to its face at opposite ends of a diameter. The card was to be so placed that the sun shone through the pin-holes, and the elevation was read off on the circle. It is also remembered that on the window sill at Lidcot he had made lines or notches to mark the positions of shadows at noon.

He showed such signs of mathematical power that in 1837 the idea of his going to Cambridge was entertained. He accordingly entered St John's College, Cambridge, in October, 1839. During his undergraduate career he was invariably the first man of his year in the college examinations, and in 1843 he graduated as Senior Wrangler, being also first Smith's Prizeman. In the same year he was elected Fellow of his college.

His attention was drawn to the irregularities in the motion of Uranus by reading Airy's report upon recent progress in astronomy in the Report of the British Association for 1831–32[1], and on July 3, 1841, he made the following memorandum:—"Formed a design at the beginning of this week of investigating, as soon as possible after taking my degree, the irregularities in the motion of Uranus which are yet unaccounted for, in order to find whether they may be attributed to the action of an undiscovered planet beyond it; and, if possible, thence to determine the elements of its orbit &c. approximately, which would probably lead to its discovery." This memorandum was made at the beginning of his second long vacation, when he had just entered upon his twenty-third year[2].

In 1843, the year in which he took his B.A. degree, he attempted a first rough solution of the problem on the assumption that the orbit was a circle with a radius equal to twice the mean distance of Uranus from the Sun. The result showed that a good general agreement between theory and observation might be obtained. In order to make the data employed more complete, application was made through Professor Challis, to Mr Airy, the Astronomer Royal, in February 1844, for the errors of the tabular geocentric longitudes of Uranus for 1818—1826, with the factors for reducing them to errors of heliocentric longitude. The Astronomer Royal at once supplied all the results of the Greenwich observations of Uranus from 1754 to 1830. Adams now undertook a new solution of the problem, taking into account the most important terms depending on the first power of the eccentricity of the orbit of the supposed disturbing planet, but retaining the same assumption as before with respect to the mean distance. In September, 1845, he gave to Professor Challis a paper containing numerical values of the mean longitude at a given epoch, longitude of perihelion, eccentricity of orbit, mass, and geocentric longitude for September 30, of the assumed planet. On September 22, 1845, Challis wrote a letter of introduction to the Astronomer Royal beginning, "My friend Mr Adams, who will probably deliver this note to you, has completed his calculations respecting the perturbation of the orbit of Uranus by a supposed ulterior planet, and has arrived at results which he would be glad to communicate to you, if you could spare him a few moments of your valuable time." Adams called at the Royal Observatory, Greenwich, in September, but the Astronomer Royal was absent in France. In the following month, on October 21, 1845, Adams called again at the Royal Observatory, and not being successful in seeing the Astronomer Royal, left a paper giving the following values of the mass and orbit of the new planet:—

Mean distance (assumed nearly in accordance with Bode's law)	38·4
Mean sidereal motion in 365·25 days	1° 30′ 9″
Mean longitude, 1st October, 1845	323° 34′
Longitude of perihelion	315° 55′
Eccentricity	0·1610
Mass (that of the Sun being unity)	0·0001656

The paper which he left on this occasion also contained a list of the residual

[1] This report does not contain any reference to the possibility of the irregularities being due to an undiscovered exterior planet. It is merely mentioned that it seems impossible to unite all the observations in one elliptic orbit, and that Bouvard was therefore obliged to reject the ancient observations entirely (Report, p. 154).

[2] The original memorandum, written by itself on a slip of paper, is reproduced in facsimile facing p. liv.

errors of the mean longitude of Uranus, after taking account of the disturbing effect of the new planet, the errors being small except in the case of Flamsteed's observation of 1690[1].

On November 10, 1845, Le Verrier presented to the French Academy an elaborate investigation of the perturbations of Uranus produced by Jupiter and Saturn, in which he pointed out several small inequalities which had previously been neglected. After taking these into account he still found that the theory was quite incapable of explaining the observed irregularities of the motion of Uranus.

On June 1, 1846, Le Verrier presented to the French Academy his second memoir on the theory of Uranus. After reducing afresh nearly all the existing observations, he came to the conclusion that there was no other possible explanation of the discordances except that of a disturbing planet exterior to Uranus. He investigated the elements of the orbit of such a planet, and assuming its mean distance to be double that of Uranus, and its orbit to be in the plane of the ecliptic, he gave as the most probable result that the value of the true longitude of the disturbing body for January 1, 1847 was about 325°, and that it was not likely that this place was in error by so much as 10°. Neither the elements of the orbit nor the mass of the planet were given.

The position thus assigned by Le Verrier to the disturbing planet differed by only 1° from that given by Adams in the paper which he had left at the Royal Observatory more than seven months before. As will be mentioned subsequently, Le Verrier's third memoir, containing the elements of the orbit, was communicated to the French Academy on August 31, 1846.

On July 9, 1846 the Astronomer Royal, who was then staying with Dean Peacock at Ely, wrote a letter to Challis suggesting that search should be made for the new planet with the Northumberland Equatorial at Cambridge, and offering to supply him with an assistant if he were unable himself to make the examination; and on July 13 he transmitted to Challis a paper of suggestions with respect to the proposed sweep for the planet, which was to extend over a part of the heavens 30° long in the direction of the ecliptic, and 10° broad, having the theoretical place of the planet as its centre. On July 18, Challis, who had been absent from Cambridge, replied to these communications, stating that he had determined to sweep for the hypothetical planet himself, and that he should therefore not require the services of an assistant. The actual search for the planet was commenced by Challis with the Northumberland telescope on July 29, 1846, three weeks before the planet was in opposition, and the observations were continued steadily until September 29. The plan adopted was to make three sweeps over the whole zone, completing one sweep before commencing the next, and mapping the positions of the stars. When the observations were completed, a planet could be at once detected by its motion in the interval. For the first few nights the telescope was directed to the part of the zone in the immediate neighbourhood of the place indicated for the planet by theory.

On September 2, in a letter to the Astronomer Royal, Challis said that he had lost no opportunity of searching for the planet, and that the nights being pretty good he had

[1] A facsimile of this paper is given after p. liv.

taken a considerable number of observations, but that his progress was slow as he thought it right to include all stars to the 10–11 magnitude. He found that to scrutinise thoroughly, according to his plan, the proposed part of the heavens would require more observations than he could take in the year. On the same day Adams wrote to the Astronomer Royal a letter, the opening paragraphs of which are as follows: "In the investigation, the results of which I communicated to you last October, the mean distance of the supposed disturbing planet is assumed to be twice that of Uranus. Some assumption is necessary in the first instance, and Bode's law renders it probable that the above distance is not very remote from the truth: but the investigation could scarcely be considered satisfactory while based on anything arbitrary; and I therefore determined to repeat the calculation, making a different hypothesis as to the mean distance. The eccentricity also resulting from my former calculations was far too large to be probable; and I found that although the agreement between theory and observation continued very satisfactory down to 1840, the difference in subsequent years was becoming very sensible, and I hoped that these errors as well as the eccentricity might be diminished by taking a different mean distance. Not to make too violent a change, I assumed this distance to be less than the former value by about $\frac{1}{30}$th part of the whole. The result is very satisfactory, and appears to show that, by still further diminishing the distance, the agreement between the theory and the later observations may be rendered complete, and the eccentricity reduced at the same time to a very small quantity. The mass and the elements of the orbit of the supposed planet, which result from the two hypotheses, are as follows:—

	Hypothesis I. $\left(\frac{a}{a'}=0\cdot5\right)$	Hypothesis II. $\left(\frac{a}{a'}=0\cdot515\right)$
Mean Longitude of Planet, 1st October, 1846 ...	325° 8′	323° 2′
Longitude of Perihelion	315° 57′	299° 11′
Eccentricity	0·16103	0·12062
Mass (that of Sun being 1)	0·00016563	0·00015003."

Adams also gave the errors of mean longitude, exhibiting the difference between theory and observation on the two hypotheses, and, after pointing out that the errors given by the Greenwich Observations of 1843 are very sensible on both hypotheses, he proceeds: "By comparing these errors it may be inferred that the agreement of theory and observation would be rendered very close by assuming $\frac{a}{a'}=0\cdot57$, and the corresponding mean longitude on October 1, 1846, would be about 315° 20′, which I am inclined to think is not far from the truth. It is plain, also, that the eccentricity corresponding to this value of $\frac{a}{a'}$ would be very small." In consequence of the divergence of the results of the two hypotheses, Adams asked for two normal places near the oppositions of 1844 and 1845. In the Astronomer Royal's absence on the Continent, these were sent by Mr Main; and on September 7 Adams wrote: "I hope by to-morrow to have obtained approximate values of the inclination and longitude of the node."

Two days earlier, on August 31, 1846, Le Verrier had presented to the French

Academy his third paper on the motion of Uranus, in which he gave the following elements of the disturbing planet:

Semi-axis Major	36·154 $\left(\text{or } \frac{a}{a'} = 0·531\right)$
Periodic Time	217·387
Eccentricity	0·10761
Longitude of Perihelion	284° 45′
Mean Longitude, 1st January, 1847	318° 47′
Mass	$= \frac{1}{9300} = 0·0001075$
True Heliocentric Longitude, 1st January, 1847	326° 32′
Distance from the Sun	33·06

and also comparisons between theory and observation. The paper also contained a detailed investigation, the object of which was to restrict as far as possible the limits within which the planet should be sought. Le Verrier concluded that it would have a visible disc and sufficient light to make it conspicuous in ordinary telescopes. The number of the *Comptes Rendus* containing this paper could not reach this country until the third or fourth week in September. Le Verrier communicated his principal conclusions to Dr Galle, of the Berlin Observatory, in a letter which was received by him on September 23, 1846. The same evening Dr Galle examined the heavens, comparing the stars with Bremiker's map (Hora XXI of the Berlin Academy's star maps). He soon found a star of about the eighth magnitude, nearly in the place pointed out by Le Verrier, which did not exist on the map. There could be little doubt that this was the new planet, and the observations made on the following day showed that its motion was nearly the same as that of the predicted planet. The discovery of the planet was due, not to its disc, but to its absence as a star on Bremiker's map. The existence of this map, which had been but lately published, was unknown to the English astronomers. On October 1 Challis heard of the discovery of the planet at Berlin. He then found that he had actually observed it on August 4 and August 12, the third and fourth nights of his search, so that if the observations had been compared with each other as the work proceeded, the planet might have been discovered by him before the middle of August. When the search was discontinued, on October 1, Challis had recorded 3150 positions of stars and was making preparations for mapping them[1].

Adams's researches, therefore, preceded Le Verrier's by a considerable interval; and, in spite of the delay in commencing the search, it had been carried on at Cambridge

[1] Even as it was, the planet was nearly discovered by the middle of August. Challis used two methods of observation, one with telescope fixed and the other with telescope moving. On July 30, the second day of the search, he observed by the second of these methods, and on August 12, the fourth day of the search, he observed the same zone by the first method. Shortly afterwards he compared the observations of these days, in order to verify the adequacy of his course of procedure, and as far as the comparison was carried, he found that the positions of July 30 included all those of August 12. After the discovery of the planet, Challis, continuing this comparison, found that No. 49, a star of the 8th magnitude in the series of August 12, was wanting in the series of July 30. This was the planet, which had entered the zone between July 30 and August 12. The former comparison had not bsen continued beyond No. 39 "probably from the accidental circumstance that a line was there drawn in the memorandum-book in consequence of the interruption of the observations by a cloud."

for eight weeks before the planet was found at Berlin. Adams's first complete investigation may be regarded as having been finished on October 21, 1845, when he left his paper at the Royal Observatory. This was three weeks before Le Verrier presented to the French Academy his first memoir, in which it was shown that the irregularities in the motion of Uranus could not be attributed to the known planets, and seven months before the date of presentation of his second memoir in which he first investigated the orbit of the supposed disturbing planet. As we know, Adams had resolved to undertake the work in 1841, and his first rough solution was effected, as soon as he had leisure, in 1843. We may presume that Le Verrier did not attempt to determine the position or orbit of the disturbing planet until after the completion of his memoir of November 10, 1845.

The discovery of the actual planet by Dr Galle, in consequence of Le Verrier's prediction, was received with the greatest enthusiasm by astronomers of all countries, and the planet was at once called "Le Verrier's Planet." Adams's work was only known to the Astronomer Royal, Challis, and a few other persons, chiefly private friends. The first public mention of Adams's name occurred in a letter to the *Athenæum* from Sir J. Herschel, which appeared under the heading "Le Verrier's Planet" in the number for October 3, 1846. In this letter, which is dated October 1, Herschel refers to the address he had delivered on September 10, on the occasion of resigning the Presidential Chair of the British Association at Southampton, in which, after referring to the astronomical events of the year, which included the discovery of a new minor planet, he added: "It has done more. It has given us the probable prospect of the discovery of another. We see it as Columbus saw America from the shores of Spain. Its movements have been felt, trembling along the far-reaching line of our analysis, with a certainty hardly inferior to that of ocular demonstration."

To justify the confidence which these words express, Herschel first describes a conversation with Bessel in 1842, in which the latter had said that it was highly probable that the deviations of Uranus might be due to an unknown planet (being systematic, and such as an exterior planet would produce), and then proceeds:—

"The remarkable calculations of M. Le Verrier, which have pointed out, as now appears, nearly the true situation of the new planet by resolving the inverse problem of the perturbations—if uncorroborated by repetition of the numerical calculations by another hand, or by independent investigation from another quarter—would hardly justify so strong an assurance as that conveyed by my expressions above alluded to. But it was known to me at that time (I will take the liberty to cite the Astronomer Royal as my authority) that a similar investigation had been independently entered into, and a conclusion as to the situation of the new planet very nearly coincident with M. Le Verrier's arrived at (in entire ignorance of his conclusions) by a young Cambridge mathematician, Mr Adams, who will, I hope, pardon this mention of his name (the matter being one of great historical moment), and who will doubtless in his own good time and manner, place his calculations before the public."

This passage seems to have passed almost unnoticed by astronomers, in the excitement produced by Le Verrier's discovery, and it was not till October 17, when a letter from Challis appeared in the *Athenæum*, giving an account of the proceedings at Cambridge in connexion with the new planet, that general attention was directed to Adams's calculations. It was then known for the first time that his conclusions had been in the hands of the Astronomer Royal and Challis since 1845, and that the latter had actually been engaged in searching for the planet. There was naturally a disinclination to give full credit to facts thus suddenly brought to light at such a time. It was startling to realise that the Astronomer Royal had had in his possession the data which would have enabled the planet to have been discovered nearly a year before. On the other hand, it seemed extraordinary that a competent mathematician, who had determined the orbit of the disturbing planet, should have been content to refrain for so long from making public his results. No time was now lost in bringing the evidence before the world. On November 13, 1846, the Astronomer Royal communicated to the Royal Astronomical Society an "Account of some Circumstances historically connected with the Discovery of the Planet exterior to Uranus"; and Challis also described the observations which he had undertaken in search of the planet. At the same meeting Adams communicated a memoir containing an account of his mathematical investigations in connexion with the determination of the mass, orbit, and position of the new planet, by which he had obtained the elements communicated to the Astronomer Royal on October 21, 1845, and September 2, 1846. All of these papers are published in Vol. XVI. of the *Memoirs* of the Society; but as it was felt that the immediate publication of Adams's memoir was a matter of national interest, it was at once printed separately by Lieut. Stratford, superintendent of the *Nautical Almanac* Office, as a special appendix to the *Nautical Almanac* for 1851, and widely circulated at the beginning of 1847. This appendix was also issued as a supplement to No. 593 (March 2, 1847) of the *Astronomische Nachrichten.*

Having thus given in chronological order an outline of the main facts relating to the discovery of the new planet, it remains to describe in more detail some of the incidents which, apart from their historical interest, are of importance in connexion with the discussions which have taken place on the subject.

At the time of Adams's first visit to the Royal Observatory, in September, 1845, the Astronomer Royal was abroad. On the occasion of the second visit, on October 21, 1845, he was engaged, and was unable to see Adams, who therefore left at the Observatory the paper containing the elements of the planet. Fifteen days afterwards, on November 5, 1845, the Astronomer Royal wrote to Adams, "I am very much obliged by the paper of results which you left here a few days since, showing the perturbations on the place of Uranus produced by a planet with certain assumed elements. The latter numbers are all extremely satisfactory: I am not enough acquainted with Flamsteed's observations about 1690 to say whether they bear such an error, but I think it extremely probable. But I should be very glad to know whether this assumed perturbation will explain the error of the radius vector of Uranus. This error is now very considerable, as you will be able to ascertain by comparing the normal equations, given in the Greenwich observations for each year, for the times *before* opposition with the times *after* opposition." Unfortunately Adams did not reply to this enquiry or communicate again with the Astronomer Royal until September 2, 1846, when he forwarded to him the results of his second investigation.

Le Verrier's memoir of June 1, 1846, reached the Astronomer Royal about the 23rd or 24th of June, and on June 26th the latter addressed to Le Verrier the following letter, containing the same question with respect to the radius vector which he had previously put to Adams: "I have read with very great interest the account of your investigation on the probable place of a planet disturbing the motions of Uranus, which is contained in the *Compte Rendu de l'Académie* of June 1; and I now beg leave to trouble you with the following question. It appears, from all the later observations of Uranus made at Greenwich (which are most completely reduced in the Greenwich observations of each year so as to exhibit the effect of an error either in the tabular heliocentric longitude, or the tabular radius vector), that the tabular radius vector is considerably too small. And I wish to inquire of you whether this would be a consequence of the disturbance produced by an exterior planet, now in the position which you have indicated? I imagine that it would not be so, because the principal term of the inequality would probably be analogous to the moon's variation, or would depend on $\sin 2(v - v')$; and in that case the perturbation in radius vector would have the sign $-$ for the present relative position of the planet and Uranus. But this analogy is worth little until it is supported by proper symbolical computations."

Le Verrier replied to the Astronomer Royal's enquiry on June 28. In this letter he says, "Je compte avoir terminé la rectification des éléments de la planète troublante avant l'opposition qui va arriver; et parvenir à connaître ainsi les positions du nouvel astre avec une grande précision. Si je pouvais espérer que vous aurez assez de confiance dans mon travail pour chercher cette planète dans le ciel je m'empresserais, Monsieur, de vous envoyer sa position exacte, dès que je l'aurai obtenue." He then explains that the errors in radius vector are well accounted for by the disturbing planet.

On June 29, before Le Verrier's reply had been received, a meeting of the Board of Visitors of the Royal Observatory took place, at which Sir J. Herschel and Challis, among others, were present. In the course of a discussion the Astronomer Royal referred to the probability of shortly discovering a new planet, giving as his reason the very close coincidence between the results of Adams's and Le Verrier's positions of the supposed disturbing planet. It was in consequence of this opinion that Herschel felt justified in speaking so confidently of the approaching discovery in his address at Southampton on September 10.

When the planet was discovered at Berlin, the Astronomer Royal was on the continent, and on his return to Greenwich he wrote to Le Verrier, on October 14, 1846: "I was in Germany at the latter part of the month of September, when I received the intelligence of the actual discovery of the new planet whose place had been so clearly pointed out by you. And I beg you to accept my sincere congratulations on this successful termination to your vast and skilfully directed labours. Not many days past, I was in company with Professor Schumacher of Altona, and there I had the pleasure of reading the manuscript paper which you have transmitted to him. I was exceedingly struck with the completeness of your investigations. May you enjoy the honours which await you! and may you undertake other work with the same skill and the same success, and receive from all the enjoyment which you merit! I do not know whether you are aware that collateral researches had been going on in England, and that they had led to precisely the same result as yours. I think it probable that I shall be called on to give an account of these. If in this I shall give praise to others, I beg that you will not consider it as at all interfering with my acknowledgment of your claims. You are to be recognised beyond doubt as the real predicter of the planet's place. I may add that the

English investigations, as I believe, were not quite so extensive as yours. They were known to me earlier than yours." The rest of the letter relates to the name proposed for the new planet.

Le Verrier's reply, of October 16, was written under a sense of injustice and irritation produced by Herschel's letter in the *Athenæum*, which he considers "bien mauvaise et bien injuste pour moi." He feels very much hurt that Herschel should have said that he should not have felt justified in expressing himself so confidently at Southampton if his results had not been independently corroborated by Adams's work. He gives a succinct account in historical order of his own publications on the subject, and, in connexion with the paper of June 1, 1846, refers to Airy's letter of June 26, 1846, which he says shows that at that time Airy had no precise information with respect to the position of the planet, and that he was even surprised that he (Le Verrier) had placed it where he had, "parce qu'ainsi située elle ne lui paraissait pas rendre compte des inexactitudes du rayon vecteur." With reference to Adams he writes, "Pourquoi Mr Adams aurait-il gardé le silence depuis quatre mois? Pourquoi n'aurait-il parlé dès le mois de juin s'il eût eu de bonnes raisons à donner? Pourquoi attend-on que l'astre ait été vu dans les lunettes?" He appeals to Airy to defend his rights, and states that he has documents to prove that on September 28 and 29 Challis was still searching for the planet "sur mes indications." The Astronomer Royal's reply to this letter contained a statement of the facts with regard to Adams's work and the search for the planet.

The French astronomers were at first very unwilling to admit that Adams had any rights whatever in connexion with the planet, either as an independent discoverer or otherwise: and Arago, the secretary of the Academy, was especially violent in his denunciations. Le Verrier, who had at first inclined to the name of Neptune for the planet, delegated the right to name it to Arago, who insisted that it should be called Le Verrier. It is unnecessary to enter further into the discussions which took place on this subject: a very fair view of the whole matter was taken by Biot, and ultimately the name of Neptune was adopted by general consent.

Strange as it may seem, the course of events in this country was somewhat similar, it being contended by some English astronomers that the fact that Adams's results had not been publicly announced deprived him of all claims in relation to the discovery. The recognition of the merit of Adams's researches was mainly due to the warm and generous advocacy of two Cambridge men, Sedgwick and Sheepshanks.

Adams's determination of the orbit of the new planet was completed by October 1845, and by this date his results were in the possession of Challis and the Astronomer Royal, and yet no announcement whatever was made with respect to them until October 3, 1846. It is a most striking fact in the history of science that researches of such novelty and importance could have been known to two official astronomers besides their author for nearly a year without any steps being taken to make them public. The causes which produced this result are necessarily peculiar, and require to be examined in some detail.

Adams, having completed his determination, took the results in person to the Royal Observatory, in the hope that steps would forthwith be taken to find the planet. He was disappointed at not seeing the Astronomer Royal, and probably had expected more encouragement than the letter he received a fortnight afterwards with the enquiry relative to the radius vector. Regarding this as a matter of trifling importance, he delayed to reply to it, and applied himself to his second calculation with a different mean distance. With respect to Challis, he has explained in his report to the Cambridge Observatory Syndicate[1] that it might reasonably be supposed that the position of the planet was only roughly determined, and that a search for it must necessarily be long and laborious. In 1845, when Adams had completed his calculations, the planet was considerably past opposition, and Challis had no thought of commencing the search then. The succeeding interval until June 1846 was occupied with observations of the planet Astræa, Biela's double comet, and several other comets, and during this period he had little communication with Adams respecting the new planet. Attention was again called to the matter by Le Verrier's paper of June 1, and, as has been stated, the search was commenced on July 29.

From the Astronomer Royal's "Account &c." we learn that he attached great importance to the explanation of the error in radius vector. After giving the letter which he addressed to Adams on this subject he states that he considered the establishment of the error of the radius vector of Uranus to be a very important determination and proceeds, "I therefore considered that the trial, whether the error of radius vector would be explained by the same theory which explained the error of longitude, would be truly an *experimentum crucis*. And I waited with much anxiety for Mr Adams's answer to my query. Had it been in the affirmative I should have exerted all the influence which I might possess, either directly, or indirectly through my friend Professor Challis, to procure the publication of Mr Adams's theory. From some cause with which I am unacquainted, probably an accidental one, I received no immediate answer to this enquiry. I regret this deeply for many reasons. While I was expecting more complete information on Mr Adams's theory, the results of a new and most important investigation reached me from another quarter." This refers to Le Verrier's paper of June 1, 1846, after giving an account of which, the Astronomer Royal proceeds: "This memoir reached me about the 23rd or 24th of June. I cannot sufficiently express the feeling of delight and satisfaction which I received from it. The place which it assigned to the disturbing planet was the same, to one degree, as that given by Mr Adams's calculations which I had perused seven months earlier. To this time I had considered that there was still room for doubt of the accuracy of Mr Adams's investigations...But now I felt no doubt of the accuracy of both calculations, as applied to the perturbation in longitude. I was however still desirous, as before, of learning whether the perturbation in radius vector was fully explained."

Le Verrier replied to this enquiry in a letter from which some passages have already been quoted. With reference to Le Verrier's explanations regarding the error of radius vector the Astronomer Royal writes: "It is impossible, I think, to read this letter without being struck with its clearness of explanation, with the writer's extraordinary command, not only of the physical theories of perturbation, but also of the geometrical theories of the deduction of orbits from observation, and with his perception that his theory *ought* to explain all the phenomena, and his firm belief that it had done so. I had no longer any doubt upon the reality and general exactness of the prediction of the planet's place." After describing the contents of Le Verrier's third paper, of August 31, 1846, the Astronomer Royal proceeds: "My analysis of this paper has necessarily been exceedingly imperfect, as regards the astronomical and mathematical parts of it; but I am sensible that in regard to another part it fails totally. I cannot attempt to convey to you the

[1] This report, on account of its importance, is reprinted *in extenso* on pp. xlix—liv.

impression which was made on me by the author's undoubting confidence in the general truth of his theory, by the calmness and clearness with which he limited the field of observation, and by the firmness with which he proclaimed to observing astronomers, 'Look in the place which I have indicated, and you will see the planet well.'...It is here, if I mistake not, that we see a character far superior to that of the able, or enterprising, or industrious mathematician: it is here that we see the philosopher."

Adams was not fortunate in the two astronomers to whom he communicated his results: neither of them gave to a young and retiring man the kind of help or advice that he should have received. Challis, a most conscientious and painstaking astronomer, had obtained for him the places of Uranus that he required, and written him a letter of introduction to the Astronomer Royal. Although quite appreciative of Adams's calculations, he was occupied with his own observatory work, and seems to have left the matter in the hands of Airy. He undertook the search for the planet when it was suggested to him by Airy, after the publication of Le Verrier's paper, and carried it out methodically and with scrupulous care, as was his practice in everything; and in course of time the planet would have been discovered: but he does not seem to have been alive to the importance of making known in a more public way than by communication to the Astronomer Royal the results which Adams had obtained. As professor in the University he should not have allowed a young Senior Wrangler, through modesty or diffidence or inexperience, to do such injustice to himself. It is evident that even if the planet had been discovered at Cambridge, the same difficulty would have had to be encountered as that which actually occurred in bringing Adams's claims before the world, as Le Verrier's work had been already published and his indications had been used in the search. Airy states that he regarded the question of the radius vector as an *experimentum crucis*, and waited with much anxiety for Adams's reply to his query. When he found that Le Verrier assigned nearly the same position to the planet as Adams, and when Le Verrier had explained to him that the error in radius vector was corrected, any doubt with respect to the quality of Adams's work, which the absence of a reply to his enquiry may have caused, must have been removed, and the time had clearly come to take some notice of the paper which had been in his possession for seven months. But though he mentioned the matter at the meeting of the Board of Visitors on June 29 and suggested the search to Challis on July 9, he took no steps, either directly or through Challis, to bring about the public announcement of Adams's results.

Of course Airy knew that Adams had Challis and possibly other Cambridge men to advise him with respect to publication. Challis was a man of gentle and kindly nature, but slow in action and wanting in initiative: Airy, however, was a man of vigorous character, and it seems unaccountable that he should have taken no steps to secure the publication of Adams's results, even after his correspondence with Le Verrier in June 1846[1]. The fact that no reply had been received to the radius vector question affords no adequate explanation; he could have written to Adams again or applied to Challis, if he still considered an answer essential.

It is easy to understand the "delight and satisfaction" which Airy as a mathematician may have received from Le Verrier's paper confirming Adams's place of the

[1] Sedgwick's letter, from which the interview with Adams is quoted on the next page, contains the following passage: "When it was found that Adams was confirmed by the fortunate Frenchman the facts ought to have been out without more delay. Was Adams ever so much as told that Le Verrier was at his heels? Our astronomers ought to have got up a flare in an instant."

planet, but one would have thought that at the same time he would have felt some regret that Adams's paper had remained so long untouched in his keeping, thus depriving this country and his own University of the merit of the first announcement. It is impossible not to contrast the admiration with which he received Le Verrier's published writings with the indifference shown towards Adams's still unpublished work. Adams was certainly as clearly convinced of the reality of the planet as Le Verrier, and whatever claims the latter has to the name of philosopher rather than mathematician apply equally to the former. It is difficult also to see how Airy could have felt justified in writing to Le Verrier, after the discovery of the planet, the words, "you are to be recognised beyond doubt as the real predicter of the planet's place."

It has been said, and truly, that it was no part of the Astronomer Royal's duty to search for a new planet, and that he had no telescope available for the purpose even if he had desired to do so: but Adams (who possibly acted on Challis's advice) cannot be much blamed for taking his paper to Greenwich, in hopes that the planet might be found in this country. Adams himself seems to have been content to leave the matter in the hands of the Astronomer Royal, and it is to be remarked that at that time he was not only the official head of Astronomy, but was much looked up to by Cambridge men as one who had recently given a great impulse to astronomical studies in the University, as professor and director of the Observatory[1].

When it became known in Cambridge that Airy and Challis had been in possession of results which would have enabled the planet to be discovered in 1845 a good deal of indignation was naturally felt at the apathy and incredulity with which Adams's work had been received. This led Sedgwick, an intimate friend of Airy, to write two letters on the subject, which are now in the archives of the Royal Observatory at Greenwich. The second of these letters, dated December 6, 1846, contains the following interesting passages.

"Adams, though a great philosopher in his way, has shown no worldly wisdom, indeed has acted like a bashful boy rather than like a man who had made a great discovery.

"Again, he was certainly wrong in not answering Airy's letter. How strange and how unfortunate! Surely he must have been ill advised on this point; but I will try to learn this from himself.

"Just as I had written so far, in came Adams, to return my call, and five minutes after in came Sheepshanks, who, after chatting for half an hour with his surplice on, went to drink tea at the Lodge. Adams remained and drank tea with me, and we have had a very long chat....

"(1) He called at the Observatory soon after his calculations were finished—the Astronomer Royal away—bad luck, but no blame anywhere—this was September 1845. (2) Called again (October, the same Autumn) and the Astronomer out—left his card—heard that Airy would return soon, and therefore left *word that he would call again.* (3) Did call again (I think in a little more than an hour) and was told that the

[1] Adams did at last contemplate publication, for he concludes his letter of September 2, 1846 to the Astronomer Royal with the words, "I have been thinking of drawing up a brief account of my investigation to present to the British Association," and in his letter of November 18, 1846 (p. xxviii) he states that he drew up such a paper but arrived at the meeting too late to present it.

Astronomer was at dinner; had no message, and therefore went away. But he added that he did *not* call by *appointment*. He only took his chance on his way back from Devonshire to Cambridge, &c. &c. I collected that he had been mortified (I am not using his own words) at receiving no message on the second call in October. 'I thought' (said he) 'that though he had been at dinner he would have sent me a message, or perhaps spoken a word or two to me: but I am now convinced that in fact he never knew of my second call—that the servant had not delivered my message along with my card.' These were mainly his words. I asked him whether the circumstances just mentioned had any influence in preventing his reply to Professor Airy's note. He said in answer, that had these not happened he possibly might have replied more readily; but assuredly had he considered the question about the radius vector as of great importance ('as an *experimentum crucis*') he should have answered the note instantly. 'But,' said he, 'I could not look on the corrections of the radius vector as an *experimentum crucis;* because any hypothesis (however wrong) which gave a correction in longitude must give a correction in the radius vector of the *same kind* as the correction deduced from the perturbations of the new planet' (I think I state this correctly). 'Again,' said he, 'I wanted to send my papers in good order to the Astronomer Royal. I went over all my calculations three times. I added a few terms, without changing my results. I was much interrupted, so it was my vacation before I could finish my last revision,' &c. &c. 'I lament very much that I did not immediately answer the first note. I ought to have answered it,' &c. &c. 'But,' he added, 'I did think that the Astronomer Royal would have communicated my results among his correspondents. I took all that for granted, and I thought it a publication,' &c. &c. He is anxious to have no misunderstanding with Airy. He spoke very earnestly on this subject, and expressed himself grieved at the ill-natured things that had been said."

The following letter from Adams to Airy was written five days after the meeting of the Royal Astronomical Society at which Airy's 'Account &c.' was read.

"St John's College,
18 *November*, 1846.

"Dear Sir,

"Allow me to thank you for your able, interesting, and impartial account of circumstances connected with the discovery of the new planet. I need scarcely say how deeply I regret the neglect of which I was guilty in delaying to reply to the question respecting the radius vector of Uranus, in your note of Nov. 5th, 1845.

"In palliation, though not in excuse of this neglect, I may say that I was not aware of the importance which you attached to my answer on this point, and I had not the smallest notion that you felt any difficulty on it, such as you subsequently mentioned to M. Le Verrier.

"For several years past, the observed place of Uranus has been falling rapidly more and more behind its tabular place. In other words, the real angular motion of Uranus is considerably *slower* than that given by the tables. This appeared to me to show clearly that the tabular radius vector would be considerably increased by any theory which represented the motion in longitudes, for the variation in the second member of the equation $r^2 \frac{d\theta}{dt} = \sqrt{\mu a (1-e^2)}$ is very small.

"Accordingly, I found that if I simply corrected the elliptic elements, so as to satisfy the modern observations as nearly as possible, without taking into account any additional perturbations, the corresponding increase in the radius vector would not be very different from that given by my actual theory. Hence it was that I was led to defer writing to you till I could find time to draw up an account of the method employed to obtain the results which I had communicated to you. More than once I commenced writing with this object, but unfortunately did not persevere. I was also much pained at not having been able to see you when I called at the Royal Observatory the second time, as I felt that the whole matter might be better explained by half-an-hour's conversation than by several letters, in writing which I have always experienced a strange difficulty.

"I entertained, from the first, the strongest conviction that the observed anomalies were due to the action of an exterior planet; no other hypothesis appeared to me to possess the slightest claims to attention.

"Of the accuracy of my calculations I was quite sure, from the care with which they were made, and the number of times I had examined them. The only point which appeared to admit of any doubt was the assumption as to the mean distance, and this I soon proceeded to correct. The work however went on very slowly throughout, as I had scarcely any time to give to these investigations except during the vacations.

"I could not expect however that practical astronomers, who were already fully occupied with important labours, would feel as much confidence in the results of my investigation as I myself did; and I therefore had our instruments put in order, with the express purpose, if no one else took up the subject, of undertaking the search for the planet myself, with the small means afforded by our Observatory at St John's.

"I remain, dear Sir,

"Yours very respectfully,

"J. C. ADAMS.

"I drew up a paper for the meeting of the British Association at Southampton, but did not arrive there in sufficient time to present it, as Section A closed its sittings one day earlier than I expected."

In connexion with Adams's researches on the new planet, and his omission to reply to Airy's enquiry[1], the following interesting extracts from a letter from Challis to Airy, of December 19, 1846, should also find a place here.

"In the *Athenæum* of Dec. 5 there was an article on the new planet, ably and fairly written in general, but so unjust with respect to Mr Adams's scientific merits, that I wrote a letter to the Editor, which is in the *Athenæum* of to-day...There is one point in the story which is in an unsatisfactory state. Why did not Adams answer your question? I know that he is extremely tardy about writing, and that he pleads guilty to this fault.

[1] In 1883, when the present writer was preparing the obituary notice of Challis for the Royal Astronomical Society, in reply to a question why he had not answered the Astronomer Royal's letter about the radius vector, Adams said, "I should have done so: but the enquiry seemed to me trivial."

He experiences also a difficulty, which all young writers feel more or less, in putting into shape and order what he has done, and well done, so as to convey an adequate idea of it to others by writing. After receiving your questions it occurred to him that it would be well for him to send you a full account of his methods of calculation, and that he might send the answer at the same time. I believe that nothing but procrastination in fulfilling this intention was the reason of his not sending an answer at all. I have always found him more ready to communicate orally than by writing. It will hardly be believed that before I began my observations I had seen nothing of his in writing respecting the new planet, except the elements which he gave me in September written on a small piece of paper without date.

"I first got an idea of the nature and value of his researches by an abstract which he drew up to produce at the meeting of the British Association at Southampton. The public would hardly take such a reason as that I have mentioned to be the true reason for his not answering your question, and I fear therefore a hiatus must remain in the history."

As the Astronomer Royal laid so much stress upon the explanation of the error of radius vector, regarding it as an *experimentum crucis* with respect to the value of Adams's calculations, and as his views upon the matter have been much criticised, it seems proper to quote the following explanatory passages which were written by him after he had received Adams's letter of November 18, and when the matter was attracting general attention. Writing to Sheepshanks on December 17, 1846, he says: "Concerning the radius vector of Uranus, the error was *certain* as to *sign*. It was determined with *reasonable accuracy* as to magnitude (perhaps the probable error might be $\frac{1}{6}$ or $\frac{1}{8}$ of the whole). Now, suppose that Adams's elements which gave longitude-corrections had given a wrong sign for the correction of the radius vector, what would his theory have been worth? The alternation of signs of errors + − in longitude does *not* exclude any other hypothesis than that of an exterior planet. If the law of force *differed slightly* from that of inverse square of the distance (of which two years ago there was great probability) and if tables were calculated strictly on the law of inverse square of distance (as was done in existing tables), then the discordances in longitude would have the alternate signs + − Le Verrier evidently attached great importance to the radius vector...The radius vector, as you say, was to be used as an indirect verification, but its error demanded explanation quite as imperatively as the other."

And writing to Challis, December 21, 1846, he says:

"I am sure that you cannot have a higher opinion of Adams's ability in the scientific parts of this matter than I have....But with regard to one part of your own published letter in the last *Athenæum*, I must make one remark[1]. There were two things to be explained, which might have existed each independently of the other, and of which one could be ascertained independently of the other: viz. error of longitude and error of radius vector. And there is no *à priori* reason for thinking that a hypothesis

[1] Challis had written: "Again, as to the error of the radius vector: it is quite impossible that its longitude could be corrected during a period of at least 130 years independently of correction of the radius vector....The investigation of one correction necessarily involves that of the other. Mr Adams actually employed a method of calculation which required him to compute the coefficients of the expression for error of radius vector, *before* computing the coefficients of the expression for error of longitude." (*Athenæum*, December 19, 1846.)

which will explain the error of longitude will *also* explain the error of radius vector. If, after Adams had satisfactorily explained the error of longitude, he had (with the numerical values of the elements of the two planets so found) converted his formulae for perturbation of radius vector into numbers, and if these numbers had been discordant with the *observed* numbers of discordance of radius vector, *then the theory would have been false, not* from any error of Adams's, *but* from a failure in the law of gravitation. On this question therefore turned the continuance or fall of the law of gravitation. This, it appears to me, has been totally overlooked in your letter. It was a question of vast importance.

"The progress of science almost always depends on questions of this kind. Thus, in Chemistry, the phlogistic theory explained the concurring facts of oxidation of metals and vitiation of air, or gaseous formation in water. But did it *also* account for the increased weight of the metal? No. Then it was false. Laplace's notion of forces gave an explanation of the course of extraordinary pencils of light. But did it or could it give an explanation also of the separation of pencils and of their polarisation? No. Then it was false.

"The theory of gravitation *might* have been in the same predicament with regard to Uranus. Adams's answer would have made this satisfactory.... What could be the reason of Adams's silence, I could not guess. It was so far unfortunate that it interposed an effectual barrier to all further communication. It was clearly impossible for me to write to him again."

Looking back now upon Adams's achievement, which, as has been truly said, belongs at once to the science and to the romance of astronomy, there are several points that stand out as very remarkable: his extreme youth when he attacked, unaided, so difficult a problem, and steadily carried it through to success; his complete faith in the Newtonian law and in the results of his own mathematics; and his extreme modesty. As soon as he took his degree in 1843 he devoted his whole leisure, in term time at Cambridge, and in vacations in Cornwall, to the new planet's orbit, without assistance or encouragement from anyone. How quietly and unassumingly he pursued his investigations is shown by the fact that at the time of the finding of the planet his name was only known to Airy, Challis, Herschel, Earnshaw, and a few intimate university friends of his own standing. He was perfectly convinced of the reality of the planet from the first, and of the approximate accuracy of the place he had assigned to it; and in the paper which he placed in the hands of Challis in September, 1845, he used the words "the new planet."

Although containing no new facts it may be well to conclude the account of Adams's researches on the new planet with the following extract from a letter written by him at the time (November 26, 1846) to Professor James Thomson:

"On considering the subject it appeared to me that by far the most probable hypothesis that could be formed to account for these irregularities was that of the existence of an exterior undiscovered planet whose action on *Uranus* produced the disturbances in question. None of the other hypotheses that had been thrown out seemed to possess the slightest claims to attention, as they were all improbable in themselves, and incapable of being tested by any exact calculation. Some had even supposed that, at the great distance of *Uranus* from the Sun, the law of attraction became different from that of the inverse square of the distance, but the law of gravitation was too firmly established for this to be admitted till every other hypothesis had failed to account for the observed irregularities; and I felt convinced that in this, as in all previous instances of the kind, the discrepancies which had for a time thrown doubts on the truth of the law would eventually afford it the most striking confirmation. In contrast with all these vague hypotheses, the supposition that the irregularities were caused by the action of an unknown planet appeared to be thoroughly in accordance with the present state of our knowledge, could be tested by calculation, and would probably lead to important practical results—viz. the approximate determination of the position of the disturbing body." After quoting the memorandum of July 3, 1841, he proceeds:—"Accordingly, in 1843, I commenced my calculations, and in the course of that year I arrived at a first solution of the problem, which, though incomplete in itself, fully convinced me that the hypothesis which I had formed was quite adequate to account for the observed irregularities, and that the place of the disturbing body might be very approximately determined by a more extended investigation. Having received from the Astronomer Royal, in February 1844, the whole of the Greenwich observations of *Uranus*, I accordingly attacked the problem afresh, and in a much more complete manner than before, and, after obtaining several solutions, differing little from each other, by gradually taking into account more and more terms in the series expressing the perturbations, I communicated my final results to Professor Challis in September 1845, and the same, slightly corrected, to the Astronomer Royal in the following month. The near agreement of the several solutions which I had obtained gave me great confidence in my results, which included a determination of the mass, position and elements of the orbit of the supposed planet."

Adams took no part whatever in the controversies or discussions which arose with regard to the discovery of the planet, either publicly or privately, and at no time in his life did he ever criticise the conduct of anyone, or say an unkind word in connexion with the matter. Fortunately all the facts relating to the calculations of Adams and Le Verrier and the discovery of the planet are undisputed, and any discussions that may take place in the future can have reference only to the conclusions to be drawn from them[1].

On the discovery of the planet the Royal Society at once awarded their highest honour, the Copley Medal, to Le Verrier (1846), and it was not till two years afterwards that it was awarded to Adams. The Royal Astronomical Society was saved from expressing a similar preference by the by-law requiring that the award of the medal should be confirmed by a majority of three-quarters of the Council. A sufficient minority were of opinion that "an award to M. Le Verrier, unaccompanied by another to Mr Adams, would be drawing a greater distinction between the two than fairly represents the proper inference from facts, and would be an injustice to the latter[2]."

[1] The principal contemporary publications relating to the new planet are to be found in Vol. XVI. of the *Memoirs of the Royal Astronomical Society*, in the *Comptes Rendus*, in the *Athenæum*, in the *Astronomische Nachrichten*, and in Vol. VII. (1847) of the *North British Review*, which contains an article by Brewster. A number of letters bearing upon the subject are contained in the Archives of the Royal Observatory, and Sheepshanks's correspondence is in the possession of the Royal Astronomical Society. Free use has been made of these documents in writing the account in the text. Challis's report to the Observatory Syndicate at Cambridge, which contains an account of his own proceedings relative to the new planet, is added as an appendix to this notice (pp. xlix—liv). References to the discovery of the planet occur in the *Life and Letters of Adam Sedgwick*, by Clark and Hughes, 1890, Vol. II. pp. 107 and 287.

[2] In an interesting letter to Schumacher, in the possession of the Royal Astronomical Society, Sheepshanks wrote as follows, under date April 7, 1847:—"You will be

The honours so freely and deservedly bestowed upon Le Verrier in France and other countries form a striking contrast to the general want of appreciation with which Adams's work was at first received. But there were conspicuous exceptions. In 1847, on the occasion of the Queen's visit to Cambridge, the honour of knighthood was offered to Adams, but this offer he felt obliged to decline. The members of St John's College, also, were not slow in showing their sense of the honour he had conferred upon his college and the University, for in a very short time a fund, producing about £80 per annum, was raised for establishing a prize to be connected with his name. This fund was offered to the University, and accepted on April 7, 1848. The Adams Prize, which is biennial, is awarded for the best essay on some subject of pure mathematics, astronomy, or other branch of natural philosophy.

A French translation of Adams's memoir on the motion of Uranus was published in Liouville's *Journal de Mathématiques pures et appliquées* for 1875. The editor, M. Résal, stated that he had been led to undertake this republication by the pressing solicitations of several eminent mathematicians. In introducing the memoir he writes:—"Le problème fut résolu simultanément, en Angleterre par M. Adams, et en France par M. Leverrier, qui, ainsi que le reconnaît M. Adams, a publié le premier les résultats de ses recherches. ...Il est impossible de rencontrer, dans l'histoire des sciences, une découverte qui fasse plus d'honneur au génie humain. Les lois de Newton recevaient ainsi la plus éclatante des confirmations, et l'Astronomie, désormais indiscutable dans ses principes, était arrivée à l'état de science parfaite. Le Mémoire de M. Adams a valu, à juste titre, à son auteur la plus glorieuse célébrité: il est digne, en effet, de figurer à côté des plus beaux mémoires de Laplace et Lagrange." This republication of the memoir, after an interval of thirty years, in a purely mathematical journal, derives additional interest from the fact that Adams added a few notes at the end, some of which relate to the objections made by Professor Benjamin Peirce to the legitimacy of the methods pursued by himself and Le Verrier. In Peirce's paper, which was published in 1847, it was contended that the period of Neptune differed so considerably from that of the hypothetical planet that the modes of procedure adopted were unreliable, so that the finding of the planet was partly due to a happy accident. In reply to this, Adams points out that the objection would be valid if the object in view had been to represent the perturbations of Uranus during

surprised when I tell you that the *strongest* opponents to Mr Adams's claims to consideration are to be found in England, of course with the exception of France. All acknowledge M. Le Verrier's merits, and all admit his *undoubted* claim to *independent* discovery. All are agreed, too, that in *making public his results and investigations* in the masterly and confident way he did, he deserves the highest praise. As to *national* feeling (which, by the way, is too often national injustice) there is absolutely none whatever, so far as I know, or among astronomers. In England at present the current runs the other way, and though I very much prefer this failing of the two, yet it is provoking too. I assure you that it was with difficulty that one could get a hearing, while pointing out the fact that Mr Adams had deduced the elements and place of the planet in October, 1845. I have been told repeatedly by those who should have known better that this was nothing at all, simply because the over-modest man communicated his results to Airy and Challis, that the planet might be looked for, instead of bringing his investigation before the world as he ought to have done. Surely it is a greater honour to science that two men should independently have come to the same conclusion from the same data than that one should have hit on it, as it were, accidentally. Thanks to Struve and Biot, &c. our anti-Adamites are calmer, and as there never was any opposition to Le Verrier, we are quite satisfied at present, and so I hope are the two discoverers. I think there is a hope that Mr Adams will continue his astronomical researches. In any other country there could be no doubt of it, but in England there is no *carrière* for men of science. The Law or the Church seizes on all talent which is not independently rich or careless about wealth."

two or three synodic periods, but that the case is different when, as in this instance, it was only required to represent the perturbations for a fraction of a synodic period.

Before leaving the subject of Neptune, it should be stated that Adams always expressed the warmest appreciation of Le Verrier's work. It was a great pleasure to him when they met at Oxford in 1847. In the same year Le Verrier visited Adams at Cambridge. The honorary degree of LL.D. was conferred upon Le Verrier in 1874 by the University of Cambridge, and it cannot be doubted that this was owing to the action of Adams. In 1876, when Adams was President of the Royal Astronomical Society for the second time, the gold medal was awarded to Le Verrier for his planetary researches. In delivering the medal Adams spoke of "the admiration we feel for the skill and perseverance by which he has succeeded in binding all the principal planets of our system from Mercury to Neptune in the chains of his Analysis."

In 1847 Adams communicated to the Royal Astronomical Society a paper on an important error in Bouvard's tables of Saturn. Having been engaged upon a comparison of the theory of Saturn with the Greenwich observations, he was struck with the magnitude of the tabular errors in heliocentric latitude, which could not be attributed to imperfections in the theory. He found that the error was one of computation, two terms of different arguments having been, in effect, united into one.

In 1848 he was occupied with the determination of the constants in Gauss's theory of terrestrial magnetism. This investigation he afterwards resumed, and the calculations connected with it, upon which he was engaged in the later years of his life, were left unfinished at the time of his death. When failing health prevented him from any longer giving his personal attention to the work, he placed the manuscripts in the hands of his brother, Professor W. G. Adams, for completion.

In 1851 he was elected President of the Royal Astronomical Society, and held the office for the usual term of two years. As president he delivered the addresses on the presentation of the medal to Peters and to Hind. In 1852 he communicated to the Society new tables of the Moon's parallax, to be substituted for those of Burckhardt. Henderson had compared the parallaxes deduced from observation with those derived by calculation from the tables both of Damoiseau and of Burckhardt, finding a difference of no less than $1''\cdot 3$, according as one set of tables or the other was employed. The parallax in Damoiseau's tables is given at once in the form in which it is furnished by theory, but that in Burckhardt's tables is adapted to his peculiar form of the arguments, and requires transformation in order to be compared with the former. When this was done, Adams found that several of the minor equations of parallax deduced from Burckhardt differed completely from their theoretical values as given by Damoiseau. He discovered that these errors were due to Burckhardt's transformations of Laplace's formula, and he succeeded in tracing them to their sources. He also examined carefully the theories of Damoiseau, Plana, and Pontécoulant, with respect to the same subject, and supplied a number of defects and omissions. Burckhardt's value of the parallax having been employed in the *Nautical Almanac*, Adams gave, in addition to the new tables, a table of corrections to be applied to the values in the *Nautical Almanac* for every day of the year from 1840 to 1855 inclusive. This contribution to astronomy is very characteristic of its author. It contains the results of a great amount of intricate and elaborate mathematical investigation, carried out with great skill and accuracy in all its details, both analytical and numerical, but no part of the work itself is given. The method of procedure is briefly sketched, and the final conclusions are stated in the fewest words and simplest manner possible. No one unacquainted with the subject would imagine how much careful research was represented by these few pages of results. The tables were printed as a supplement to the *Nautical Almanac* for 1856.

As Adams had not taken holy orders, his Fellowship at St John's College came to an end in 1852, but he continued to reside in the college until February 1853, when he was elected to a Fellowship at Pembroke College, which he retained till his death. In the autumn of 1858 he was appointed Professor of Mathematics in the University of St Andrews, and shortly afterwards, in the same year, he was elected Lowndean Professor of Astronomy and Geometry at Cambridge, in succession to Peacock. He continued his lectures at St Andrews, however, until the end of the session in May 1859. In 1861 he succeeded Challis as Director of the Cambridge Observatory. In 1863 he married Eliza, daughter of Haliday Bruce, Esq., of Dublin, who survives him.

In 1853 Adams communicated to the Royal Society his celebrated memoir on the secular acceleration of the Moon's mean motion. Halley was the first to detect this acceleration by comparing the Babylonian observations of eclipses with those of Albategnius and of modern times, and Newton referred to his discovery in the second edition of the *Principia*. The first numerical determination of the value of the acceleration is due to Dunthorne, who found it to be about 10″ in a century. Tobias Mayer obtained the value 6″·7, which he afterwards increased to 9″. Lalande's value was nearly 10″. The discrepancies were due to the eclipses selected, the results derived from the different eclipses being inconsistent with one another. The history of the theoretical investigations relating to the acceleration may be summed up as follows:—In 1762 the French Academy proposed as the subject of their prize the influence of a resisting medium upon the movements of the planets. The prize was won by Bossut, who showed that the principal effect of such a medium would be an acceleration in their motions, which would be much more sensible in the case of the Moon than in that of the planets. In 1770 the question proposed was whether the theory of gravitation could alone explain the acceleration. Euler obtained the prize, but he was unable to discover any term of a secular character, and concluded that the force of gravitation would not account for this inequality. The subject was proposed again in 1772, Euler and Lagrange sharing the prize between them. The former came to the same conclusion as before, attributing the acceleration to a resisting medium; the latter did not carry the application of his formulæ so far as to complete the investigation. The prize was again offered for the same subject in 1774, the competitors being required to examine whether the fact that the Moon appeared to have a secular acceleration, while there was no sensible effect of this kind in the case of the Earth, could be explained by the theory of gravitation alone, taking into account not only the action of the Sun and the Earth upon the Moon, but also the action of the other planets, and even the non-spherical figure of the Moon and Earth. The prize was awarded to Lagrange, who, after showing that none of the causes proposed would suffice to explain the secular variation of the Moon, concluded that, *if this variation is real*, it must be produced in some other manner, such as by a resisting medium. But as the existence of such a medium was not confirmed by the motions of the other planets, and was even contradicted by the motion of Saturn, which seemed to show a retardation, Lagrange expressed doubts with respect to the reality of the lunar acceleration, resting as it does on observations of eclipses in very remote ages. The next investigation relating to the subject is by Laplace, who showed that the acceleration could be accounted for by supposing that the transmission of the force of gravitation was not instantaneous, but that the rate of propagation was about eight million times that of light. Some years later, however, Laplace unexpectedly discovered the true gravitational cause of the acceleration. While working at the theory of Jupiter's satellites, he remarked that the secular variation of the eccentricity of Jupiter's orbit produced secular terms in their mean motions. Applying this result to the Moon, he found that the secular variation of the eccentricity of the Earth's orbit produced on the Moon's motion a secular term which agreed very well with the value assigned to it by observation; he found also that the same cause produced secular terms in the motion of the Moon's node and perigee. This result was communicated to the French Academy in November, 1787, and the memoir containing the details of the calculation was published in the following year. The Stockholm Academy of Sciences had already proposed in 1787 the secular variations of the Moon, Jupiter and Saturn as the prize subject for 1791, but no essays being sent in, the prize was adjudged to Laplace for his memoir published in 1788.

Laplace's discovery was received with general satisfaction, and the complete explanation of so intractable a variation by means of the Newtonian principles, after so many years of fruitless attempt, was an important event in the history of astronomy. The honour of the discovery might very easily have belonged to Lagrange, for the formulæ given by him in a memoir published in 1783 would at once, if applied to the Moon, have produced Laplace's result. But Lagrange had found that, in the case of Jupiter and Saturn, these formulæ gave nearly insensible values, so that he did not extend the investigation to the other planets, or to the Moon, although the latter application would only have involved easy numerical substitutions, much simpler than those required for the principal planets.

In 1820, at the instigation of Laplace, the lunar theory was taken in hand afresh by Plana and Damoiseau, the approximations being carried to an immense extent, especially by the former. Damoiseau calculated the acceleration numerically, and found it to be 10″·72. Plana's process was algebraical, and he carried the series, of which Laplace had only calculated the first term, as far as to quantities of the seventh order. By reducing to numbers the twenty-eight terms of this series he found 10″·58 as the complete value of the acceleration, the first term, which alone had been included by Laplace, giving 10″·18. Subsequently Hansen gave the values 11″·93 (1842), 11″·47 (1847); and in his tables published in 1857 he used the value 12″·18. It does not seem clear, however, to what extent these values are to be regarded as theoretical determinations.

Thus when Adams published his memoir in the *Philosophical Transactions* for 1853 no suspicion had arisen that Laplace's discovery was not absolutely complete, and that the question of the acceleration had not been finally set at rest. In this short paper of only ten pages Adams showed that the condition of variability of the solar eccentricity introduces into the solution of the differential equations a system of additional terms which affect the value of the acceleration. He found that the second term of the series on which the acceleration depends was really equal to $\frac{3771}{64}m^4$, instead of $\frac{2187}{128}m^4$, as found by Plana. The former is more than three times as great as the latter, and the amount of the acceleration is greatly decreased by the correction of this error. For some time the paper seems to have attracted no attention, but it then became the object of a long and bitter controversy. Plana, who was the person most concerned in the matter, published, in 1856, a memoir in which he admitted that his own theory was wrong upon this point, and he deduced Adams's result from his own equations. But shortly afterwards he retracted his admission, and, rejecting some of the new terms which he had obtained, arrived at a result which differed both from his original value and from Adams's. The question was in this state when Delaunay, by employing his own special method of treating the Lunar Theory and extending the investigation only to the fourth order, had the satisfaction of obtaining Adams's coefficient $\frac{3771}{64}$, a result which he brought before the French Academy in January, 1859. This caused Adams to communicate to the Academy, in the same month, the values which he had obtained some time before for the terms in m^5, m^6, and m^7; and he pointed out at the same time that, when these terms were included, the value of the acceleration was reduced to $5''\cdot78$, and, inferring that the remainder of the series would be nearly equal to $0''\cdot08$, he concluded that the total value of the acceleration was about $5''\cdot70$. Soon afterwards Delaunay carried his approximation as far as terms of the eighth order, and by reducing the forty-two terms in the analytical expression to numbers he obtained the value $6''\cdot11$. Delaunay's result, which was communicated to the Academy in April, 1859, confirmed the accuracy of Adams's values of the terms in m^5, m^6, and m^7, and also those of m^2e^2, and $m^2\gamma^2$, which Adams had communicated to him privately. A month after the publication of this paper Pontécoulant made a vigorous attack on the new terms introduced by Adams, which he said had been rightly ignored by Laplace, Damoiseau, Plana, and himself, as they had no real existence. He also objected that if the result of Adams were admitted, it would "call in question what was regarded as settled, and would throw doubt on the merit of one of the most beautiful discoveries of the illustrious author of the *Mécanique Céleste.*" Shortly afterwards he communicated a paper to the *Monthly Notices* of the Royal Astronomical Society on "the new terms introduced by Mr Adams into the expression for the coefficient of the secular equation of the Moon," in which he characterised the mathematical process by which these terms had been obtained as "une véritable supercherie analytique[1]." It would appear that Le Verrier did not accept Adams's value, for in presenting a note by Hansen to the Academy in 1860 he states that Hansen's tables afford an irrefragable proof of the accuracy of the value $12''$ which is there attributed to the acceleration. Referring then to the fact that according to Delaunay the secular acceleration should be reduced to $6''$ he proceeds: "Pour un astronome, la première condition est que ses théories satisfassent aux observations. Or la théorie de M. Hansen les représente toutes, et l'on prouve à M. Delaunay qu'avec ses formules on ne saurait y parvenir. Nous conservons donc des doutes et plus que des doutes sur les formules de M. Delaunay. Très certainement la vérité est du côté de M. Hansen[1]."

[1] Hansen stated in 1866 (*Monthly Notices*, xxvi. p. 187) that he had never disputed the correctness of Adams's theory, but that he was not satisfied with "the development of the divisors into series." If this refers to the expansion of the acceleration-coefficient in powers of m, it should be noticed that Adams stated (Vol. xxi. p. 15) that he had calculated the value of the acceleration by a method that did not require any expansion in powers of m, and found the result to be $5''\cdot70$. Hansen says that Adams's theory appeared too late to permit of his using it; "and it was well that it so happened, for I had already found by my own theory a coefficient which represents ancient eclipses as well as could be desired." It is therefore to be inferred that in this theory the new terms were omitted by Hansen, as they had been by Plana and Damoiseau.

In the *Monthly Notices* for April, 1860, Adams replied to his objectors, pointing out simply and clearly the errors into which they had fallen. He mentions that before publishing his memoir of 1853 he had obtained his result by two different methods, and that he had subsequently confirmed and extended it by a third. In a series of letters addressed to Lubbock in June, 1860, Plana began by objecting to Adams's value of the term in m^4, but he soon admitted its accuracy. Lubbock also was led to apply his own formulæ to the question, and he too arrived at Adams's result. Another calculation was made by Cayley, who, by an entirely different method, also obtained the same result. As Pontécoulant still continued his reiterated attacks upon the accuracy of the new terms, Cayley's calculation was printed *in extenso* in the *Monthly Notices*, where it occupies fifty-six pages. Delaunay had also made another calculation, in which, by following the method indicated by Poisson in 1833, he was led to the same value. The coefficient of m^4 had also been verified in 1861 by Donkin, who used Delaunay's method of the variation of the elements. Thus Adams's value of the term in m^4 was obtained by himself in three ways, by Delaunay in two ways, and by Lubbock, Plana, Donkin, and Cayley. Pontécoulant continued his attacks with no abatement of violence in the *Comptes Rendus*. Ultimately he abandoned Plana's value and obtained one of his own, which differed both from Adams's and Plana's.

The whole controversy forms a very extraordinary episode in the history of physical astronomy; the indifference with which the memoir of 1853 was at first received, in spite of the interest and importance of the subject, being followed by the violent controversy which resulted in so many independent investigations by which Adams's result was confirmed. It is not known why Laplace did not carry the calculation beyond the term in m^2; but it may be supposed that he regarded the subsequent terms as not likely to modify the value of the first term to any considerable extent. Damoiseau's and Plana's theories passed under the review of Laplace, and may be regarded as having received his sanction. Thus Adams's result not only unsettled a matter which after years of difficulty and struggling had apparently received its full and final explanation, but it detracted from the completeness of a discovery which had long been regarded as one of the greatest triumphs of Laplace's genius. Although the point in dispute relates entirely to the mathematical solution of differential equations, in which observation in no way entered, there can be no doubt that the fact that Plana's result agreed with observation, while Adams's did not, created in the minds of many a presumption against the accuracy of the latter. This view was certainly taken by Le Verrier in the passage quoted above, and it seems also to have influenced Hansen. It is curious that it should have been possible for so much difference of opinion to exist upon a matter relating only to pure mathematics, and with which all the combatants were fully qualified to deal, as is clearly shown by their previous publications. The whole controversy illustrates the peculiar nature of the lunar problem, and of the analysis by means of which the results are reached. The complete solution being unattainable by any of the methods which have as yet been applied, the skill of the mathematician is shown in selecting from a vast number of terms those which will produce a sensible influence in that particular portion of the complete solution which is under consideration.

A most admirable account of the whole discussion was given by Delaunay in the

Additions to the *Connaissance des Temps* for 1864, in which the place occupied by Adams's memoir in the history of gravitational astronomy is so well summed up that it may be permissible to quote the passage in its entirety:—

"L'apparition du mémoire de M. Adams a été un véritable événement: c'était toute une révolution qu'il opérait dans cette partie de l'astronomie théorique. Aussi le résultat qu'il renfermait fut-il vivement attaqué; on ne voulait pas l'admettre, et on ne manquait pas de raisons à donner pour cela. Il est, disait-on, en désaccord complet avec les observations; il ne tend à rien moins qu'à enlever à Laplace l'honneur d'une de ses plus belles découvertes; il est basé d'ailleurs sur une analyse fautive et erronée. Mais parmi toutes ces raisons il n'y en avait pas une bonne; et la persistance avec laquelle elles ont été présentées et soutenues a produit un effet diamétralement opposé à celui qu'on en attendait: les confirmations de ce résultat tant contesté se sont accumulées à un tel point, qu'il serait difficile de trouver dans les sciences une vérité mieux établie que ne l'est maintenant celle que M. Adams a mise en avant le premier dans son mémoire de 1853. Toutes les objections qui avaient été formulées sont tombées d'elles-mêmes. L'analyse déclarée *fautive et erronée* a été reconnue exacte. L'accord ou le désaccord du résultat théorique avec les indications fournies par les observations n'a plus été regardé comme un moyen de contrôler l'exactitude de ce résultat théorique. Si le désaccord annoncé existe bien réellement, on en conclut simplement que la cause assignée par Laplace à l'accélération séculaire du moyen mouvement de la Lune ne produit pas seule la totalité du phénomène et on ne trouve dans ce désaccord rien qui soit de nature à amoindrir la découverte de l'illustre géomètre français."

These sentences derive additional interest from the fact that they were written by one who was himself the author of the most comprehensive and elegant method by which the lunar problem has ever been treated, and who was the first to recognise the accuracy of Adams's result. In 1866 the Gold Medal of the Society was awarded to Adams for his contributions to the development of the Lunar Theory, the address on the occasion being delivered by Mr De la Rue. In the preparation of this very able address, which contains an excellent history of the problem of the secular acceleration, Mr De la Rue had the invaluable assistance of Delaunay. To complete the account of Adams's connexion with the secular acceleration, it should be stated that in 1880, thirty-seven years after Adams's memoir, Airy communicated to the Society a paper on the theoretical value of the acceleration (*Monthly Notices*, vol. xl. p. 368), in which he obtained the value of 10″·1477. At the next meeting of the Society Adams pointed out that in Airy's method of treatment certain terms were omitted, the effect being that the expression for the coefficient was reduced to its first term, so that the result necessarily agreed with Laplace's. Subsequently, taking into account these terms, Airy obtained the value 5″·4773. Adams took the occasion of the matter being thus again raised to communicate to the Society the investigation of the acceleration which he had been in the habit of giving in his lectures.

In the *Monthly Notices* for April 1867 Adams published an account of the results he had obtained with respect to the orbit of the November meteors. Professor H. A. Newton had concluded that these meteors belong to a system of small bodies describing an elliptic orbit about the Sun, and extending in the form of a stream along an arc of that orbit of such a length that the whole stream occupies about one-tenth or one-fifteenth of the periodic time in passing any particular point. He showed that the periodic time of this group must be either 180·0 days, 185·4 days, 354·6 days, 376·6 days, or 33·25 years, and that the node of the orbit must have a mean motion of 52″·4 with respect to the fixed stars. Soon after the remarkable display of the November meteors in 1866 Adams undertook the examination of this question. From the position of the radiant-point observed by himself he calculated the elements of the orbit of the meteors, starting with the supposition that the periodic time was 354·6 days, the value which Professor Newton considered to be the most probable one. The orbit which corresponds to this period is very nearly circular, and he found that the action of Venus would produce an annual increase of about 5″ in the longitude of the node, that of Jupiter about 6″, and that of the Earth about 10″. Thus the three planets, which alone could sensibly affect the motion of the node, would produce an increase of about 12′ in 33·25 years. The observed motion of the node is about 29′ in 33·25 years, which is therefore inconsistent with a periodic time of the meteors about the Sun of 354·6 days. If the periodic time were supposed to be about 377 days, the calculated motion of the node would differ very little from that in the case already considered, while if the periodic time were a little greater or a little less than half a year, the calculated motion of the node would be still smaller. Hence, of the five possible periods indicated by Professor Newton, four were incompatible with the observed motion of the node, and it only remained to examine whether the fifth period of 33·25 years would give a motion in accordance with observation. In order to determine the secular motion of the node in this orbit the method given by Gauss in his memoir *Determinatio Attractionis &c.* was employed. By dividing the orbit of the meteors into a number of small portions, and summing up the changes corresponding to these portions, the total secular changes of the elements produced in a complete period of the meteors was determined, the result being that during a period of 33·25 years, the longitude of the node is increased by 20′ by the action of Jupiter, nearly 7′ by the action of Saturn, and about 1′ by that of Uranus. The other planets were found to produce scarcely any sensible effects, so that the entire calculated increase of the longitude of the node is about 28′, agreeing very closely with the observed amount of 29′, and leaving no doubt as to the correctness of the period of 33·25 years. In order to obtain a sufficient degree of approximation it was requisite to break up the orbit of the meteors into a considerable number of portions, for each of which the attractions of the elliptic rings corresponding to the several disturbing planets had to be determined. These calculations were therefore of necessity very long, although a modification of Gauss's formula was devised which greatly facilitated its application to the actual problem. Subsequently certain parts of the orbit of the meteors were subdivided into still smaller portions, with the view of obtaining a closer approximation. Unfortunately the mathematical investigations which Adams carried out on this subject have not been published. They exist among his papers, together with a great amount of numerical work connected with the calculations.

In 1877 Mr G. W. Hill published a memoir on the motion of the Moon's perigee, in which he calculated that part of c which depends only upon m to fifteen places of decimals by a new method in which the expansion in powers of m was avoided, the numerical value of c being obtained by means of an infinite determinant. The publication of this memoir led Adams to communicate to the Royal Astronomical Society in November 1877 a brief notice of his own work in the same field, in which, after congratulating Mr Hill upon his investigation, he mentions that his own researches had followed in some respects a parallel course. In particular he remarks that the differential equation for z, the Moon's coordinate perpendicular to the ecliptic, presents itself naturally in the same form as that to which Mr Hill had so skilfully reduced his differential equations. In solving this equation, which was therefore of Mr Hill's standard form, he fell upon the same infinite determinant as that considered by Mr Hill, and developed it in a similar manner in a series of powers and products of small quantities, the coefficient of each such term being given in a finite form. This development was continued as far as the terms of the fourth order in 1868; and in 1875, when he resumed the subject, the approximation was extended to terms of the twelfth order, which is the same degree of accuracy as that to which Mr Hill had carried his researches. On making the reductions requisite in order to render the two results comparable, he found that they were in agreement with the exception of one of the terms of the twelfth order, and that this discrepancy was due to a simple error of transcription. He states that the calculations by which he had found the value of the determinant were very different in detail from those required by Mr Hill's method, but that he had not had time to copy them out from his old papers and put them in order. In this communication, therefore, he confined himself to making known the result which he had obtained for the motion of the Moon's node. After giving an outline of the method pursued, including the equation derived from the infinite determinant, he arrives at the formulæ by means of which the value of g, as dependent only upon m, was obtained to fifteen places of decimals.

It is difficult to appreciate too highly the mathematical ability shown by Adams and Hill in devising methods which did not require expansion in powers of m, and which yielded with such wonderful accuracy these values of g and c. Apart, however, from the mathematical and astronomical interest of the researches themselves, the coincidence of methods and ideas is very striking. But for the publication of Hill's memoir it is probable that no account of these results of Adams's would have been published in his lifetime, and it is not unlikely that he would never have put into writing his views on the mathematical treatment of the lunar problem which give additional interest to this short paper. As far back as 1853, in his memoir upon the secular acceleration, he mentioned that the new terms in the expression of the Moon's coordinates occurred to him some time before, when he was engaged in thinking over a new method of treating the lunar theory, and it is well known that the theory itself, or problems connected with it, constantly occupied his attention. In this paper of 1877 he states that he had long been convinced that the most advantageous mode of treatment is by first determining with all possible accuracy the inequalities which are independent of e, e', and γ, and then in succession finding the inequalities which are of one dimension, two dimensions, and so on with respect to these quantities. Thus, the coefficient of any inequality in the Moon's coordinates would be represented by a series arranged in powers and products of e, e', and γ; and each term in this series would involve a numerical coefficient which is a function of m alone, and which admits of calculation for any given value of m without the necessity of developing it in powers of m. This method is particularly advantageous when the results are to be compared with those of an analytical lunar theory such as Delaunay's, in which the eccentricities and the inclination are left indeterminate, since each numerical coefficient admits of a separate comparison with its analytical development in powers of m. He mentions also that, many years before, he had obtained the values of the inequalities independent of the eccentricities and inclination to a great degree of approximation, the coefficients of the longitude and those of the reciprocal of the radius vector, or of the logarithm of the radius vector, being found to ten or eleven places of decimals. Adams always preferred to treat the lunar theory as far as possible by means of its special problems; and this was also the method which he followed in his Cambridge lectures.

In 1878 he published a short paper on a property of the analytical expression for the constant term in the reciprocal of the Moon's radius vector. Plana had found that the coefficients of e^2 and γ^2 in this term vanished when account was taken of terms involving m^2 and m^3, and Pontécoulant, who carried the development further, had found that this destruction of the terms in the coefficients still continued when the terms involving m^4 and m^5 were included. Thinking it probable that these cases in which the coefficient had been observed to vanish were merely particular cases of some more general property, Adams was led to consider the subject from a new point of view, and, so far back as 1859, he succeeded in proving that not only did these coefficients necessarily vanish identically, but that the same held good also for coefficients which were much more general, so that the coefficients of $e^2e'^2$, $e^2e'^4$, &c. $\gamma^2e'^2$, $\gamma^2e'^4$, &c. were also identically equal to zero. Further reflection on the subject led him in 1868 to obtain a simpler and more elegant proof of the property in question. He also obtained subsequently, in 1877, some very simple relations connecting the coefficients of e^4, $e^2\gamma^2$, and γ^4. Of this theorem he says himself that it "is remarkable for a degree of simplicity and generality of which the lunar theory affords very few examples." We thus see that a striking result—and one moreover which admitted of being isolated from the rest of the lunar theory—was obtained in 1859, but was not published till nearly twenty years afterwards, although in the meantime he had obtained another and more satisfactory proof. This illustrates the disinclination that Adams seems always to have felt to prepare his work for publication; a disinclination which was mainly due to his desire to obtain a still higher degree of simplification or perfection. The discovery of the additional relations in 1877 shows that his attention was at that time still occupied with the theorem of 1859.

It may be remarked that Adams's shorter papers deserve more attention than their mere length might seem to entitle them to, not only because they frequently consist wholly of results derived from laborious researches, but also because they afford glimpses of the nature and extent of the work with which he was occupied. For forty-five years his mind was constantly directed to mathematical research relating principally to astronomy; and it is evident that what he had accomplished is very inadequately represented by what has been published. It is also noticeable that so few of his papers should have appeared quite spontaneously: it frequently happened that he was incited to give an account of something which he had done himself—probably years before—by the publication of a paper in which the same ground was partially covered by another investigator, and in several cases he was called upon to correct misapprehensions which were leading others astray.

As already stated, there can be no doubt that he constantly allowed himself to postpone the immediate publication of his researches, with the intention of effecting improvements in the processes and mode of representing the subject, or of attaining to an even more accurate result. A striking instance of this innate craving for perfection is afforded, even as early as 1845, by his calculation of the second orbit of the new planet. No able mathematician who is engaged upon a fruitful research can continually defer publication with impunity: the subject opens before him; his views expand; the earlier results, so interesting at the moment of discovery, lose their charm in comparison with the problems still unsolved and the novel vistas of thought opened out by them; and the rearrangement and rewriting of the old work—always an irksome task—become intolerable when later and still unfinished developments on the same subject are exciting the mind. In Adams's case the difficulty of satisfying himself, and reaching his own standard of completeness, also contributed to his apparent reluctance to publish his work. Those who knew him will remember his words when pressed, "I have still some finishing touches to put to it." It was well known that he made important researches upon the motion of Jupiter's satellites, and their publication was anxiously awaited. It does not appear that he ever made any serious attempt to put his longer investigations in order for the press, though occasionally, as his manuscripts on the different subjects increased in bulk, the feeling would come over him strongly that it was time for him to do so. Although there is no similarity between the simple and easy style of Adams's writings and the cold severity of Gauss's, there is a certain resemblance in their mode of work. Each had the same dislike to early or incomplete publication, and "Pauca sed matura" might have been the motto of both. In beginning a new research, Adams rarely put pen to paper until he had carefully thought out the subject, and when he proceeded to write out the investigation he developed it rapidly and without interruption. His accuracy and power of mind enabled him to map out the course of the work beforehand in his head, and his mathematical instinct, combined with perfect familiarity with astronomical ideas and methods, guided him with ease and safety through the intricacies and dangers of the analytical treatment[1]. He scarcely ever destroyed anything he wrote, or performed rough calculations; and the manuscripts which he has left are written so carefully and clearly that it is difficult to believe that they are not finished work which has been copied out fairly. The sheets are generally dated, and during many years he kept a diary of the work he had done each day.

His contributions to pure mathematics show the same power and excellence, and, as the subject affords greater opportunities for the display of elegance and style, they indicate even more plainly the attention he bestowed upon the form of his results, as well as upon the substance. A paper communicated to the Royal Society in 1878 may be specially noticed, in which an expression is given for the product of two Legendrian coefficients, and for the integral of the product of three. The extent of his mathematical interests is perhaps best seen by looking over the series of papers which he set in the Smith Prize Examination. These questions, which cover a wide

[1] This method of working characterised him from the first, for in his Tripos Examination it was noticed that "in the problem papers, when everyone was writing hard, Adams spent the first hour in looking over the questions, scarcely putting pen to paper the while. After that he wrote out rapidly the problems he had already solved 'in his head'." It may be mentioned here that in this examination he received more than double the marks of the Second Wrangler. This affords striking evidence of Adams's mental powers, for he was not a rapid writer.

field of mathematics, clearly indicate the bent of his mind and his favourite subjects of study: they are also noticeable for a high degree of finish, which is very unusual in examination questions.

Like Euler and Gauss, he took very great pleasure in the numerical calculation of exact mathematical constants. We owe to him the calculation of thirty-one Bernoullian numbers, in addition to the first thirty-one which were previously known. The first fifteen were calculated by Euler, and the next sixteen by Rothe, the whole thirty-one being given in vol. xx. of *Crelle's Journal*. Making use of Staudt's very curious theorem with respect to the fractional part of a Bernoullian number, Adams calculated all the numbers from B_{32} to B_{62}. The results were communicated to the British Association at the Plymouth meeting in 1877, and were also published in vol. lxxxv. of *Crelle's Journal*. A much fuller account of the work, which was very considerable in amount, appeared in an appendix to vol. XXII. of the *Cambridge Observations*, where the process of calculation of the first, B_{32}, and of the last, B_{62}, is given in detail. Adams proved that if n be a prime number other than 2 or 3, then the numerator of the nth Bernoullian number is divisible by n. This afforded a good test of the accuracy of the work.

Having thus at his command the values of sixty-two Bernoullian numbers, he was tempted to apply them to the calculation of Euler's constant. For this purpose, not only the Bernoullian numbers, but also the values of certain logarithms and sums of reciprocals were required. He accordingly calculated the values of the logarithms of 2, 3, 5, and 7 to 263 (afterwards extended to 273) decimal places, and by their means obtained the value of Euler's constant to 263 places. He also calculated the value of the modulus of the common logarithms to 273 places. The papers containing these results appeared in the *Proceedings* of the Royal Society for 1878 and 1887. Anyone who has had experience of calculations extending to a great many decimal places is aware of the difficulty of manipulating with absolute accuracy the long lines of figures; but this was an enjoyment to Adams, and the work, as carried out with consummate care and neatness, in his beautiful figures, is an interesting memorial of the patience and skill that he devoted to any work upon which he was engaged.

Some may think that the portion of his own time occupied by these calculations might have been more advantageously spent: but there is a charm of its own in carrying still further the determination of the historic constants of mathematics, which has exercised its attraction over the greatest minds. Those who feel the least possible interest in calculation for its own sake, and even dislike ordinary arithmetical computations, have been unable to resist the fascination of doing their share towards the calculation of the absolute numerical magnitudes which are so intimately connected with the foundations of the sciences dealing with abstract quantity. There is a special pleasure also in applying the resources of modern mathematics to obtain the values of these incommensurable constants to such an incredible degree of accuracy, and in verifying the distant figures by methods depending upon subtle principles and complicated symbolic processes, of the absolute truth of which we thus obtain so striking an assurance.

Adams had the greatest possible admiration for Newton, and perhaps no one has ever devoted more careful and critical attention to Newton's mathematical writings, especially the *Principia*. When Lord Portsmouth presented to the University, in 1872, the large mass of scientific papers which Newton left at his death, the arrangement and cataloguing of the mathematical portion of the collection was willingly undertaken by Adams. It was a difficult and laborious task, extending over years, but one which intensely interested him, and upon which he spared no pains. He found that these papers threw light upon the remarkable extent to which Newton had carried the lunar theory, the method by which he had obtained his table of refractions (showing that the formula known as Bradley's was really due to Newton), and the manner in which he had determined the form of the solid of least resistance. In several instances he succeeded in tracing the methods that Newton must have used in order to obtain the numerical results which occurred in the papers. The solution of the enigmas presented by these numbers written on stray papers, without any clue to the source from which they were derived, was the kind of work in which all Adams's skill, patience, and industry found full scope, and his enthusiasm for Newton was so great that he had no thought of time when so employed. His mind bore naturally a great resemblance to Newton's in many marked respects, and he was so penetrated with Newton's style of thought that he was peculiarly fitted to be his interpreter. Only a few intimate friends were aware of the immense amount of time he devoted to these manuscripts or the pleasure he derived from them. In 1888 the Cambridge University Press published a catalogue of the papers, the mathematical portion of which was wholly written by Adams[1].

In 1887, on the occasion of the bicentenary of the publication of the *Principia*, he was asked by Trinity College to deliver a commemorative address. Unfortunately the state of his health prevented him from undertaking a task which he alone could have adequately performed; but, with the kindness which all who sought his help invariably received, he most freely placed all the stores of his knowledge at the disposal of the present writer, who was appointed in his stead.

He was frequently asked to undertake calculations in connexion with eclipses or other astronomical phenomena, and he never hesitated to lay aside his own work in order to comply with such requests. Mr Downing has written: "His readiness to help, and his magnificent ability to help, will long be remembered at the Nautical Almanac Office," and similar words might be used with reference to the invaluable assistance which he so willingly gave in other quarters. For more than forty years he rendered constant

[1] After proving a general proposition from which it follows that the disturbing action of the Sun necessarily produces a continual advance of the Moon's perigee, Newton gave a numerical example which has been generally regarded as his calculation of the theoretical amount of this advance in the case of the Moon (*Lib.* I. *Sect.* IX. *Prop.* xlv. *Cor.* 2). The concluding words "Apsis lunæ est duplo velocior circiter," which have been quoted in support of the view that the motion of the lunar apsides is the question considered in the corollary, were however intended to have exactly the opposite meaning, as can be shown by comparing the three editions of the *Principia*. Adams found that some of the papers in the Portsmouth Collection afforded further confirmation on this point, and he referred to the matter in a communication on the lunar theory which he made to the Plymouth meeting of the British Association in 1877. His remarks on the subject were not put into writing by himself, but a verbatim report appeared in the *Athenæum* for August 25, 1877. He also referred to Newton's explanation of the motion of the perigee, and to his theory of astronomical refraction, in a communication to the Montreal meeting in 1884. The catalogue referred to in the text, which was published subsequently to the dates of these communications, contains a brief statement of all the principal results which he derived from the examination of the manuscripts.

service to the Royal Astronomical Society, both as a referee and as a contributor to the annual reports. These references and notices often cost him much time and thought.

He was President of the Royal Astronomical Society for the second time in 1874–76, when the medal was awarded to D'Arrest and to Le Verrier. In 1870, as Vice-President, he delivered the address on the presentation of the medal to Delaunay, of whose general method of treating the lunar theory he had the greatest possible admiration. In 1881 he was offered the position of Astronomer Royal, which he declined. In 1884 he was one of the delegates for Great Britain to the International Prime Meridian Conference at Washington. He was also present at the meetings of the British Association at Montreal and of the American Association at Philadelphia in the same year. This visit to America afforded him great enjoyment and gratification.

He received the honorary degree of D.C.L. from Oxford, of LL.D. from Dublin and Edinburgh, and of Doctor in Science from Bologna and from his own university. He was a correspondent of the French Academy, of the Academy of Sciences of St Petersburg, and of numerous other societies.

As Lowndean Professor he lectured during one term in each year, generally on the lunar theory, but sometimes on the theory of Jupiter's satellites, or the figure of the Earth. His lectures on these subjects have been prepared for press by Professor Sampson, who has also examined Adams's other mathematical manuscripts and arranged for publication those which were sufficiently complete.

During Adams's tenure of the directorship of the Cambridge Observatory in 1870 a fine transit circle by Simms was added to its equipment. This instrument has been employed in observing one of the zones of the "Astronomische Gesellschaft" programme. The zone assigned to the observatory was that lying between 25° and 30° of north declination.

Adams was a man of learning as well as a man of science, and his thoughts and interests were far from being restricted to astronomy and mathematics. He was an omnivorous reader, and his memory being exact and retentive, there were few subjects upon which he was not possessed of accurate information. Botany, geology, history, and divinity, all had their share of his eager attention. He derived great enjoyment also from novels, and when engaged in severe mental work always had one on hand. Among his more marked tastes may be mentioned his love of early printed books. His collection, containing about eight hundred volumes, eighty of which belong to the fifteenth century, was bequeathed by him to the University Library. The works relate principally to mathematics or astronomy, theology, medicine, and the occult sciences; but he seems always to have bought any fine old book that took his fancy. He was so little given to talk about himself or his pursuits that probably but few of his friends were aware of his affection for black-letter books. It may be mentioned that his other mathematical books were bequeathed to the Libraries of St John's College and Pembroke College.

No one who knew him superficially, or who judged only by his quiet manner, could have imagined how deeply he was affected by great political questions or passing events. In times of public excitement (such as during the Franco-German war) his interest was so intense that he could scarcely work or sleep. His love of nature in all its forms was a source of never-failing delight to him, and he was never happier than when wandering over the cliffs and moors of his native county. Strangers who first met him were invariably struck by his simple and unaffected manner. He was a delightful companion, always cheerful and genial, showing in society but few traces of his really shy and retiring disposition. His nature was sympathetic and generous, and in few men have the moral and intellectual qualities been more perfectly balanced. An attempt to sketch his character cannot be more fitly closed than in the words of Dr Donald MacAlister, who knew him well, and attended him in his last illness:—"His earnest devotion to duty, his simplicity, his perfect self-lessness, were to all who knew his life at Cambridge a perpetual lesson, more eloquent than speech. From the time of his first great discovery scientific honours were showered upon him, but they left him as they found him—modest, gentle, and sincere. Controversies raged for a time around his name, national and scientific rivalries were stirred up concerning his work and its reception, but he took no part in them, and would generously have yielded to others' claims more than his greatest contemporaries would allow to be just. With a single mind for pure knowledge he pursued his studies, here bringing a whole chaos into cosmic order, there vindicating the supremacy of a natural law beyond the imagined limits of its operation; now tracing and abolishing errors that had crept into the calculations of the acknowledged masters of his craft, and now giving time and strength to resolving the self-made difficulties of a mere beginner, and all the while with so little thought of winning recognition or applause that much of his most perfect work remained for long, or still remains, unpublished."

He was suddenly attacked by severe illness at the end of October 1889, but he recovered sufficiently to resume his mathematical work in the usual way for several months. In June of the following year he was again attacked by an illness from which he never completely recovered, and he passed away on the early morning of January 21, 1892, after being confined to his bed for ten weeks. The funeral service took place in Pembroke College Chapel, and he was interred in St Giles's Cemetery, on the Huntingdon Road. There were many who thought that his resting-place should have been in Westminster Abbey, and a royal wish was expressed to this effect; but it is perhaps more fitting that he should lie in this quiet graveyard close to the Observatory where he passed so many happy and peaceful years.

On February 20, 1892, a public meeting was held at St John's College, with the view of taking steps to place a bust or other memorial of him in Westminster Abbey. The proceedings on this representative occasion bore eloquent testimony to the admiration and affection in which he was held by his friends, and to the widespread wish throughout the country for such a memorial to one who was not only a great but a good man[1]. No suitable site for a bust could be found in the Abbey, but a medallion has been placed in an admirable position close to the grave of Newton. This medallion, executed by Mr Bruce Joy, was unveiled on May 9, 1895, after a ceremony in the Jerusalem Chamber, at which addresses were delivered by leading members of the University and others. A bust, also executed by Mr Bruce Joy, which represents Adams in the later years of his life, was presented to St John's College by Mrs Adams in the same

[1] A report of this meeting was published in a special number of the *Cambridge University Reporter*, March 10, 1892, p. 607.

year. In 1888 an excellent portrait was painted by Herkomer, which is now in the Combination Room of Pembroke College; a replica is in the possession of Mrs Adams. The portrait in the Combination Room of St John's College was painted by Mogford in 1850—51. The Royal Astronomical Society also possesses a bust of Adams which was executed when he was a young man.

J. W. L. G.

PROFESSOR CHALLIS'S FIRST REPORT TO THE CAMBRIDGE OBSERVATORY SYNDICATE UPON THE NEW PLANET[1].

At a meeting of the Observatory Syndicate, held at the Observatory on December 4, for the despatch of ordinary business, a strong desire having been expressed by the Vice-Chancellor and the members of the Syndicate generally, to receive from me a Special Report of Observatory proceedings relating to the newly-discovered Planet, drawn up in such a manner, and in such detail, as would enable them to lay complete information on the subject before the members of the Senate, I considered it to be my duty at once to comply with this request. A new body of the solar system has been discovered, by means depending on the farthest advances hitherto made in theoretical and practical astronomy, and confirming, in a most remarkable manner, the theory of universal gravitation. It is, therefore, on every account desirable that the members of the Senate should be made fully acquainted with the part which has been taken by the Cambridge Observatory, relatively to this important extension of astronomical science. The observations I shall have to speak of, and the reasons for undertaking them, are so closely connected with theoretical calculations performed by a member of this University, to account for anomalies in the motion of the planet Uranus, that the history of the former necessarily involves that of the latter. I hope that for this reason, and because of the peculiar nature of the circumstances, I may be allowed to make a communication less formal and restricted in its character, than a mere Report of Observatory proceedings.

The tables with which the observations of the planet Uranus have been uniformly compared, were published by A. Bouvard in 1821. They are founded on a continued series of observations extending from 1781, the year of its discovery, to 1821. Previous to 1781, it had been accidentally observed seventeen times as a fixed star, the earliest observation of this kind being one by Flamsteed in 1690. Bouvard met with a difficulty in forming his Tables. On an attempt to found them upon the ancient, as well as the modern, observations, it appeared that the theoretical did not agree with the observed course of the planet. He thought this might be attributed to the imperfection of the ancient observations, and consequently rejected all previous to 1781, in the formation of the Tables finally published. These Tables represent well enough the observations in the forty years from 1781 to 1821; but very soon after the latter year, new errors began to show themselves, which have gone on increasing to the present time. It

[1] This report, which is headed 'Special Report of Proceedings in the Observatory relative to the new Planet,' is signed by Challis and dated December 12, 1846. It is preceded by the following introductory remarks. "The syndicate appointed to visit the Observatory, conceiving the subject at the present time to possess peculiar interest, beg leave to submit to the Senate the following statement of Professor Challis, describing the course of observations, founded on the theoretical calculations of Mr Adams, of St John's College, and made at the Observatory with a view to the discovery of the new planet." This preamble is signed by the syndics, H. Philpott (Vice-Chancellor), John Graham, B. Chapman, W. Whewell, Joshua King, Geo. Peacock, James Cartmell, Chas. W. Goodwin, W. C. Mathison, G. G. Stokes. Professor Challis issued a second report to the Syndicate, dated March 22, 1847, relating to the subsequent observations of the new planet. This second report was reprinted in the *Astronomische Nachrichten* (Vol. xxv. col. 309).

was now evident that the ancient observations had been rejected on insufficient grounds, and that from some unknown cause the theory was in fault. Were the Tables calculated inaccurately? The difference between observation and theory (amounting in 1841 to 96″ of geocentric longitude) was too great, and Bouvard's calculations were made with too much care to allow of this explanation. The effect of small terms neglected in the calculation of the perturbations caused by Jupiter and Saturn, could not be supposed to bear any considerable proportion to the observed amount of error. This state of the theory suggested to several astronomers the idea of disturbances, caused by an undiscovered planet more distant than Uranus. But there is no evidence of this hypothesis having been put to the test of calculation previous to 1843. The usual problem of perturbations is to find the disturbing action of one body on another, by knowing the positions of both. Here an inverse problem, hitherto untried, was to be solved; viz. from known disturbances of a planet in known positions, to find the place of the disturbing body at a given time. Mr Adams, Fellow of St John's College, showed me a memorandum made in 1841, recording his intention of attempting to solve this problem as soon as he had taken his degree of B.A. Accordingly, after graduating in January 1843, he obtained an approximate solution by supposing the disturbing body to move in a circle at twice the distance of Uranus from the Sun. The result so far satisfied the observed anomalies in the motion of Uranus, as to induce him to enter upon an exact solution. For this purpose he required reduced observations made in the years 1818—1826, and requested my intervention to obtain them from Greenwich. The Astronomer Royal, on my application, immediately supplied (February 15, 1844) all the heliocentric errors of Uranus in longitude and latitude, from 1754 to 1830, completely reduced. Mr Adams was now furnished with ample data from observation, and his next care was to ascertain whether Bouvard's theoretical calculations were correct enough for his purpose. He tested the accuracy of the principal terms of the perturbations caused by Jupiter and Saturn, and concluded that the small terms which Bouvard had not taken into account would not sensibly affect the final results, the chief of them being either of long period or of a period nearly equal to that of Uranus. Besides which he introduced into the theory several corrections which had been derived from observation and calculation by different astronomers since 1821. The calculations were completed in 1845. In September of that year, Mr Adams placed in my hands a paper containing numerical values of the mean longitude at a given epoch, longitude of perihelion, eccentricity of orbit, mass, and geocentric longitude, September 30, of the supposed disturbing planet, which he calls by anticipation "The New Planet," evidently showing the conviction in his own mind of the reality of its existence. Towards the end of the next month, a communication of results slightly different was made to the Astronomer Royal, with the addition of what was far more important, viz. a list of the residual errors of the mean longitude of Uranus, for a period extending from 1690 to 1840, after taking account of the disturbing effect of the supposed planet. This comparison of observation with the theory implied the determination of *all* the unknown quantities of the problem, both the corrections of the elements of Uranus and the elements of the disturbing body. The smallness of the residual errors proved that the new theory was adequate to the explanation of the observed anomalies in the motion of Uranus, and that as the error of longitude was corrected for a period of at least 130 years, the error of radius vector was also corrected. As the calculations rested on an assumption, made according to Bode's law, that the mean distance of the disturbing planet was double that of Uranus, without the above-mentioned numerical verification, no proof was given that the problem was solved or that the elements of the supposed planet were not mere speculative results. The earliest evidence of the complete solution of an inverse problem of perturbations is to be dated from October 1845.

Although the comparison of the theory with observation proved synthetically that the assumed mean distance was not very far from the truth, it was yet desirable to try the effect of an alteration of the mean distance. Mr Adams accordingly went through the same calculations as before, assuming a mean distance something less than the double of that of Uranus, and obtained results which indicated a better accordance of the theory with observation, and led him to the conclusion, which has since been confirmed by observation, that the mean distance should be still farther diminished. This second solution taken in conjunction with the first may be considered to relieve the question of every kind of assumption. The new elements of the disturbing body, and the results of comparing the observed with the theoretical mean longitudes of Uranus, were communicated to the Astronomer Royal at the beginning of September 1846. These were accompanied by numerical values of errors of the radius vector, the Astronomer Royal having inquired, after the reception of the first solution, whether the error of radius vector, known to exist from observation, was explained by this theory. It would be wrong to infer that Mr Adams was not prepared to answer this question till he had gone through the second solution. Errors of radius vector were as readily deducible from the first solution as from the other.

The preceding details are intended to point out the circumstances which led astronomers to suspect the existence of an additional body of the solar system, and the theoretical reasons there were for undertaking to search for it. No one could have anticipated that the place of the unknown body was indicated with any degree of exactness by a theory of this kind. It might reasonably be supposed, without at all mistrusting the evidence which the theory gave of the *existence* of the planet, that its position was determined but roughly, and that a search for it must necessarily be long and laborious. This was the view I took, and consequently I had no thought of commencing the search in 1845, the planet being considerably past opposition at the time Mr Adams completed his calculations. The succeeding interval to midsummer of 1846 was a period of great astronomical activity, the planet Astræa, Biela's double comet, and several other comets. successively demanding attention. During this time I had little communication with Mr Adams respecting the new planet. Attention was again called to the subject by the publication of M. Le Verrier's first researches in the *Comptes Rendus* for June 1, 1846. At a meeting of the Greenwich Board of Visitors held on June 29, at which I was present, Mr Airy announced that M. Le Verrier had obtained very nearly the same longitude of the supposed planet as that given by Mr Adams. On July 9 I received a letter from Mr Airy, in which he suggested employing the Northumberland Telescope in a systematic search for the planet, offering at the same time to send an assistant from Greenwich, in case I declined undertaking the observations. This letter was followed by another dated July 13, containing suggestions respecting the mode of conducting the observations, and an estimation of the amount of work they might be expected to require.

In my answer, dated July 18, I signified the determination I had come to of undertaking the search. Various reasons led me to this conclusion. I had already, as Mr Adams can testify, entertained the idea of making these observations; the most convenient time for commencing them was now approaching; and the confirmation of Mr Adams's theoretical position by the calculations of M. Le Verrier appeared to add very greatly to the probability of success. I had no answer to make to Mr Airy's offer of sending an assistant, as I understood the acceptance of it to imply the relinquishing on my part of the undertaking.

I have now to speak of the observations. The plan of operations was formed mainly on the suggestions contained in Mr Airy's note of July 13. It was recommended to sweep over, three times at least, a zodiacal belt 30° long and 10° broad, having the theoretical place of the planet at its centre; to complete one sweep before commencing the next; and to map the positions of the stars. The three sweeps, it was calculated, would take 300 hours of observing. This extent of work, which will serve to show the idea entertained of the difficulty of the undertaking before the planet was discovered, did not appear to me greater than the case required. It will be seen that the plan did not contemplate the use of hour XXI. of the Berlin Star Maps, the publication of which was equally unknown at that time to Mr Airy and myself. It may be proper here to explain that the construction of a good star-map requires a great amount of time and labour both in observing and calculating, and that precisely this sort of labour must be gone through to conduct a search of the kind I had undertaken. The stars must first be mapped before the search can properly be said to begin. With a map ready made, the detection of a moving body, as it happened in this instance, might be effected on a comparison of the heavens with the map by mere inspection. Not having the advantage of such a map, I proceeded as follows. I noted down very approximately the positions of all the stars to the 11th magnitude that could be conveniently taken as they passed through the field of view of the telescope, the breadth of the field with a magnifying power of 166 being 9′, and the telescope being in a fixed position. When the stars came thickly, some were necessarily allowed to pass without recording their places. Wishing to include *all* stars of the 11th magnitude, I proposed, in going over the same region a second time, to avail myself of an arrangement peculiar to the Northumberland Equatorial, the merit of inventing which is due to Mr Airy. The Hour-circle, Telescope, and Polar Frame are movable by clockwork, which may be regulated to sidereal time nearly. While this motion is going on, the Telescope and Polar Frame are movable *relatively to the Hour-circle*, by a tangent-screw apparatus, and a handle extending to the observer's seat. This contrivance enables the observer to measure at his leisure differences of Right Ascension however small, and therefore meets the case of stars coming in groups. The observations made by this method might include all the stars it was thought desirable to take, and therefore might include *all* the stars taken in the first sweep. The discovery of the planet would result from finding that any star in the first sweep was not in its position in the second sweep. If two sweeps failed in detecting the planet among the stars of the first sweep, it might be among the stars of the second, which would be decided by taking a third sweep of the same kind as the second. It will appear that this plan carried out would not only detect the planet if it were in the region explored, but would also, in case of failure, enable the observer to pronounce that it was not in that region. The second mode of observing required the aid of my two assistants, Mr Morgan and Mr Breen, in reading off and recording the observations.

I commenced observing July 29, employing on that day the first method, with telescope fixed. The next day I observed according to the second method, with telescope moving. On August 4, the telescope was fixed as to Right Ascension, but was moved in Declination in a zone of about 70′ breadth, the intention of the observations of that day being to record points of reference for the zones of 9′ breadth. On August 12, the fourth day of observing, I went over the same zone, telescope fixed, as on July 30 with telescope moving. Soon after August 12, I compared, to a certain extent, the observations of that day, with the observations of July 30, taken with telescope moving; and finding, as far as I carried the comparison, that the positions of July 30 included *all* those of August 12, I felt convinced of the adequacy of the method of search I had adopted. The observations were continued with diligence to September 29, chiefly with telescope fixed, and were made early in Right Ascension for the purpose of exploring as large a space as possible before I should be compelled to desist by the approach of daylight. On October 1, I heard that the planet was discovered by Dr Galle, at Berlin, on September 23. I had then recorded 3150 positions of stars, and was making preparations for mapping them. The following results were obtained by a discussion of the observations after the announcement of the discovery.

On continuing the comparison of the observations of July 30 and August 12, I found that No. 49, a star of the 8th magnitude in the series of August 12, *was wanting in the series of July* 30. According to the principle of the search, this was the planet. It had wandered into the zone in the interval between July 30 and August 12. I had not continued the former comparison beyond No. 39, probably from the accidental circumstance that a line was there drawn in the memorandum-book in consequence of the interruption of the observations by a cloud. After ascertaining the place of the planet on August 12, I readily inferred that it was also among the reference stars taken on August 4. Thus, after four days of observing, two positions of the planet were obtained. This is entirely to be attributed to my having, on those days, directed the telescope towards the planet's theoretical place, according to instructions given in a paper Mr Adams had the kindness to draw up for me. I would also beg to call attention to the fact that, after August 12, the planet was discoverable by a closet-comparison of the observations, a method of observing, depending on novel and ingenious mechanism, having been adopted by which I could say of each star, to No. 48, "This is not a planet," and of No. 49, "This *is* a planet." I lost the opportunity of announcing the discovery by deferring the discussion of the observations, being much occupied with reductions of comet observations, and little suspecting that the indications of theory were accurate enough to give a chance of discovery in so short a time. On September 29, I saw, for the first time, the communication presented by M. Le Verrier to the Paris Academy on August 31. I was much struck with the manner in which the author limits the field of observation; and with his recommending the endeavour to detect the planet by its disk. Mr Adams had already told me that, according to his estimation, the planet would not be less bright than a star of the ninth magnitude. On the same evening I swept a considerable breadth in Declination, between the limits of Right Ascension marked out by M. Le Verrier, and I paid particular attention to the physical appearance of the brighter stars. Out of

300 stars, whose positions I recorded that night, I fixed on one which appeared to have a disk, and which proved to be the planet. This was the third time it was observed before the announcement of the discovery reached me. This last observation may be regarded as a discovery of the planet, due to the good definition of the noble instrument which we owe to the munificence of our Chancellor.

From the reduced places of the planet, on August 4 and August 12, and from observations since its discovery extending to October 13, Mr Adams calculated, at my request, values of its heliocentric longitude at a given epoch, its actual distance from the Sun, longitude of the node, and inclination of the orbit, which were published as early as October 17. I am now diligently observing the planet with the meridian instruments, and when daylight prevents its being seen on the meridian, I propose carrying on the observations as long as possible with the Northumberland Equatorial, for the purpose of obtaining data for a further approximation to the elements of the orbit.

My report of proceedings relating to the planet here terminates. I beg permission to add a few remarks, which the facts I have stated seem to call for. It will appear by the above account, that my success might have been complete, if I had trusted more implicitly to the indications of the theory. It must, however, be remembered, that I was in quite a novel position: the history of astronomy does not afford a parallel instance of observations undertaken entirely in reliance upon deductions from theoretical calculations, and those too of a kind before untried. As the case stands, a very prominent part has been taken in the University of Cambridge, with reference to this extension of the boundaries of astronomical science. We may certainly assert to be facts, for which there is documentary evidence, that the problem of determining, from perturbations, the unknown place of the disturbing body, was first solved here; that the planet was here first sought for; that places of it were here first recorded; and that approximate elements of its orbit were here first deduced from observation. And that all this may be said, is entirely due to the talents and labours of one individual among us, who has at once done honour to the University, and maintained the scientific reputation of the country. It is to be regretted that Mr Adams was more intent upon bringing his calculations to perfection, than on establishing his claims to priority by early publication. Some may be of opinion, that, in placing before the first astronomer of the kingdom results which showed that he had completed the solution of the problem, and by which he was, in a manner, pledged to the production of his calculations, there was as much publication as was justifiable on the part of a mathematician whose name was not yet before the world, the theory being one by which it was possible the practical astronomer might be misled. Now that success has attended a different course, this will probably not be the general opinion. I should consider myself to be hardly doing justice to Mr Adams, if I did not take this opportunity of stating, from the means I have had of judging, that it was impossible for any one to have comprehended more fully and clearly all the parts of this intricate problem; that he carefully considered all that was necessary for its exact solution; and that he had a firm conviction, from the results of his calculations, that a planet was to be found.

Memoranda.

1841. July 3. Formed a design, in the beginning of this week, of investigating, as soon as possible after taking my degree, the irregularities in the motion of Uranus, wh. are yet unaccounted for; in order to find whether they may be attributed to the action of an undiscovered planet beyond it; and if possible thence to determine the elements of its orbit, &c. approximately, wh w.d probably lead to its discovery.

1845 October

According to my calculations the obs^d irregularities in the motion of Uranus may be accounted for by supposing the existence of an ext^r planet the mass & orbit of wh. are as follows

Mean Dist. (assumed nearly in accordance with Bode's law)

38.4

Mean sid^l mot^n in 365.25 days

1°.30'.9

Mean Long. 1st Oct^r 1845

323°.34'

Long. Perih^n

315.55

Eccent^y

0.1610

Mass (that of Sun being unity)

0.0001656

For the modern obs^ns I have used the method of Normal places, taking the mean of the Tabular errors as given by obs^ns near 3 consecutive opp^ns to correspond with the mean of the times & the Greenw. obs^ns have been used down to 1830 since wh. the Cambridge & Greenwich obs^ns and those given in the Ashon. Nachr. have been made use of. The foll^g are the resul^g errors of *mean* Longitude

	Obs^n – Theory		Obs^n – Theory		Obs^n – Theory
1780	+0.″27	1801	–0.″04	1822	+0.″30
1783	–0.23	1804	+1.76	1825	+1.92
1786	–0.96	1807	–0.21	1828	+2.25
1789	+1.82	1810	+0.56	1831	–1.06
1792	–0.91	1813	–0.94	1834	–1.44
1795	+0.09	1816	–0.31	1837	–1.62
1798	–0.99	1819	–2.00	1840	+1.73

The error for 1780 is concluded from that for 1781 given by obs^n compared with those of 4 or 5 following years & also with Lemonnier's obs^ns in 1769 & 1771

For the ancient obs^ns the foll^g are the rem^g errors

	Obs^n – Theory		Obs^n – Theory
1690	+ 44.″4	1756	− 4.″0
1712	+ 6.7	1763	− 5.1
1715	− 6.8	1769	+ 0.6
1750	− 1.6	1771	+ 11.8
1753	+ 5.7		

The errors are small except for Flamsteed's obs^n of 1690. This being an isolated obs^n very distant from the rest, I thought it best not to use it in forming the =^ns of cond^n. It is not improbable however that this error might be destroy^d by a small change in the assumed mean motion of the new planet.

J. C. Adams

1.

RESULTS OF CALCULATIONS OF THE ELEMENTS OF AN EXTERIOR PLANET, WHICH WILL ACCOUNT FOR THE OBSERVED IRREGULARITIES IN THE MOTION OF URANUS.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. VII. (1846). Papers delivered to the Astronomer Royal Oct. 21, 1845 and Sept. 2, 1846.]

I.

ACCORDING to my calculations, the observed irregularities in the motion of *Uranus* may be accounted for by supposing the existence of an exterior planet, the mass and orbit of which are as follows:—

Mean Distance (assumed nearly in accordance with Bode's law)	38·4
Mean Sidereal Motion in 365·25 days	1° 30′·9
Mean Longitude, 1st October, 1845	323° 34′
Longitude of Perihelion	315° 55′
Eccentricity	0·1610
Mass (that of the Sun being unity)	0·0001656.

For the modern observations I have used the method of normal places, taking the mean of the tabular errors, as given by observations near three consecutive oppositions, to correspond with the mean of the times; and the Greenwich observations have been used down to 1830: since which, the Cambridge and Greenwich observations, and those given in the *Astronomische Nachrichten*, have been made use of. The following are the remaining errors of mean longitude :—

OBSERVATION – THEORY.

1780	+0″·27	1801	−0″·04	1822	+0″·30
1783	−0·23	1804	+1·76	1825	+1·92
1786	−0·96	1807	−0·21	1828	+2·25
1789	+1·82	1810	+0·56	1831	−1·06
1792	−0·91	1813	−0·94	1834	−1·44
1795	+0·09	1816	−0·31	1837	−1·62
1798	−0·99	1819	−2·00	1840	+1·73

The error for 1780 is concluded from that for 1781 given by observation, compared with those of four or five following years, and also with Lemonnier's observations in 1769 and 1771.

For the ancient observations, the following are the remaining errors :—

OBSERVATION – THEORY.

1690	+44″·4	1750	−1″·6	1763	− 5″·1
1712	+ 6·7	1753	+5·7	1769	+ 0·6
1715	− 6·8	1756	−4·0	1771	+11·8

The errors are small, except for Flamsteed's observation of 1690. This being an isolated observation, very distant from the rest, I thought it best not to use it in forming the equations of condition. It is not improbable, however, that this error might be destroyed by a small change in the assumed mean motion of the planet.

II.

In the investigation, the results of which I communicated to you last October, the mean distance of the supposed disturbing planet is assumed to be twice that of *Uranus*. Some assumption is necessary in the first instance, and Bode's law renders it probable that the above distance is not very remote from the truth: but the investigation could scarcely be considered satisfactory while based on anything arbitrary; and I therefore determined to repeat the calculation, making a different hypothesis as to the mean distance. The eccentricity also resulting from my former calculations was far too large to be probable; and I found that, although the agreement between theory and observation continued very satisfactory down to 1840, the difference in subsequent years was becoming very sensible, and I hoped that these errors, as well as the eccentricity, might be diminished by taking a different mean distance. Not to make too violent a change, I assumed this distance to be less than the former value by about $\frac{1}{30}$th part of the whole. The result is very satisfactory, and appears to shew that, by still further diminishing the distance, the agreement between the theory and the later observations may be rendered complete, and the eccentricity reduced at the same time to a very small quantity. The mass and the elements of the orbit of the supposed planet, which result from the two hypotheses, are as follows:—

	Hypothesis I. $\left(\frac{a}{a^1}=0{\cdot}5\right)$	Hypothesis II. $\left(\frac{a}{a^1}=0{\cdot}515\right)$
Mean longitude of Planet, 1st Oct. 1846...	325° 8′	323° 2′
Longitude of Perihelion	315° 57′	299° 11′
Eccentricity	0·16103	0·12062
Mass (that of Sun being 1)	0·00016563	0·00015003

The investigation has been conducted in the same manner in both cases, so that the differences between the two sets of elements may be considered as wholly due to the variation of the fundamental hypothesis. The following table exhibits the differences between the theory and the observations which were used as the basis of calculation. The quantities given are the errors of *mean* longitude, which I found it more convenient to employ in my investigations than those of the *true* longitude.

Ancient Observations.

Date.	(Obs. – Theory.) Hypoth. I.	Hypoth. II.	Date.	(Obs. – Theory.) Hypoth. I.	Hypoth. II.
	″	″		″	″
1712	+6·7	+6·3	1756	− 4·0	− 4·0
1715	−6·8	−6·6	1764	− 5·1	− 4·1
1750	−1·6	−2·6	1769	+ 0·6	+ 1·8
1753	+5·7	+5·2	1771	+11·8	+12·8

MODERN OBSERVATIONS.

Date.	(Obs. – Theory.) Hypoth. I.	Hypoth. II.	Date.	(Obs. – Theory.) Hypoth. I.	Hypoth. II.
1780	$+0''.27$	$+0''.54$	1813	$-0''.94$	$-1''.00$
1783	-0.23	-0.21	1816	-0.31	-0.46
1786	-0.96	-1.10	1819	-2.00	-2.19
1789	$+1.82$	$+1.63$	1822	$+0.30$	$+0.14$
1792	-0.91	-1.06	1825	$+1.92$	$+1.87$
1795	$+0.09$	$+0.04$	1828	$+2.25$	$+2.35$
1798	-0.99	-0.93	1831	-1.06	-0.82
1801	-0.04	$+0.11$	1834	-1.44	-1.17
1804	$+1.76$	$+1.94$	1837	-1.62	-1.53
1807	-0.21	-0.08	1840	$+1.73$	$+1.31$
1810	$+0.56$	$+0.61$			

The greatest difference in the above table, viz. that for 1771, is deduced from a single observation, whereas the difference immediately preceding, which is deduced from the mean of several observations, is much smaller. The error of the tables for 1780 is found by interpolating between the errors given by the observations of 1781, 1782, and 1783, and those of 1769 and 1771. The differences between the results of the two hypotheses are exceedingly small till we come to the last years of the series, and become sensible precisely at the point where both sets of results begin to diverge from the observations; the errors corresponding to the second hypothesis being, however, uniformly smaller. The errors given by the *Greenwich Observations* of 1843 are very sensible, being for the first hypothesis $+6''.84$, and for the second $+5''.50$. By comparing these errors, it may be inferred that the agreement of theory and observation, would be rendered very close by assuming $\frac{a}{a^1} = 0.57$, and the corresponding mean longitude on the 1st October, 1846, would be about 315° 20′, which I am inclined to think is not far from the truth. It is plain also that the eccentricity corresponding to this value of $\frac{a}{a^1}$, would be very small. In consequence of the divergence of the results of the two hypotheses, still later observations would be most valuable for correcting the distances, and I should feel exceedingly obliged if you would kindly communicate to me two normal places near the oppositions of 1844 and 1845.

As Flamsteed's first observation of *Uranus* (in 1690) is a single one, and the interval between it and the rest is so large, I thought it unsafe to employ this observation in forming the equations of condition. On comparing it with the theory, I find the difference to be rather large, and greater for the second hypothesis than for the first, the errors being $+44''\cdot5$ and $+50''\cdot0$ respectively. If the error be supposed to change in proportion to the change of mean distance, its value corresponding to $\frac{a}{a^1} = 0\cdot57$, will be about $+70''$, and the error in the time of transit will be between 4^s and 5^s. It would be desirable to ascertain whether Flamsteed's manuscripts throw any light on this point.

The corrections of the tabular radius vector of *Uranus*, given by the theory for some late years, are as follows:—

Date.	Hypoth. I.	Hypoth. II.
1834	$+0\cdot005051$	$+0\cdot004923$
1840	$+0\cdot007219$	$+0\cdot006962$
1846	$+0\cdot008676$	$+0\cdot008250$

The correction for 1834 is very nearly the same as that which you have deduced from observation, in the *Astronomische Nachrichten;* but the increase in later years is more rapid than the observations appear to give it: the second hypothesis, however, still having the advantage.

I am at present employed in discussing the errors in latitude, with the view of obtaining an approximate value of the inclination and position of the node of the new planet's orbit; but the perturbations in latitude are so very small that I am afraid the result will not have great weight. According to a rough calculation made some time since, the inclination appeared to be rather large, and the longitude of the ascending node to be about 300°; but I am now treating the subject much more completely, and hope to obtain the result in a few days.

I have been thinking of drawing up a brief account of my investigation to present to the British Association.

NOTE. The mass was found to be three times that of *Uranus*, and it was thence inferred and stated to Professor Challis that the brightness would not be below that of a star of the ninth magnitude.

2.

AN EXPLANATION OF THE OBSERVED IRREGULARITIES IN THE MOTION OF URANUS, ON THE HYPOTHESIS OF DISTURBANCES CAUSED BY A MORE DISTANT PLANET; WITH A DETERMINATION OF THE MASS, ORBIT, AND POSITION OF THE DISTURBING BODY.

[From the *Memoirs of the Royal Astronomical Society*, Vol. XVI. (1847). Appendix to *Nautical Almanack* (1851). Read November 13, 1846.]

1. THE irregularities in the motions of *Uranus* have for a long time engaged the attention of Astronomers. When the path of the planet became approximately known, it was found that, previously to its discovery by Sir W. Herschel in 1781, it had several times been observed as a fixed star by Flamsteed, Bradley, Mayer, and Lemonnier. Although these observations are doubtless very far inferior in accuracy to the modern ones, they must be considered valuable, in consequence of the great extension which they give to the observed arc of the planet's orbit. Bouvard, however, to whom we owe the tables of *Uranus* at present in use, found that it was impossible to satisfy these observations without attributing much larger errors to the modern observations than they admit of, and consequently founded his Tables exclusively on the latter. But, in a very few years, sensible errors began again to shew themselves, and, though the tables were formed so recently as 1821, their error at the present time exceeds two minutes of space, and is still rapidly increasing. There appeared, therefore, no longer any sufficient reason for rejecting the ancient obser-

vations, especially since, with the exception of Flamsteed's first observation, which is more than twenty years anterior to any of the others, they are mutually confirmatory of each other.

2. Now that the discovery of another planet has confirmed in the most brilliant manner the conclusions of analysis, and enabled us with certainty to refer these irregularities to their true cause, it is unnecessary for me to enter at length upon the reasons which led me to reject the various other hypotheses which had been formed to account for them. It is sufficient to say, that they all appeared to be very improbable in themselves, and incapable of being tested by any exact calculation. Some had even supposed that, at the great distance of *Uranus* from the sun, the law of attraction becomes different from that of the inverse square of the distance. But the law of gravitation was too firmly established for this to be admitted till every other hypothesis had failed, and I felt convinced that in this, as in every previous instance of the kind, the discrepancies which had for a time thrown doubts on the truth of the law, would eventually afford the most striking confirmation of it.

3. My attention was first directed to this subject several years since, by reading Mr Airy's valuable Report on the recent progress of Astronomy. I find among my papers the following memorandum, dated July 3, 1841: "Formed a design, in the beginning of this week, of investigating, as soon as possible after taking my degree, the irregularities in the motion of *Uranus*, which are yet unaccounted for, in order to find whether they may be attributed to the action of an undiscovered planet beyond it, and, if possible, thence to determine approximately the elements of its orbit, &c., which would probably lead to its discovery." Accordingly, in 1843, I attempted a first solution of the problem, assuming the orbit to be a circle, with a radius equal to twice the mean distance of *Uranus* from the sun. Some assumption as to the mean distance was clearly necessary in the first instance, and Bode's law appeared to render it probable that the above would not be far from the truth. This investigation was founded exclusively on the modern observations, and the errors of the tables were taken from those given in the equations of condition of Bouvard's tables as far as the year 1821, and subsequently from the observations given in the *Astronomische Nachrichten*, and from the Cambridge and Greenwich Observations. The result shewed that a good general agreement between theory and observation might be obtained; but the larger differences occurring in years where the observations used were deficient in number, and the Greenwich Planetary

Observations being then in process of reduction, I applied to Mr Airy, through the kind intervention of Professor Challis, for the observations of some years in which the agreement appeared least satisfactory. The Astronomer Royal, in the kindest possible manner, sent me in February 1844 the results of all the Greenwich Observations of *Uranus*.

4. Meanwhile the Royal Academy of Sciences of Göttingen had proposed the theory of *Uranus* as the subject of their mathematical prize, and although the little time which I could spare from important duties in my college prevented me from attempting the complete examination of the theory which a competition for the prize would have required, yet this fact, together with the possession of such a valuable series of observations, induced me to undertake a new solution of the problem. I now took into account the most important terms depending on the first power of the eccentricity of the disturbing planet, retaining the same assumption as before with respect to the mean distance. For the modern observations, the errors of the tables were taken exclusively from the Greenwich Observations as far as the year 1830, with the exception of an observation by Bessel in 1823; and subsequently from the Cambridge and Greenwich Observations, and those given in various numbers of the *Astronomische Nachrichten*. The errors of the tables for the ancient observations were taken from those given in the equations of condition of Bouvard's tables. After obtaining several solutions differing little from each other, by gradually taking into account more and more terms of the series expressing the perturbations, I communicated to Professor Challis, in September 1845, the final values which I had obtained for the mass, heliocentric longitude, and elements of the orbit of the assumed planet. The same results, slightly corrected, I communicated in the following month to the Astronomer Royal. The eccentricity coming out much larger than was probable, and later observations shewing that the theory founded on the first hypothesis as to the mean distance was still sensibly in error, I afterwards repeated my investigation, supposing the mean distance to be about $\frac{1}{30}$th part less than before. The result, which I communicated to Mr Airy in the beginning of September of the present year, appeared more satisfactory than my former one, the eccentricity being smaller, and the errors of theory, compared with late observations, being less, and led me to infer that the distance should be still further diminished.

5. In November 1845, M. Le Verrier presented to the Royal Academy of Sciences, at Paris, a very complete and elaborate investigation of the

Theory of *Uranus*, as disturbed by the action of *Jupiter* and *Saturn*, in which he pointed out several small inequalities which had previously been neglected; and in June, of the present year, he followed up this investigation by a memoir, in which he attributed the residual disturbances to the action of another planet at a distance from the sun equal to twice that of *Uranus*, and found a longitude for the new planet agreeing very nearly with the result which I had obtained on the same hypothesis. On the 31st of August, he presented to the Academy a more complete investigation, in which he determined the mass and the elements of the orbit of the new planet, and also obtained limiting values of the mean distance and heliocentric longitude. I mention these dates merely to shew that my results were arrived at independently, and previously to the publication of those of M. Le Verrier, and not with the intention of interfering with his just claims to the honours of the discovery; for there is no doubt that his researches were first published to the world, and led to the actual discovery of the planet by Dr Galle, so that the facts stated above cannot detract, in the slightest degree, from the credit due to M. Le Verrier.

6. In order not to have an inconvenient number of equations of condition, I divided the modern observations into groups, each including a period of three years, and as Mr Airy had shewn that the error of the tabular radius vector was sometimes considerable, I either selected those observations which were made near opposition, or combined the others in such a manner that the results should be nearly free from the effects of this error. From the observations of each group, the error of the tables in heliocentric longitude was found, corresponding to the time of mean opposition in the middle year of the group. Thus were formed 21 normal errors of the tables, corresponding to as many equidistant periods between 1780 and 1840. The error for 1780 was found by interpolating between the errors of 1781, 1782, and 1783, and those given by the ancient observations of 1769 and 1771; and though not entitled to the same weight as the others, cannot, I think, be liable to much uncertainty. In my last calculations I might have used more recent observations, but in order to obtain the effect due to the change of mean distance, it was necessary that the investigation should be founded on the same elements as before, and the later observations might be used as a test of the theory.

7. In order to satisfy myself that there was no important error in Bouvard's tables, I re-computed all the principal inequalities produced by the action of *Jupiter* and *Saturn*, and found no difference of any consequence,

except in the equation depending on the mean longitude of *Saturn* minus twice that of *Uranus*, the error of which had been already pointed out by Bessel. The principal equation depending on the action of *Jupiter* also required correction, in consequence of the increased value which has been lately obtained for the mass of that planet. The corrections to be applied to Bouvard's tables on these accounts are the following:—

$$+1''\!\cdot\!918 \sin\{\phi_1 - 2\phi_2 - 13^\circ\, 1'\!\cdot\!5\}$$
$$+1\cdot085 \sin\{\phi - \phi_2\}$$

ϕ, ϕ_1, ϕ_2 being the mean longitudes of *Jupiter*, *Saturn*, and *Uranus*, respectively. In the reduction of the Greenwich Observations, the latter correction was already taken into account. M. Hansen having also found some new inequalities in the motion of *Uranus*, depending on the square of the disturbing force, I re-computed the values of these, following the same method as that given by M. Delaunay in the *Conn. des Temps* for 1845, and my results agreed very closely with his, the terms to be added to the longitude being

$$+32''\!\cdot\!00 \sin\{3\phi_2 - 6\phi_1 + 2\phi + 22^\circ\, 18'\!\cdot\!8\}$$
$$-\ \ 8\cdot35 \sin\{2\phi_2 - 6\phi_1 + 2\phi + 39\ 10\cdot5\}$$
$$-\ \ 1\cdot49 \sin\{4\phi_2 - 6\phi_1 + 2\phi + 34\ 48\cdot4\}.$$

With respect to the inequalities of higher orders neglected by Bouvard, I considered that the most important of them would be, either those of long period, or those whose period was nearly equal to that of *Uranus*. During three-fourths of a revolution of the planet, the effects of the former class would be nearly confounded with those arising from a change in the epoch and mean motion, and those of the latter class with the effects produced by a constant change in the eccentricity and longitude of the perihelion. The position of the planet to be determined would, therefore, be little affected by these terms, and the others would probably be much smaller than those which would necessarily be neglected in a first approximation to the perturbations produced by the new planet.

8. Taking into account the several corrections above-mentioned, the residual differences between the theoretical and observed heliocentric longitudes were the following:—

ANCIENT OBSERVATIONS.

Year.	Observation – Theory.
1690	+61″·2
1712	+92·7
1715	+73·8
1750	−47·6
1753	−39·5
1756	−45·7
1764	−34·9
1769	−19·3
1771	− 2·3

MODERN OBSERVATIONS.

Year.	Observation – Theory.	Year.	Observation – Theory.
1780	+ 3″·46	1813	+22″·00
1783	+ 8·45	1816	+22·88
1786	+12·36	1819	+20·69
1789	+19·02	1822	+20·97
1792	+18·70	1825	+18·16
1795	+21·38	1828	+10·82
1798	+20·95	1831	− 3·98
1801	+22·21	1834	−20·80
1804	+24·16	1837	−42·66
1807	+22·07	1840	−66·64
1810	+23·16		

9. It is easily seen, that the series expressing the correction of the *mean* longitude in terms of the corrections applied to the elements of the orbit, is more convergent than that which gives the correction of the *true* longitude, and the same thing is true for the perturbations of the mean longitude, as compared with those of the true. The corrections found above were accordingly converted into corrections of mean longitude by multiplying each of them by the factor $\frac{r^2}{ab}$, r being the radius vector, and a and b the semi-axes of the orbit. Hence these latter corrections were found to be the following:—

ANCIENT OBSERVATIONS.

Year.	Observation – Theory.
1690	+62″·6
1712	+84·5
1715	+67·2
1750	−51·8
1753	−43·2
1756	−50·1
1764	−37·8
1769	−20·5
1771	− 2·4

MODERN OBSERVATIONS.

Year.	Observation – Theory.	Year.	Observation – Theory.
1780	+ 3″·42	1813	+21″·19
1783	+ 8·19	1816	+22·50
1786	+11·74	1819	+20·78
1789	+17·75	1822	+21·50
1792	+17·22	1825	+18·97
1795	+19·52	1828	+11·50
1798	+19·06	1831	− 4·29
1801	+20·24	1834	−22·63
1804	+22·19	1837	−46·70
1807	+20·52	1840	−73·09
1810	+21·89		

These numbers form the basis of the subsequent investigations.

10. Let $\delta\epsilon$, δa, δe, and $\delta\varpi$ denote the corrections to be applied to the tabular elements of *Uranus*, then the correction of the mean longitude at any time t is

$$= \delta\epsilon + 2e^2\delta\varpi + t\,\delta n - \left\{2\cos(nt+\epsilon-\varpi) + \frac{e}{2}\cos 2(nt+\epsilon-\varpi)\right\} e\,\delta\varpi$$

$$+ \left\{2\sin(nt+\epsilon-\varpi) + \frac{e}{2}\sin 2(nt+\epsilon-\varpi)\right\} \delta e.$$

If we include the small term $2e^2\delta\varpi$ in the quantity $\delta\epsilon$, this correction may be put under the following form:—

$$\delta\epsilon + t\,\delta n + \cos nt\,\delta x_1 + \sin nt\,\delta y_1 + \cos 2nt\,\delta x_2 + \sin 2nt\,\delta y_2$$

in which expression

$$\delta x_2 = \frac{1}{4} e\{\cos(\epsilon-\varpi)\,\delta x_1 + \sin(\epsilon-\varpi)\,\delta y_1\}$$

$$\delta y_2 = -\frac{1}{4} e\{\sin(\epsilon-\varpi)\,\delta x_1 + \cos(\epsilon-\varpi)\,\delta y_1\}.$$

11. Also, adopting the notation of Pontécoulant's *Théorie Analytique*, the perturbations of mean longitude

$$= \frac{m'}{2}\Sigma F_i \sin i(nt-n't+\epsilon-\epsilon')$$

$$+ m'e\,\Sigma G_i \sin\{i(nt-n't+\epsilon-\epsilon')-(nt+\epsilon-\varpi)\}$$

$$+ m'e'\,\Sigma H_i \sin\{i(nt-n't+\epsilon-\epsilon')-(nt+\epsilon-\varpi')\}.$$

Where the accented letters belong to the disturbing planet, i takes all integral values, positive and negative, except zero, and if we put $i(n-n')=z$, the values of F_i, G_i and H_i are the following:—

$$F_i = \left\{\frac{3\,in^4}{z^2(z^2-n^2)} + \frac{in^2}{z^2-n^2}\right\} aA_i + \frac{2n^3}{z(z^2-n^2)} a^2\frac{dA_i}{da},$$

$$G_i = \left\{-\frac{3i(i-1)n^4}{(z-n)^2 z(z-2n)} - \frac{i(i+1)n^2}{z(z-2n)} + \frac{in^2}{z^2-n^2} + \frac{3\,in^3}{z(z-n)(z-2n)}\right\} aA_i$$

$$+ \left\{-\frac{3}{2}\frac{(i-1)n^4}{(z-n)^2 z(z-2n)} - \frac{1}{2}\frac{(i-1)n^2}{z(z-2n)} - \frac{1}{2}\frac{n^2}{z^2-n^2} - \frac{2\,in^3}{z(z-n)(z-2n)}\right\} a^2\frac{dA_i}{da}$$

$$- \frac{n^3}{z(z-n)(z-2n)} a^3\frac{d^2A_i}{da^2},$$

$$H_i = \left\{\frac{3}{2}\frac{(i-1)(2i-1)\,n^4}{(z-n)^2 z(z-2n)} + \frac{1}{2}\frac{(i-1)(2i-1)\,n^2}{z(z-2n)}\right\} aA_{i-1}$$
$$+\left\{\frac{3}{2}\frac{(i-1)\,n^4}{(z-n)^2 z(z-2n)} + \frac{1}{2}\frac{(i-1)\,n^2}{z(z-2n)} + \frac{2in^3}{z(z-n)(z-2n)}\right\} a^2\frac{dA_{i-1}}{da}$$
$$+\frac{n^3}{z(z-n)(z-2n)}\,a^3\frac{d^2A_{i-1}}{da^2}.$$

12. Now, if we assume $\dfrac{a}{a'}$ or $\alpha = \sin 30° = 0{\cdot}5$, the values of the fundamental quantities b, $\alpha\dfrac{db}{d\alpha}$, $\alpha^2\dfrac{d^2b}{d\alpha^2}$, will be

$$\log b_0 = 0{\cdot}33170 \qquad \log \alpha\frac{db_0}{d\alpha} = 9{\cdot}53765 \qquad \log \alpha^2\frac{d^2b_0}{d\alpha^2} = 9{\cdot}77848$$

$$\log b_1 = 9{\cdot}74497 \qquad \log \alpha\frac{db_1}{d\alpha} = 9{\cdot}83868 \qquad \log \alpha^2\frac{d^2b_1}{d\alpha^2} = 9{\cdot}70857$$

$$\log b_2 = 9{\cdot}32425 \qquad \log \alpha\frac{db_2}{d\alpha} = 9{\cdot}68012 \qquad \log \alpha^2\frac{d^2b_2}{d\alpha^2} = 9{\cdot}87776$$

$$\log b_3 = 8{\cdot}94670 \qquad \log \alpha\frac{db_3}{d\alpha} = 9{\cdot}46315 \qquad \log \alpha^2\frac{d^2b_3}{d\alpha^2} = 9{\cdot}86253$$

Hence the principal inequalities of mean longitude, produced by the action of a planet whose mass is $\dfrac{m'}{5000}$, that of the Sun being unity, and the eccentricity of whose orbit is $\dfrac{e'}{20}$ will be the following:—

$$\begin{array}{llll}
 & '' & & \\
- & 36{\cdot}99\, m' & \sin & \{nt - n't + \epsilon - \epsilon'\} \\
+ & 58{\cdot}97\, m' & \sin 2 & \{nt - n't + \epsilon - \epsilon'\} \\
+ & 5{\cdot}80\, m' & \sin 3 & \{nt - n't + \epsilon - \epsilon'\} \\
+ & 2{\cdot}06\, m' & \sin & \{n't + \epsilon' - \varpi\} \\
- & 4{\cdot}30\, m'e' & \sin & \{n't + \epsilon' - \varpi'\} \\
+ & 31{\cdot}25\, m' & \sin & \{nt - 2n't + \epsilon - 2\epsilon' + \varpi\} \\
- & 12{\cdot}14\, m'e' & \sin & \{nt - 2n't + \epsilon - 2\epsilon' + \varpi'\} \\
+ & 48{\cdot}55\, m' & \sin & \{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi\} \\
- & 93{\cdot}01\, m'e' & \sin & \{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi'\}.
\end{array}$$

To these may be added the following, which are of two dimensions in terms of the eccentricities:—

$$+ 0''\!\cdot\!57\, m' \quad \sin 3\,\{nt - n't + \epsilon - \epsilon'\}$$
$$- 1\cdot 08\, m'e' \sin \quad \{3\,(nt - n't + \epsilon - \epsilon') - \varpi + \varpi'\}.$$

These expressions may be put under the following form:—

$$\begin{aligned} &h_1 \cos (n - n')\, t + h_2 \cos 2\,(n - n')\, t + h_3 \cos 3\,(n - n')\, t \\ &+ k_1 \sin (n - n')\, t + k_2 \sin 2\,(n - n')\, t + k_3 \sin 3\,(n - n')\, t \\ &+ p_1 \cos n't + p_2 \cos (n - 2n')\, t + p_3 \cos (2n - 3n')\, t \\ &+ q_1 \sin n't + q_2 \sin (n - 2n')\, t + q_3 \sin (2n - 3n')\, t. \end{aligned}$$

13. Let the time of the mean opposition in 1810 be taken as the epoch from which t is reckoned; this date, expressed in decimal parts of a year, will be 1810·328. Also, let 3 synodic periods of *Uranus*, $= 3\cdot 0362$ years, be taken for the unit of time; then the change of the mean anomaly in an unit of time will be $13^\circ\, 0'\cdot 5$; also $n = 13^\circ\, 0'\cdot 6$, $n' = 4^\circ\, 36'\cdot 0$

$$\therefore\; n - n' = 8^\circ\, 24'\cdot 6, \qquad n - 2n' = 3^\circ\, 48'\cdot 6, \qquad 2n - 3n' = 12^\circ\, 13'\cdot 2.$$

Hence the equations of condition given by the modern observations will be of the form

$$\begin{aligned} c'' = \;& \delta\epsilon + \delta x_1 \cos \{13^\circ\; 0'\cdot 5\}\, t + \delta x_2 \cos \{26^\circ\; 1'\cdot 0\}\, t \\ & + t\, \delta n + \delta y_1 \sin \{13\;\; 0\cdot 5\}\, t + \delta y_2 \sin \{26\;\; 1\cdot 0\}\, t \\ & + h_1 \cos \{\;8^\circ\, 24'\cdot 6\}\, t + h_2 \cos \{16^\circ\, 49'\cdot 2\}\, t + h_3 \cos \{25^\circ\, 13'\cdot 8\}\, t \\ & + k_1 \sin \{\;8\; 24\cdot 6\}\, t + k_2 \sin \{16\; 49\cdot 2\}\, t + k_3 \sin \{25\; 13\cdot 8\}\, t \\ & + p_1 \cos \{\;4^\circ\, 36'\cdot 0\}\, t + p_2 \cos \{\;3^\circ\, 48'\cdot 6\}\, t + p_3 \cos \{12^\circ\, 13'\cdot 2\}\, t \\ & + q_1 \sin \{\;4\; 36\cdot 0\}\, t + q_2 \sin \{\;3\; 48\cdot 6\}\, t + q_3 \sin \{12\; 13\cdot 2\}\, t \end{aligned}$$

in which t assumes all integral values from -10 to $+10$ in succession, and the several values of c'' are contained in the table given in Article 9.

14. The final equations for the corrections of the elliptic elements will be found by multiplying each equation successively by the coefficients of $\delta\epsilon$, δn, δx_1, and δy_1, which occur in it, and adding the several results.

Let the equations be treated in a similar manner with reference to the quantities h_1, k_1, h_2, k_2, h_3, k_3, p_2, q_2, p_3, q_3.

It will be seen that, in consequence of the arrangement which has been given to the equations of condition, the equations thus formed naturally separate themselves into two groups, one of which involves only $\delta\epsilon$, δx_1, δx_2, with the quantities h and p, while the other involves δn, δy_1, δy_2, with the quantities k and q.

Also the coefficients in these equations are easily calculated by the following formulæ, putting $t=10$ in their right-hand members:—

$$\Sigma 2\cos mt = \frac{\sin m\,(t+\frac{1}{2})}{\sin\frac{1}{2}m}$$

$$\Sigma 2t\sin mt = \frac{(t+1)\sin mt - t\sin m\,(t+1)}{2\sin^2\frac{1}{2}m}$$

$$\Sigma 2\cos mt\cos nt = \frac{1}{2}\left\{\frac{\sin(m-n)(t+\frac{1}{2})}{\sin\frac{1}{2}(m-n)} + \frac{\sin(m+n)(t+\frac{1}{2})}{\sin\frac{1}{2}(m+n)}\right\}$$

$$\Sigma 2\sin mt\sin nt = \frac{1}{2}\left\{\frac{\sin(m-n)(t+\frac{1}{2})}{\sin\frac{1}{2}(m-n)} - \frac{\sin(m+n)(t+\frac{1}{2})}{\sin\frac{1}{2}(m+n)}\right\}$$

$$\Sigma 2\cos^2 mt = t+\frac{1}{2}+\frac{1}{2}\frac{\sin m\,(2t+1)}{\sin m}$$

$$\Sigma 2\sin^2 mt = t+\frac{1}{2}-\frac{1}{2}\frac{\sin m\,(2t+1)}{\sin m}.$$

15. By performing the calculations, the equations of the first group are found to be the following:—

$$\begin{array}{lrl}
(\epsilon) & 151''\!\cdot\!48 = & 21\cdot0000\,\delta\epsilon + 6\cdot0670\,\delta x_1 - 4\cdot4358\,\delta x_2 \\
 & & +13\cdot6320\,h_1 + 0\cdot4043\,h_2 - 4\cdot5608\,h_3 \\
 & & +18\cdot6046\,p_1 + 19\cdot3384\,p_2 + 7\cdot3721\,p_3 \\
(x) & 246\cdot48 = & 6\cdot0670\,\delta\epsilon + 8\cdot2821\,\delta x_1 + 4\cdot1762\,\delta x_2 \\
 & & +\ 7\cdot4041\,h_1 + 8\cdot2523\,h_2 + 4\cdot6963\,h_3 \\
 & & +\ 6\cdot5389\,p_1 + 6\cdot3978\,p_2 + 8\cdot1831\,p_3 \\
(h_1) & 209\cdot74 = & 13\cdot6320\,\delta\epsilon + 7\cdot4041\,\delta x_1 - 0\cdot2337\,\delta x_2 \\
 & & +10\cdot7022\,h_1 + 4\cdot5356\,h_2 - 0\cdot0018\,h_3 \\
 & & +12\cdot7013\,p_1 + 12\cdot9883\,p_2 + 8\cdot0038\,p_3 \\
(h_2) & 242\cdot68 = & 0\cdot4043\,\delta\epsilon + 8\cdot2523\,\delta x_1 + 7\cdot5650\,\delta x_2 \\
 & & +\ 4\cdot5356\,h_1 + 10\cdot2960\,h_2 + 8\cdot1944\,h_3 \\
 & & +\ 1\cdot7866\,p_1 + 1\cdot3667\,p_2 + 7\cdot6671\,p_3
\end{array}$$

$$(h_3) \quad 86''\!\cdot67 = -\ 4\cdot5608\,\delta\epsilon + 4\cdot6963\,\delta x_1 + 10\cdot5023\,\delta x_2 - 0\cdot0018\,h_1 + 8\cdot1944\,h_2 + 10\cdot7071\,h_3 - 3\cdot0812\,p_1 - 3\cdot5347\,p_2 + 3\cdot8855\,p_3$$

$$(p_2) \quad 165\cdot99 = 19\cdot3384\,\delta\epsilon + 6\cdot3978\,\delta x_1 - 3\cdot4948\,\delta x_2 + 12\cdot9883\,h_1 + 1\cdot3667\,h_2 - 3\cdot5347\,h_3 + 17\cdot2795\,p_1 + 17\cdot9106\,p_2 + 7\cdot5423\,p_3$$

$$(p_3) \quad 242\cdot56 = 7\cdot3721\,\delta\epsilon + 8\cdot1831\,\delta x_1 + 3\cdot4071\,\delta x_2 + 8\cdot0038\,h_1 + 7\cdot6671\,h_2 + 3\cdot8855\,h_3 + 7\cdot6127\,p_1 + 7\cdot5423\,p_2 + 8\cdot2019\,p_3.$$

16. By means of (ϵ) eliminate $\delta\epsilon$ from each of the other equations, and these latter become

$$(x) \quad 202''\!\cdot72 = 6\cdot5294\,\delta x_1 + 5\cdot4577\,\delta x_2 + 3\cdot4658\,h_1 + 8\cdot1355\,h_2 + 6\cdot0139\,h_3 + 1\cdot1640\,p_1 + 0\cdot8109\,p_2 + 6\cdot0533\,p_3$$

$$(h_1) \quad 111\cdot41 = 3\cdot4658\,\delta x_1 + 2\cdot6458\,\delta x_2 + 1\cdot8531\,h_1 + 4\cdot2731\,h_2 + 2\cdot9588\,h_3 + 0\cdot6243\,p_1 + 0\cdot4349\,p_2 + 3\cdot2183\,p_3$$

$$(h_2) \quad 239\cdot76 = 8\cdot1355\,\delta x_1 + 7\cdot6504\,\delta x_2 + 4\cdot2731\,h_1 + 10\cdot2882\,h_2 + 8\cdot2822\,h_3 + 1\cdot4284\,p_1 + 0\cdot9944\,p_2 + 7\cdot5252\,p_3$$

$$(h_3) \quad 119\cdot57 = 6\cdot0139\,\delta x_1 + 9\cdot5389\,\delta x_2 + 2\cdot9588\,h_1 + 8\cdot2822\,h_2 + 9\cdot7166\,h_3 + 0\cdot9593\,p_1 + 0\cdot6652\,p_2 + 5\cdot4866\,p_3$$

$$(p_2) \quad 26\cdot50 = 0\cdot8109\,\delta x_1 + 0\cdot5900\,\delta x_2 + 0\cdot4349\,h_1 + 0\cdot9944\,h_2 + 0\cdot6652\,h_3 + 0\cdot1470\,p_1 + 0\cdot1024\,p_2 + 0\cdot7535\,p_3$$

$$(p_3) \quad 189\cdot38 = 6\cdot0533\,\delta x_1 + 4\cdot9643\,\delta x_2 + 3\cdot2183\,h_1 + 7\cdot5252\,h_2 + 5\cdot4866\,h_3 + 1\cdot0815\,p_1 + 0\cdot7535\,p_2 + 5\cdot6139\,p_3.$$

17. Again, by means of (x) eliminate δx_1 from each of the other equations, and we find

$$(h_1) \quad 3''\!\cdot807 = -0\cdot2512\,\delta x_2 + 0\cdot0135\,h_1 - 0\cdot0452\,h_2 - 0\cdot2334\,h_3 + 0\cdot0065\,p_1 + 0\cdot0045\,p_2 + 0\cdot0052\,p_3$$

$$(h_2) \quad -12\cdot821 = 0\cdot8502\,\delta x_2 - 0\cdot0452\,h_1 + 0\cdot1515\,h_2 + 0\cdot7890\,h_3 - 0\cdot0219\,p_1 - 0\cdot0160\,p_2 - 0\cdot0171\,p_3$$

$$(h_3) \qquad -67''{\cdot}149 = 4{\cdot}5120\,\delta x_2 - 0{\cdot}2334\,h_1 + 0{\cdot}7890\,h_2 + 4{\cdot}1775\,h_3$$
$$- 0{\cdot}1128\,p_1 - 0{\cdot}0817\,p_2 - 0{\cdot}0888\,p_3$$

$$(p_2) \qquad 1{\cdot}327 = -0{\cdot}0878\,\delta x_2 + 0{\cdot}0045\,h_1 - 0{\cdot}0160\,h_2 - 0{\cdot}0817\,h_3$$
$$+ 0{\cdot}0024\,p_1 + 0{\cdot}0017\,p_2 + 0{\cdot}0018\,p_3$$

$$(p_3) \qquad 1{\cdot}448 = -0{\cdot}0955\,\delta x_2 + 0{\cdot}0052\,h_1 - 0{\cdot}0171\,h_2 - 0{\cdot}0888\,h_3$$
$$+ 0{\cdot}0024\,p_1 + 0{\cdot}0018\,p_2 + 0{\cdot}0020\,p_3$$

18. Similarly, the equations of the second group are found to be

$$(n) \qquad -171''{\cdot}27 = 77{\cdot}0000\,\delta n + 9{\cdot}3938\,\delta y_1 - 1{\cdot}2183\,\delta y_2$$
$$+ 8{\cdot}8463\,k_1 + 7{\cdot}3034\,k_2 - 0{\cdot}5927\,k_3$$
$$+ 5{\cdot}7519\,q_1 + 4{\cdot}8755\,q_2 + 9{\cdot}5583\,q_3$$

$$(y) \qquad -166{\cdot}33 = 93{\cdot}9380\,\delta n + 12{\cdot}7179\,\delta y_1 + 1{\cdot}8907\,\delta y_2$$
$$+ 11{\cdot}2022\,k_1 + 11{\cdot}0848\,k_2 + 2{\cdot}6731\,k_3$$
$$+ 7{\cdot}0956\,q_1 + 5{\cdot}9913\,q_2 + 12{\cdot}7441\,q_3$$

$$(k_1) \qquad -182{\cdot}87 = 88{\cdot}4630\,\delta n + 11{\cdot}2022\,\delta y_1 - 0{\cdot}3210\,\delta y_2$$
$$+ 10{\cdot}2978\,k_1 + 9{\cdot}0964\,k_2 + 0{\cdot}4061\,k_3$$
$$+ 6{\cdot}6370\,q_1 + 5{\cdot}6163\,q_2 + 11{\cdot}3346\,q_3$$

$$(k_2) \qquad -89{\cdot}07 = 73{\cdot}0340\,\delta n + 11{\cdot}0848\,\delta y_1 + 4{\cdot}8266\,\delta y_2$$
$$+ 9{\cdot}0964\,k_1 + 10{\cdot}7040\,k_2 + 5{\cdot}4376\,k_3$$
$$+ 5{\cdot}5855\,q_1 + 4{\cdot}6976\,q_2 + 10{\cdot}9375\,q_3$$

$$(k_3) \qquad +124{\cdot}80 = -5{\cdot}9270\,\delta n + 2{\cdot}6731\,\delta y_1 + 10{\cdot}4253\,\delta y_2$$
$$+ 0{\cdot}4061\,k_1 + 5{\cdot}4376\,k_2 + 10{\cdot}2929\,k_3$$
$$- 0{\cdot}2497\,q_1 - 0{\cdot}2643\,q_2 + 2{\cdot}1788\,q_3$$

$$(q_2) \qquad -107{\cdot}02 = 48{\cdot}7550\,\delta n + 5{\cdot}9913\,\delta y_1 - 0{\cdot}6614\,\delta y_2$$
$$+ 5{\cdot}6163\,k_1 + 4{\cdot}6976\,k_2 - 0{\cdot}2643\,k_3$$
$$+ 3{\cdot}6475\,q_1 + 3{\cdot}0894\,q_2 + 6{\cdot}0897\,q_3$$

$$(q_3) \qquad -175{\cdot}89 = 95{\cdot}5830\,\delta n + 12{\cdot}7441\,\delta y_1 + 1{\cdot}3845\,\delta y_2$$
$$+ 11{\cdot}3346\,k_1 + 10{\cdot}9375\,k_2 + 2{\cdot}1788\,k_3$$
$$+ 7{\cdot}2084\,q_1 + 6{\cdot}0897\,q_2 + 12{\cdot}7981\,q_3$$

19. By means of (n) eliminate δn from each of the other equations, and we have

$$
\begin{array}{lrl}
(y) & 42''\!\cdot\!61 = & 1\cdot2578\,\delta y_1 + 3\cdot3771\,\delta y_2 + 0\cdot4100\,k_1 + 2\cdot1748\,k_2 \\
 & & + 3\cdot3962\,k_3 + 0\cdot0785\,q_1 + 0\cdot0433\,q_2 + 1\cdot0833\,q_3 \\
(k_1) & 13\cdot90 = & 0\cdot4100\,\delta y_1 + 1\cdot0787\,\delta y_2 + 0\cdot1346\,k_1 + 0\cdot7057\,k_2 \\
 & & + 1\cdot0871\,k_3 + 0\cdot0288\,q_1 + 0\cdot0150\,q_2 + 0\cdot3534\,q_3 \\
(k_2) & 73\cdot38 = & 2\cdot1748\,\delta y_1 + 5\cdot9822\,\delta y_2 + 0\cdot7057\,k_1 + 3\cdot7767\,k_2 \\
 & & + 5\cdot9998\,k_3 + 0\cdot1298\,q_1 + 0\cdot0732\,q_2 + 1\cdot8715\,q_3 \\
(k_3) & 111\cdot62 = & 3\cdot3962\,\delta y_1 + 10\cdot3315\,\delta y_2 + 1\cdot0871\,k_1 + 5\cdot9998\,k_2 \\
 & & + 10\cdot2473\,k_3 + 0\cdot1930\,q_1 + 0\cdot1110\,q_2 + 2\cdot9145\,q_3 \\
(q_2) & 1\cdot42 = & 0\cdot0433\,\delta y_1 + 0\cdot1100\,\delta y_2 + 0\cdot0150\,k_1 + 0\cdot0732\,k_2 \\
 & & + 0\cdot1110\,k_3 + 0\cdot0055\,q_1 + 0\cdot0023\,q_2 + 0\cdot0375\,q_3 \\
(q_3) & 36\cdot72 = & 1\cdot0833\,\delta y_1 + 2\cdot8969\,\delta y_2 + 0\cdot3534\,k_1 + 1\cdot8715\,k_2 \\
 & & + 2\cdot9145\,k_3 + 0\cdot0684\,q_1 + 0\cdot0375\,q_2 + 0\cdot9330\,q_3
\end{array}
$$

20. Again, eliminating δy_1 by means of (y) we find

$$
\begin{array}{lrl}
(k_1) & 0''\!\cdot\!009 = & -0\cdot0221\,\delta y_2 + 0\cdot0010\,k_1 - 0\cdot0032\,k_2 - 0\cdot0200\,k_3 \\
 & & + 0\cdot0032\,q_1 + 0\cdot0009\,q_2 + 0\cdot0003\,q_3 \\
(k_2) & -0\cdot301 = & 0\cdot1430\,\delta y_2 - 0\cdot0032\,k_1 + 0\cdot0162\,k_2 + 0\cdot1274\,k_3 \\
 & & - 0\cdot0059\,q_1 - 0\cdot0017\,q_2 - 0\cdot0016\,q_3 \\
(k_3) & -3\cdot443 = & 1\cdot2129\,\delta y_2 - 0\cdot0200\,k_1 + 0\cdot1274\,k_2 + 1\cdot0769\,k_3 \\
 & & - 0\cdot0189\,q_1 - 0\cdot0059\,q_2 - 0\cdot0105\,q_3 \\
(q_2) & -0\cdot045 = & -0\cdot0062\,\delta y_2 + 0\cdot0009\,k_1 - 0\cdot0017\,k_2 - 0\cdot0059\,k_3 \\
 & & + 0\cdot0028\,q_1 + 0\cdot0008\,q_2 + 0\cdot0002\,q_3 \\
(q_3) & +0\cdot017 = & -0\cdot0116\,\delta y_2 + 0\cdot0003\,k_1 - 0\cdot0016\,k_2 - 0\cdot0105\,k_3 \\
 & & + 0\cdot0008\,q_1 + 0\cdot0002\,q_2 + 0\cdot0000\,q_3
\end{array}
$$

21. From the equations remaining in the two groups after the elimination of $\delta\epsilon$, δn, δx_1, δy_1, it will be easy, when approximate values of the mass and mean longitude of the disturbing planet have been found, to deduce the final equations for determining these quantities more accurately by the method of least squares.

It may be observed, however, that the equations in each group are very nearly identical with each other, and therefore two final equations may be formed by simply adding together the several equations of each group, after giving the unknown quantities the same sign in them all. Thus we find

$$
\begin{aligned}
86''\!\cdot\!552 = &- 5\cdot7967\ \delta x_2 + 0\cdot3018\ h_1 - 1\cdot0188\ h_2 - 5\cdot3704\ h_3 \\
&+ 0\cdot1460\ p_1 + 0\cdot1056\ p_2 + 0\cdot1149\ p_3 \\
3\cdot725 = &- 1\cdot3958\ \delta y_2 + 0\cdot0254\ k_1 - 0\cdot1501\ k_2 - 1\cdot2407\ k_3 \\
&+ 0\cdot0316\ q_1 + 0\cdot0095\ q_2 + 0\cdot0127\ q_3
\end{aligned}
$$

22. If in the expressions before given for δx_2 and δy_2 we substitute $e = 0\cdot046679$ and $\epsilon - \varpi = 50^\circ\ 15'\cdot8$, we obtain

$$
\begin{aligned}
\delta x_2 &= 0\cdot007460\ \delta x_1 + 0\cdot008974\ \delta y_1 \\
\delta y_2 &= -0\cdot008974\ \delta x_1 + 0\cdot007460\ \delta y_1
\end{aligned}
$$

Substituting these values in the equations (x) and (y), and in those just found, it may be seen that by adding to the latter equations

$$0\cdot006768\,(x) + 0\cdot040287\,(y)$$

and $$-0\cdot001869\,(x) + 0\cdot008187\,(y) \text{ respectively,}$$

δx_1 and δy_1 will be eliminated, and we shall obtain the following equations:

$$
\begin{aligned}
(1)\quad 89''\!\cdot\!641 = &\ 0\cdot3252\ h_1 - 0\cdot9637\ h_2 - 5\cdot3297\ h_3 \\
&+ 0\cdot0165\ k_1 + 0\cdot0876\ k_2 + 0\cdot1368\ k_3 \\
&+ 0\cdot1539\ p_1 + 0\cdot1111\ p_2 + 0\cdot1559\ p_3 \\
&+ 0\cdot0032\ q_1 + 0\cdot0017\ q_2 + 0\cdot0436\ q_3 \\
(2)\quad 3\cdot695 = &- 0\cdot0065\ h_1 - 0\cdot0152\ h_2 - 0\cdot0112\ h_3 \\
&+ 0\cdot0288\ k_1 - 0\cdot1323\ k_2 - 1\cdot2129\ k_3 \\
&- 0\cdot0022\ p_1 - 0\cdot0015\ p_2 - 0\cdot0113\ p_3 \\
&+ 0\cdot0323\ q_1 + 0\cdot0099\ q_2 + 0\cdot0215\ q_3
\end{aligned}
$$

23. These equations would be sufficient for determining the mass of the disturbing planet and its longitude at the epoch, if the eccentricity of the orbit were neglected. We will now proceed to find equations from the ancient observations for determining the eccentricity and longitude of the perihelion.

The equations of condition given by the ancient observations are the following:—

$$
\begin{aligned}
62''\!\cdot\!6 = \ & \delta\epsilon - 0\cdot8776\ \delta x_1 + 0\cdot5402\ \delta x_2 + 0\cdot8712\ h_1 + 0\cdot5180\ h_2 \\
&- 39\cdot31\ \delta n - 0\cdot4795\ \delta y_1 + 0\cdot8415\ \delta y_2 + 0\cdot4909\ k_1 + 0\cdot8554\ k_2 \\
&+ 0\cdot0314\ h_3 - 0\cdot9999\ p_1 - 0\cdot8640\ p_2 - 0\cdot5055\ p_3 \\
&+ 0\cdot9995\ k_3 + 0\cdot0145\ q_1 - 0\cdot5035\ q_2 - 0\cdot8628\ q_3
\end{aligned}
$$

$$\begin{aligned}
84''.5 = {} & \delta\epsilon + 0.4975\,\delta x_1 - 0.5050\,\delta x_2 + 0.0288\,h_1 - 0.9984\,h_2 \\
& - 32.30\,\delta n - 0.8675\,\delta y_1 - 0.8631\,\delta y_2 + 0.9996\,k_1 + 0.0573\,k_2 \\
& - 0.0860\,h_3 - 0.8534\,p_1 - 0.5456\,p_2 + 0.8220\,p_3 \\
& - 0.9963\,k_3 - 0.5213\,q_1 - 0.8380\,q_2 - 0.5695\,q_3
\end{aligned}$$

$$\begin{aligned}
67.2 = {} & \delta\epsilon + 0.6732\,\delta x_1 - 0.0935\,\delta x_2 - 0.1120\,h_1 - 0.9749\,h_2 \\
& - 31.34\,\delta n - 0.7394\,\delta y_1 - 0.9956\,\delta y_2 + 0.9937\,k_1 - 0.2227\,k_2 \\
& + 0.3305\,h_3 - 0.8105\,p_1 - 0.4912\,p_2 + 0.9206\,p_3 \\
& - 0.9438\,k_3 - 0.5857\,q_1 - 0.8711\,q_2 - 0.3905\,q_3
\end{aligned}$$

$$\begin{aligned}
-51.8 = {} & \delta\epsilon - 0.2616\,\delta x_1 - 0.8631\,\delta x_2 - 0.9649\,h_1 + 0.8618\,h_2 \\
& - 19.59\,\delta n + 0.9652\,\delta y_1 - 0.5050\,\delta y_2 - 0.2627\,k_1 + 0.5073\,k_2 \\
& - 0.6982\,h_3 - 0.0023\,p_1 + 0.2650\,p_2 - 0.5090\,p_3 \\
& - 0.7159\,k_3 - 1.0000\,q_1 - 0.9642\,q_2 + 0.8607\,q_3
\end{aligned}$$

$$\begin{aligned}
-43.2 = {} & \delta\epsilon - 0.4741\,\delta x_1 - 0.5505\,\delta x_2 - 0.9154\,h_1 + 0.6758\,h_2 \\
& - 18.58\,\delta n + 0.8805\,\delta y_1 - 0.8348\,\delta y_2 - 0.4025\,k_1 + 0.7371\,k_2 \\
& - 0.3220\,h_3 + 0.0787\,p_1 + 0.3291\,p_2 - 0.6814\,p_3 \\
& - 0.9467\,k_3 - 0.9969\,q_1 - 0.9443\,q_2 + 0.7319\,q_3
\end{aligned}$$

$$\begin{aligned}
-50.1 = {} & \delta\epsilon - 0.6430\,\delta x_1 - 0.1731\,\delta x_2 - 0.8543\,h_1 + 0.4599\,h_2 \\
& - 17.68\,\delta n + 0.7659\,\delta y_1 - 0.9849\,\delta y_2 - 0.5198\,k_1 + 0.8879\,k_2 \\
& + 0.0686\,h_3 + 0.1510\,p_1 + 0.3848\,p_2 - 0.8085\,p_3 \\
& - 0.9976\,k_3 - 0.9885\,q_1 - 0.9230\,q_2 + 0.5885\,q_3
\end{aligned}$$

$$\begin{aligned}
-37.8 = {} & \delta\epsilon - 0.9492\,\delta x_1 + 0.8021\,\delta x_2 - 0.6189\,h_1 - 0.2340\,h_2 \\
& - 15.25\,\delta n + 0.3145\,\delta y_1 - 0.5972\,\delta y_2 - 0.7855\,k_1 + 0.9722\,k_2 \\
& + 0.9085\,h_3 + 0.3396\,p_1 + 0.5287\,p_2 - 0.9939\,p_3 \\
& - 0.4179\,k_3 - 0.9406\,q_1 - 0.8488\,q_2 + 0.1100\,q_3
\end{aligned}$$

$$\begin{aligned}
-20.5 = {} & \delta\epsilon - 0.9985\,\delta x_1 + 0.9942\,\delta x_2 - 0.4128\,h_1 - 0.6591\,h_2 \\
& - 13.60\,\delta n - 0.0538\,\delta y_1 + 0.1074\,\delta y_2 - 0.9108\,k_1 + 0.7520\,k_2 \\
& + 0.9571\,h_3 + 0.4607\,p_1 + 0.6182\,p_2 - 0.9711\,p_3 \\
& + 0.2899\,k_3 - 0.8875\,q_1 - 0.7860\,q_2 - 0.2385\,q_3
\end{aligned}$$

$$\begin{aligned}
-2.4 = {} & \delta\epsilon - 0.9633\,\delta x_1 + 0.8560\,\delta x_2 - 0.2807\,h_1 - 0.8424\,h_2 \\
& - 12.64\,\delta n - 0.2684\,\delta y_1 + 0.5170\,\delta y_2 - 0.9598\,k_1 + 0.5388\,k_2 \\
& + 0.7536\,h_3 + 0.5279\,p_1 + 0.6670\,p_2 - 0.9023\,p_3 \\
& + 0.6574\,k_3 - 0.8493\,q_1 - 0.7451\,q_2 - 0.4310\,q_3
\end{aligned}$$

24. From each of these equations eliminate $\delta\epsilon$, δn, δx_1, and δy_1, by means of the equations (ϵ), (n), (x), and (y), before found, and we have the following:—

$$
\begin{aligned}
-142''\!\cdot 0 = {} & 1\cdot 7265\,\delta x_2 + 0\cdot 8412\,h_1 + 1\cdot 9521\,h_2 + 1\cdot 3230\,h_3 \\
& -11\cdot 3691\,\delta y_2 + 3\cdot 6001\,k_1 - 2\cdot 8793\,k_2 - 10\cdot 9578\,k_3 \\
& - 1\cdot 6779\,p_1 - 1\cdot 6400\,p_2 + 0\cdot 2249\,p_3 \\
& + 2\cdot 6815\,q_1 + 1\cdot 8369\,q_2 + 0\cdot 2995\,q_3 \\
-105\cdot 2 = {} & -0\cdot 4681\,\delta x_2 - 0\cdot 7311\,h_1 - 1\cdot 2776\,h_2 - 0\cdot 0609\,h_3 \\
& -9\cdot 6249\,\delta y_2 + 3\cdot 7087\,k_1 - 2\cdot 1926\,k_2 - 9\cdot 5426\,k_3 \\
& - 1\cdot 7765\,p_1 - 1\cdot 4924\,p_2 + 0\cdot 2786\,p_3 \\
& + 1\cdot 6997\,q_1 + 1\cdot 1014\,q_2 + 0\cdot 7934\,q_3 \\
-126\cdot 1 = {} & -0\cdot 2035\,\delta x_2 - 0\cdot 9653\,h_1 - 1\cdot 4730\,h_2 + 0\cdot 1937\,h_3 \\
& -9\cdot 7719\,\delta y_2 + 3\cdot 5895\,k_1 - 2\cdot 5827\,k_2 - 9\cdot 5123\,k_3 \\
& - 1\cdot 7649\,p_1 - 1\cdot 4598\,p_2 - 0\cdot 2133\,p_3 \\
& + 1\cdot 5629\,q_1 + 1\cdot 0070\,q_2 + 0\cdot 8437\,q_3 \\
-199\cdot 1 = {} & -0\cdot 1917\,\delta x_2 - 1\cdot 3218\,h_1 + 1\cdot 5284\,h_2 + 0\cdot 0260\,h_3 \\
& -9\cdot 8232\,\delta y_2 + 0\cdot 8943\,k_1 - 3\cdot 4359\,k_2 - 9\cdot 9270\,k_3 \\
& - 0\cdot 7901\,p_1 - 0\cdot 5885\,p_2 - 0\cdot 3497\,p_3 \\
& + 0\cdot 2540\,q_1 + 0\cdot 1607\,q_2 + 0\cdot 4028\,q_3 \\
-174\cdot 7 = {} & 0\cdot 2985\,\delta x_2 - 1\cdot 1595\,h_1 + 1\cdot 6072\,h_2 + 0\cdot 5979\,h_3 \\
& -9\cdot 5788\,\delta y_2 + 0\cdot 7062\,k_1 - 2\cdot 9425\,k_2 - 9\cdot 5877\,k_3 \\
& - 0\cdot 6712\,p_1 - 0\cdot 4970\,p_2 - 0\cdot 3251\,p_3 \\
& + 0\cdot 1946\,q_1 + 0\cdot 1238\,q_2 + 0\cdot 3277\,q_3 \\
-166\cdot 7 = {} & 0\cdot 8171\,\delta x_2 - 1\cdot 0088\,h_1 + 1\cdot 6018\,h_2 + 1\cdot 1442\,h_3 \\
& -9\cdot 1122\,\delta y_2 + 0\cdot 5586\,k_1 - 2\cdot 4890\,k_2 - 9\cdot 0258\,k_3 \\
& - 0\cdot 5688\,p_1 - 0\cdot 4203\,p_2 - 0\cdot 2956\,p_3 \\
& + 0\cdot 1498\,q_1 + 0\cdot 0958\,q_2 + 0\cdot 2658\,q_3 \\
-114\cdot 2 = {} & 2\cdot 0482\,\delta x_2 - 0\cdot 6027\,h_1 + 1\cdot 2894\,h_2 + 2\cdot 2661\,h_3 \\
& -6\cdot 6781\,\delta y_2 + 0\cdot 2576\,k_1 - 1\cdot 3421\,k_2 - 6\cdot 4080\,k_3 \\
& - 0\cdot 3256\,p_1 - 0\cdot 2384\,p_2 - 0\cdot 1971\,p_3 \\
& + 0\cdot 0628\,q_1 + 0\cdot 0419\,q_2 + 0\cdot 1298\,q_3
\end{aligned}
$$

$$
\begin{aligned}
-72''\!\cdot\!4 = {} & 2\cdot2815\,\delta x_2 - 0\cdot3786\,h_1 + 0\cdot9257\,h_2 + 2\cdot3601\,h_3 \\
& - 4\cdot4181\,\delta y_2 + 0\cdot1283\,k_1 - 0\cdot7339\,k_2 - 4\cdot1495\,k_3 \\
& - 0\cdot1957\,p_1 - 0\cdot1428\,p_2 - 0\cdot1286\,p_3 \\
& + 0\cdot0283\,q_1 + 0\cdot0198\,q_2 + 0\cdot0671\,q_3
\end{aligned}
$$

$$
\begin{aligned}
-42\cdot0 = {} & 2\cdot1139\,\delta x_2 - 0\cdot2652\,h_1 + 0\cdot6985\,h_2 + 2\cdot1241\,h_3 \\
& - 3\cdot1027\,\delta y_2 + 0\cdot0772\,k_1 - 0\cdot4646\,k_2 - 2\cdot8790\,k_3 \\
& - 0\cdot1348\,p_1 - 0\cdot0984\,p_2 - 0\cdot0924\,p_3 \\
& + 0\cdot0154\,q_1 + 0\cdot0114\,q_2 + 0\cdot0412\,q_3
\end{aligned}
$$

25. The largest terms depending on the eccentricity of the disturbing planet occur in p_3, q_3; it will be proper, therefore, to combine the above equations in such a manner that these quantities may acquire the largest coefficients possible. This will be done by multiplying each equation by a quantity nearly proportional to the coefficient of each of the unknown quantities p_3 and q_3, and adding together the several results. It was thought unsafe to employ the first of the above equations, since it is derived from the single observation of Flamsteed made in 1690, twenty-two years anterior to any other observation.

Hence the equation for finding p_3 may be formed by multiplying the above equations, taken in order, by

$$-0\cdot8,\ -0\cdot6,\ +1\cdot0,\ +1\cdot0,\ +0\cdot9,\ +0\cdot6,\ +0\cdot4,\ +0\cdot3,$$

beginning with the second; and the equation for q_3 by multiplying the same equations by

$$1\cdot0,\ 1\cdot0,\ 0\cdot5,\ 0\cdot4,\ 0\cdot3,\ 0\cdot2,\ 0\cdot1,\ 0\cdot1.$$

Hence we obtain

$$
\begin{aligned}
-474''\!\cdot\!1 = {} & 4\cdot114\,\delta x_2 - 2\cdot817\,h_1 + 7\cdot837\,h_2 + 4\cdot528\,h_3 \\
& - 20\cdot745\,\delta y_2 - 2\cdot789\,k_1 - 6\cdot551\,k_2 - 20\cdot666\,k_3 \\
& + 0\cdot193\,p_1 + 0\cdot377\,p_2 - 1\cdot489\,p_3 \\
& - 1\cdot660\,q_1 - 1\cdot078\,q_2 - 0\cdot054\,q_3
\end{aligned}
$$

$$
\begin{aligned}
-485\cdot0 = {} & 0\cdot446\,\delta x_2 - 3\cdot308\,h_1 - 0\cdot442\,h_2 + 1\cdot629\,h_3 \\
& - 32\cdot961\,\delta y_2 + 8\cdot267\,k_1 - 8\cdot805\,k_2 - 32\cdot546\,k_3 \\
& - 4\cdot473\,p_1 - 3\cdot643\,p_2 + 0\cdot037\,p_3 \\
& + 3\cdot530\,q_1 + 2\cdot278\,q_2 + 2\cdot086\,q_3
\end{aligned}
$$

26. Eliminate δx_2 and δy_2 from these equations by means of (x) and (y) and they become

$$(3) \quad -476''\cdot 7 = -2\cdot 930 h_1 + 7\cdot 572 h_2 + 4\cdot 332 h_3 - 2\cdot 751 k_1 - 6\cdot 348 k_2 - 20\cdot 350 k_3 + 0\cdot 155 p_1 + 0\cdot 350 p_2 - 1\cdot 686 p_3 - 1\cdot 653 q_1 - 1\cdot 074 q_2 + 0\cdot 047 q_3$$

$$(4) \quad -485\cdot 9 = -3\cdot 463 h_1 - 0\cdot 805 h_2 + 1\cdot 360 h_3 + 8\cdot 345 k_1 - 8\cdot 391 k_2 - 31\cdot 900 k_3 - 4\cdot 525 p_1 - 3\cdot 679 p_2 - 0\cdot 233 p_3 + 3\cdot 545 q_1 + 2\cdot 286 q_2 + 2\cdot 292 q_3$$

These equations, with (1) and (2) of Article 22, suffice for the solution of our problem.

27. Eliminate the left-hand members from equations (2), (3), (4), by means of equation (1), and we have

$$0 = 0\cdot 4819 h_1 - 0\cdot 5950 h_2 - 5\cdot 0570 h_3 + 0\cdot 2063 p_1 + 0\cdot 1475 p_2 + 0\cdot 4300 p_3 - 0\cdot 6812 k_1 + 3\cdot 2982 k_2 + 29\cdot 5618 k_3 - 0\cdot 7804 q_1 - 0\cdot 2375 q_2 - 0\cdot 4789 q_3$$

$$0 = -1\cdot 2005 h_1 + 2\cdot 4466 h_2 - 24\cdot 0122 h_3 + 0\cdot 9735 p_1 + 0\cdot 9412 p_2 - 0\cdot 8575 p_3 - 2\cdot 6633 k_1 - 5\cdot 8825 k_2 - 19\cdot 6219 k_3 - 1\cdot 6362 q_1 - 1\cdot 0648 q_2 + 0\cdot 2791 q_3$$

$$0 = -1\cdot 7003 h_1 - 6\cdot 0294 h_2 - 27\cdot 5295 h_3 - 3\cdot 6908 p_1 - 3\cdot 0772 p_2 + 0\cdot 6118 p_3 + 8\cdot 4344 k_1 - 7\cdot 9162 k_2 - 31\cdot 1583 k_3 + 3\cdot 5621 q_1 + 2\cdot 2954 q_2 + 2\cdot 5285 q_3$$

28. If now we put $\epsilon - \epsilon' = \theta$ and $\epsilon - \varpi = \beta$, it is easily seen that

$$\frac{h_1}{m'} = -36''\cdot 99 \sin\theta \qquad \frac{h_2}{m'} = 58''\cdot 97 \sin 2\theta$$

$$\frac{k_1}{m'} = -36\cdot 99 \cos\theta \qquad \frac{k_2}{m'} = 58\cdot 97 \cos 2\theta$$

$$\frac{h_3}{m'} = 5\cdot 80 \sin 3\theta + 0\cdot 007460 \frac{p_3}{m'} + 0\cdot 008974 \frac{q_3}{m'}$$

$$\frac{k_3}{m'} = 5\cdot 80 \cos 3\theta - 0\cdot 008974 \frac{p_3}{m'} + 0\cdot 007460 \frac{q_3}{m'}$$

$$\frac{p_1}{m'} = 0\cdot 18 \sin(\theta - \beta) - 0\cdot 046247 \left\{\frac{p_3}{m'} \cos 2\theta - \frac{q_3}{m'} \sin 2\theta\right\}$$

$$\frac{q_1}{m'} = -\ 0''{\cdot}18\cos(\theta-\beta)\ +0{\cdot}046247\left\{\frac{p_3}{m'}\sin 2\theta+\frac{q_3}{m'}\cos 2\theta\right\}$$

$$\frac{p_2}{m'} = \quad 24{\cdot}91\sin(2\theta-\beta)+0{\cdot}13055\ \left\{\frac{p_3}{m'}\cos\theta-\frac{q_3}{m'}\sin\theta\right\}$$

$$\frac{q_2}{m'} = \quad 24{\cdot}91\cos(2\theta-\beta)+0{\cdot}13055\ \left\{\frac{p_3}{m'}\sin\theta+\frac{q_3}{m'}\cos\theta\right\}$$

29. Substituting these expressions in the equations of Article 27, and putting for β its value $50^\circ\ 15'{\cdot}8$, we obtain, after a slight reduction,

$$\begin{aligned}
0 = & -(1{\cdot}24782)\sin\theta+(1{\cdot}40248)\cos\theta-(1{\cdot}57155)\sin 2\theta+(2{\cdot}27388)\cos 2\theta \\
& -(1{\cdot}46746)\sin 3\theta+(2{\cdot}23430)\cos 3\theta+(9{\cdot}10380)\frac{p_3}{m'}-(9{\cdot}48254)\frac{q_3}{m'} \\
& +(8{\cdot}28455)\left\{\frac{p_3}{m'}\cos\theta-\frac{q_3}{m'}\sin\theta\right\}-(8{\cdot}49138)\left\{\frac{p_3}{m'}\sin\theta+\frac{q_3}{m'}\cos\theta\right\} \\
& -(7{\cdot}97958)\left\{\frac{p_3}{m'}\cos 2\theta-\frac{q_3}{m'}\sin 2\theta\right\}-(8{\cdot}55742)\left\{\frac{p_3}{m'}\sin 2\theta+\frac{q_3}{m'}\cos 2\theta\right\}
\end{aligned}$$

$$\begin{aligned}
0 = & \quad (1{\cdot}65083)\sin\theta+(1{\cdot}99378)\cos\theta+(2{\cdot}14259)\sin 2\theta-(2{\cdot}58192)\cos 2\theta \\
& -(2{\cdot}14400)\sin 3\theta-(2{\cdot}05631)\cos 3\theta-(9{\cdot}93475)\frac{p_3}{m'}-(8{\cdot}91803)\frac{q_3}{m'} \\
& +(9{\cdot}08947)\left\{\frac{p_3}{m'}\cos\theta-\frac{q_3}{m'}\sin\theta\right\}-(9{\cdot}14306)\left\{\frac{p_3}{m'}\sin\theta+\frac{q_3}{m'}\cos\theta\right\} \\
& -(8{\cdot}65341)\left\{\frac{p_3}{m'}\cos 2\theta-\frac{q_3}{m'}\sin 2\theta\right\}-(8{\cdot}87892)\left\{\frac{p_3}{m'}\sin 2\theta+\frac{q_3}{m'}\cos 2\theta\right\}
\end{aligned}$$

$$\begin{aligned}
0 = & \quad (1{\cdot}79213)\sin\theta-(2{\cdot}49403)\cos\theta-(2{\cdot}55700)\sin 2\theta-(2{\cdot}56972)\cos 2\theta \\
& -(2{\cdot}20337)\sin 3\theta-(2{\cdot}25714)\cos 3\theta+(9{\cdot}83632)\frac{p_3}{m'}+(0{\cdot}31156)\frac{q_3}{m'} \\
& -(9{\cdot}60395\left\{\frac{p_3}{m'}\cos\theta-\frac{q_3}{m'}\sin\theta\right\}+(9{\cdot}47665)\left\{\frac{p_3}{m'}\sin\theta+\frac{q_3}{m'}\cos\theta\right\} \\
& +(9{\cdot}23220)\left\{\frac{p_3}{m'}\cos 2\theta-\frac{q_3}{m'}\sin 2\theta\right\}+(9{\cdot}21679)\left\{\frac{p_3}{m'}\sin 2\theta+\frac{q_3}{m'}\cos 2\theta\right\}
\end{aligned}$$

where the numbers enclosed within parentheses denote the logarithms of the corresponding coefficients.

30. These equations may be rapidly solved by approximation. The coefficients of $\frac{p_3}{m'}$ and $\frac{q_3}{m'}$ in the first equation being small, we may find from it an approximate value of θ, the substitution of which in the second and third equations will give approximate values of $\frac{p_3}{m'}$ and $\frac{q_3}{m'}$. By means of these a more accurate value of θ may be found from the first equation, and the process being repeated, will enable us to satisfy all the equations as nearly as we please.

Thus we find $\theta = -51° \, 30'$, $\frac{p_3}{m'} = 271''{\cdot}57$, $\frac{q_3}{m'} = -207''{\cdot}24$.

Now ϵ is known and $= 217° \, 55'$ $\therefore$ $\epsilon' = 269° \, 25'$ the mean longitude of the disturbing planet at the epoch 1810·328. The sidereal motion in 36 synodic periods of *Uranus* $= 55° \, 12'$, precession $= 30'$; $\therefore$ mean longitude at the time 1846·762, or October 6, 1846, $= 325° \, 7'$.

Also, the analytical expressions for $\frac{p_3}{m'}$ and $\frac{q_3}{m'}$ are

$$\frac{p_3}{m'} = 48''{\cdot}55 \sin(3\theta - \beta) - 93''{\cdot}01 \, e' \sin(3\theta - \beta')$$

$$\frac{q_3}{m'} = 48{\cdot}55 \cos(3\theta - \beta) - 93{\cdot}01 \, e' \cos(3\theta - \beta')$$

where $\epsilon - \varpi' = \beta'$. Equating these to the values given above, we find

$$e' = 3{\cdot}2206, \quad \beta' = 262° \, 28', \text{ and } \therefore \varpi' = 315° \, 27'.$$

Hence longitude of perihelion in $1846 = 315° \, 57'$.

Lastly, substituting the values just obtained in equation (1), we find $m' = 0{\cdot}82816$.

31. Hence the values of the mass and elements of the orbit of the disturbing planet, resulting from the first hypothesis as to the mean distance, are the following:—

$$\frac{a}{a'} = 0{\cdot}5$$

Mean Long. of the Planet, October 6, 1846	325° 7′
Longitude of the Perihelion	315° 57′
Eccentricity of the Orbit	0·16103
Mass (that of the Sun being 1)	0·0001656

These are the results which I communicated to the Astronomer Royal in October, 1845.

32. I next entered upon a similar investigation, founded on the assumption that the mean distance was about $\frac{1}{30}$th part less than before, so that $\frac{a}{a'}$ or $\alpha = \sin 31^\circ = 0{\cdot}515$. The method employed was, in principle, exactly the same as that given before; but the numerical calculations were somewhat shortened by a few alterations in the process, which had been suggested by my previous solution.

33. Assuming then that $\alpha = \sin 31^\circ$, the values of the quantities b, $\alpha \frac{db}{d\alpha}$, $\alpha^2 \frac{d^2 b}{d\alpha^2}$, will be

$$\log b_0 = 0{\cdot}33385 \qquad \log \alpha \frac{db_0}{d\alpha} = 9{\cdot}57333 \qquad \log \alpha^2 \frac{d^2 b_0}{d\alpha^2} = 9{\cdot}82911$$

$$\log b_1 = 9{\cdot}76106 \qquad \log \alpha \frac{db_1}{d\alpha} = 9{\cdot}86149 \qquad \log \alpha^2 \frac{d^2 b_1}{d\alpha^2} = 9{\cdot}76573$$

$$\log b_2 = 9{\cdot}35361 \qquad \log \alpha \frac{db_2}{d\alpha} = 9{\cdot}71359 \qquad \log \alpha^2 \frac{d^2 b_2}{d\alpha^2} = 9{\cdot}92466$$

$$\log b_3 = 8{\cdot}98918 \qquad \log \alpha \frac{db_3}{d\alpha} = 9{\cdot}50854 \qquad \log \alpha^2 \frac{d^2 b_3}{d\alpha^2} = 9{\cdot}91563$$

Hence, by means of the formulæ given before, the principal inequalities of the mean longitude of *Uranus*, produced by the action of a planet whose mass is $\frac{m'}{5000}$, that of the sun being unity, and the eccentricity of whose orbit is $\frac{e'}{20}$, may be found to be the following:—

$$\begin{aligned}
&- 42''{\cdot}33\, m' \sin \{nt - n't + \epsilon - \epsilon'\}\\
&+ 76{\cdot}55\, m' \sin 2\{nt - n't + \epsilon - \epsilon'\}\\
&+ 7{\cdot}25\, m' \sin 3\{nt - n't + \epsilon - \epsilon'\}\\
&+ 2{\cdot}34\, m' \sin \{n't + \epsilon' - \varpi\}\\
&- 4{\cdot}74\, m'e' \sin \{n't + \epsilon' - \varpi'\}\\
&+ 41{\cdot}72\, m' \sin \{nt - 2n't + \epsilon - 2\epsilon' + \varpi\}\\
&- 16{\cdot}47\, m'e' \sin \{nt - 2n't + \epsilon - 2\epsilon' + \varpi'\}\\
&+ 33{\cdot}93\, m' \sin \{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi\}\\
&- 63{\cdot}41\, m'e' \sin \{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi'\}.
\end{aligned}$$

To these we may add the following, which are of two dimensions in terms of the eccentricities :—

$$
\begin{aligned}
&+0''\cdot40\, m' \ \ \sin 3\,\{nt-n't+\epsilon-\epsilon'\}\\
&-0\cdot74\, m'e' \sin \ \ \{3\,(nt-n't+\epsilon-\epsilon')-\varpi+\varpi'\}.
\end{aligned}
$$

34. Now, on our present assumption,

$n=13^\circ\,0'\cdot6,\ n'=4^\circ\,48'\cdot5,\ n-n'=8^\circ\,12'\cdot1,\ n-2n'=3^\circ\,23'\cdot6,\ 2n-3n'=11^\circ\,35'\cdot7.$

Hence the equations of condition given by the modern observations will be of the form

$$
\begin{aligned}
c''= \quad & \delta\epsilon+\delta x_1\cos\{13^\circ\ 0'\cdot5\}\,t+\delta x_2\cos\{26^\circ\ 1'\cdot0\}\,t\\
&+t\,\delta n+\delta y_1\sin\{13\ \ 0\cdot5\}\,t+\delta y_2\sin\{26\ \ 1\cdot0\}\,t\\
&\quad+\ h_1\cos\{\ 8^\circ\,12'\cdot1\}\,t+\ h_2\cos\{16^\circ\,24'\cdot2\}\,t+h_3\cos\{24^\circ\,36'\cdot3\}\,t\\
&\quad+\ k_1\sin\{\ 8\ 12\cdot1\}\,t+\ k_2\sin\{16\ 24\cdot2\}\,t+k_3\sin\{24\ 36\cdot3\}\,t\\
&\quad+\ p_1\cos\{\ 4\ 48\cdot5\}\,t+\ p_2\cos\{\ 3\ 23\cdot6\}\,t+p_3\cos\{11\ 35\cdot7\}\,t\\
&\quad+\ q_1\sin\{\ 4\ 48\cdot5\}\,t+\ q_2\sin\{\ 3\ 23\cdot6\}\,t+q_3\sin\{11\ 35\cdot7\}\,t.
\end{aligned}
$$

35. Treating these equations of condition in the same manner as before, the equations in the first group, derived from them, are found to be the following :—

$$
\begin{array}{llrlrlrl}
(\epsilon) & 151''\cdot48= & 21\cdot0000 & \delta\epsilon\ + & 6\cdot0670 & \delta x_1- & 4\cdot4358 & \delta x_2\\
 & & +13\cdot9515 & h_1+ & 0\cdot9471 & h_2- & 4\cdot5965 & h_3\\
 & & +18\cdot3916 & p_1+ & 19\cdot6752 & p_2+ & 8\cdot4184 & p_3\\
(x) & 246\cdot48= & 6\cdot0670 & \delta\epsilon\ + & 8\cdot2821 & \delta x_1+ & 4\cdot1762 & \delta x_2\\
 & & +\ 7\cdot3540 & h_1+ & 8\cdot3027 & h_2+ & 5\cdot0961 & h_3\\
 & & +\ 6\cdot5793 & p_1+ & 6\cdot3319 & p_2+ & 8\cdot0850 & p_3\\
(h_1) & 207\cdot58= & 13\cdot9515 & \delta\epsilon\ + & 7\cdot3540 & \delta x_1- & 0\cdot4177 & \delta x_2\\
 & & +10\cdot9735 & h_1+ & 4\cdot6775 & h_2- & 0\cdot0005 & h_3\\
 & & +12\cdot8697 & p_1+ & 13\cdot4050 & p_2+ & 8\cdot4781 & p_3\\
(h_2) & 245\cdot17= & 0\cdot9471 & \delta\epsilon\ + & 8\cdot3027 & \delta x_1+ & 7\cdot2362 & \delta x_2\\
 & & +\ 4\cdot6775 & h_1+ & 10\cdot0259 & h_2+ & 8\cdot3220 & h_3\\
 & & +\ 2\cdot3661 & p_1+ & 1\cdot6727 & p_2+ & 7\cdot3073 & p_3\\
(h_3) & 103\cdot48= & -\ 4\cdot5965 & \delta\epsilon\ + & 5\cdot0961 & \delta x_1+ & 10\cdot5558 & \delta x_2\\
 & & -\ 0\cdot0005 & h_1+ & 8\cdot3220 & h_2+ & 10\cdot9749 & h_3\\
 & & -\ 2\cdot8935 & p_1- & 3\cdot7316 & p_2+ & 3\cdot5852 & p_3.
\end{array}
$$

36. Similarly the equations in the second group are

$$
\begin{aligned}
(n)\quad -171''{\cdot}27 = {} & 77{\cdot}0000\,\delta n + 9{\cdot}3938\,\delta y_1 - 1{\cdot}2183\,\delta y_2 \\
& + 8{\cdot}7355\,k_1 + 7{\cdot}6213\,k_2 - 0{\cdot}0590\,k_3 \\
& + 5{\cdot}9764\,q_1 + 4{\cdot}3875\,q_2 + 9{\cdot}6152\,q_3
\end{aligned}
$$

$$
\begin{aligned}
(y)\quad -166{\cdot}33 = {} & 93{\cdot}9380\,\delta n + 12{\cdot}7179\,\delta y_1 + 1{\cdot}8907\,\delta y_2 \\
& + 11{\cdot}0393\,k_1 + 11{\cdot}3717\,k_2 + 3{\cdot}3196\,k_3 \\
& + 7{\cdot}3747\,q_1 + 5{\cdot}3825\,q_2 + 12{\cdot}6816\,q_3
\end{aligned}
$$

$$
\begin{aligned}
(k_1)\quad -181{\cdot}31 = {} & 87{\cdot}3550\,\delta n + 11{\cdot}0393\,\delta y_1 - 0{\cdot}3758\,\delta y_2 \\
& + 10{\cdot}0264\,k_1 + 9{\cdot}2740\,k_2 + 0{\cdot}9476\,k_3 \\
& + 6{\cdot}8054\,q_1 + 4{\cdot}9866\,q_2 + 11{\cdot}1971\,q_3
\end{aligned}
$$

$$
\begin{aligned}
(k_2)\quad -\ 99{\cdot}51 = {} & 76{\cdot}2130\,\delta n + 11{\cdot}3717\,\delta y_1 + 4{\cdot}4810\,\delta y_2 \\
& + 9{\cdot}2740\,k_1 + 10{\cdot}9740\,k_2 + 5{\cdot}6294\,k_3 \\
& + 6{\cdot}0523\,q_1 + 4{\cdot}3916\,q_2 + 11{\cdot}0843\,q_3
\end{aligned}
$$

$$
\begin{aligned}
(k_3)\quad 113{\cdot}14 = {} & -0{\cdot}5900\,\delta n + 3{\cdot}3196\,\delta y_1 + 10{\cdot}2112\,\delta y_2 \\
& + 0{\cdot}9476\,k_1 + 5{\cdot}6294\,k_2 + 10{\cdot}0251\,k_3 \\
& + 0{\cdot}1746\,q_1 + 0{\cdot}0454\,q_2 + 2{\cdot}4791\,q_3.
\end{aligned}
$$

37. The equations (p_2), (p_3) of the first group, and (q_2), (q_3) of the second, were not formed, as our previous solution shewed that when $\delta\epsilon$, δn, δx_1, and δy_1, were eliminated, the coefficients of the remaining unknown quantities in these equations would be extremely small. It will be preferable to combine the equations (h_1), (h_2), (h_3), and (k_1), (k_2), (k_3) before, instead of after, the elimination of $\delta\epsilon$, δn, δx_1, and δy_1, from them. If then we change the sign of the third equation in each group, and add it to the fourth and fifth, we obtain

$$
\begin{aligned}
141''{\cdot}07 = {} & -17{\cdot}6009\,\delta\epsilon + 6{\cdot}0448\,\delta x_1 + 18{\cdot}2097\,\delta x_2 \\
& - 6{\cdot}2965\,h_1 + 13{\cdot}6704\,h_2 + 19{\cdot}2974\,h_3 \\
& - 13{\cdot}3971\,p_1 - 15{\cdot}4639\,p_2 + 2{\cdot}4144\,p_3
\end{aligned}
$$

$$
\begin{aligned}
194{\cdot}94 = {} & -11{\cdot}7320\,\delta n + 3{\cdot}6520\,\delta y_1 + 15{\cdot}0680\,\delta y_2 \\
& + 0{\cdot}1951\,k_1 + 7{\cdot}3294\,k_2 + 14{\cdot}7069\,k_3 \\
& - 0{\cdot}5785\,q_1 - 0{\cdot}5496\,q_2 + 2{\cdot}3663\,q_3.
\end{aligned}
$$

38. By means of (ϵ) and (n) of Articles 35 and 36, eliminate $\delta\epsilon$ and δn from (x) and (y), and also from the equations just found, and we have

$$(x)\quad 202''.72 = 6.5294\,\delta x_1 + 5.4577\,\delta x_2 + 3.3234\,h_1 + 8.0291\,h_2 + 6.4240\,h_3 + 1.2659\,p_1 + 0.6477\,p_2 + 5.6529\,p_3$$

$$(y)\quad 42.61 = 1.2578\,\delta y_1 + 3.3771\,\delta y_2 + 0.3822\,k_1 + 2.0739\,k_2 + 3.3916\,k_3 + 0.0836\,q_1 + 0.0298\,q_2 + 0.9513\,q_3$$

$$268.02 = 11.1297\,\delta x_1 + 14.4919\,\delta x_2 + 5.3967\,h_1 + 14.4642\,h_2 + 15.4449\,h_3 + 2.0175\,p_1 + 1.0266\,p_2 + 9.4702\,p_3$$

$$168.85 = 5.0833\,\delta y_1 + 14.8824\,\delta y_2 + 1.5261\,k_1 + 8.4906\,k_2 + 14.6979\,k_3 + 0.3320\,q_1 + 0.1189\,q_2 + 3.8313\,q_3.$$

39. Substituting for δx_2, δy_2, their values in terms of δx_1, δy_1, we find

$$6.5294\,\delta x_1 + 5.4577\,\delta x_2 = 6.5700\,\delta x_1 + 0.0490\,\delta y_1$$
$$1.2578\,\delta y_1 + 3.3771\,\delta y_2 = -0.0303\,\delta x_1 + 1.2829\,\delta y_1$$
$$11.1297\,\delta x_1 + 14.4919\,\delta x_2 = 11.2378\,\delta x_1 + 0.1300\,\delta y_1$$
$$5.0833\,\delta y_1 + 14.8824\,\delta y_2 = -0.1335\,\delta x_1 + 5.1943\,\delta y_1.$$

Hence, if we add to the two latter equations

$$-1.7106\ (x) - 0.03607\ (y)$$

and $0.00165\,(x) - 4.0487\ (y)$ respectively,

δx_1 and δy_1 will be eliminated, and we shall obtain the following equations :—

$$(1)\quad 80''.28 = 0.2883\,h_1 - 0.7295\,h_2 - 4.4559\,h_3 + 0.0138\,k_1 + 0.0748\,k_2 + 0.1223\,k_3 + 0.1479\,p_1 + 0.0813\,p_2 + 0.1997\,p_3 + 0.0030\,q_1 + 0.0011\,q_2 + 0.0343\,q_3$$

$$(2)\quad 3.34 = -0.0055\,h_1 - 0.0132\,h_2 - 0.0106\,h_3 + 0.0212\,k_1 - 0.0939\,k_2 - 0.9662\,k_3 - 0.0021\,p_1 - 0.0011\,p_2 - 0.0093\,p_3 + 0.0066\,q_1 + 0.0017\,q_2 + 0.0203\,q_3.$$

40. Again, the equations of condition given by the ancient observations are

$$\begin{aligned}
62''{\cdot}6 = {} & \delta\epsilon - 0{\cdot}8776\,\delta x_1 + 0{\cdot}5402\,\delta x_2 + 0{\cdot}7923\,h_1 + 0{\cdot}2554\,h_2 \\
& - 39{\cdot}31\,\delta n - 0{\cdot}4795\,\delta y_1 + 0{\cdot}8415\,\delta y_2 + 0{\cdot}6101\,k_1 + 0{\cdot}9668\,k_2 \\
& - 0{\cdot}3875\,h_3 - 0{\cdot}9877\,p_1 - 0{\cdot}6870\,p_2 - 0{\cdot}1009\,p_3 \\
& + 0{\cdot}9219\,k_3 + 0{\cdot}1566\,q_1 - 0{\cdot}7267\,q_2 - 0{\cdot}9949\,q_3
\end{aligned}$$

$$\begin{aligned}
84{\cdot}5 = {} & \delta\epsilon + 0{\cdot}4975\,\delta x_1 - 0{\cdot}5050\,\delta x_2 - 0{\cdot}0887\,h_1 - 0{\cdot}9843\,h_2 \\
& - 32{\cdot}30\,\delta n - 0{\cdot}8675\,\delta y_1 - 0{\cdot}8631\,\delta y_2 + 0{\cdot}9961\,k_1 - 0{\cdot}1767\,k_2 \\
& + 0{\cdot}2634\,h_3 - 0{\cdot}9085\,p_1 - 0{\cdot}3355\,p_2 + 0{\cdot}9681\,p_3 \\
& - 0{\cdot}9647\,k_3 - 0{\cdot}4178\,q_1 - 0{\cdot}9420\,q_2 - 0{\cdot}2506\,q_3
\end{aligned}$$

$$\begin{aligned}
67{\cdot}2 = {} & \delta\epsilon + 0{\cdot}6732\,\delta x_1 - 0{\cdot}0935\,\delta x_2 - 0{\cdot}2243\,h_1 - 0{\cdot}8994\,h_2 \\
& - 31{\cdot}34\,\delta n - 0{\cdot}7394\,\delta y_1 - 0{\cdot}9956\,\delta y_2 + 0{\cdot}9745\,k_1 - 0{\cdot}4371\,k_2 \\
& + 0{\cdot}6277\,h_3 - 0{\cdot}8720\,p_1 - 0{\cdot}2815\,p_2 + 0{\cdot}9982\,p_3 \\
& - 0{\cdot}7785\,k_3 - 0{\cdot}4895\,q_1 - 0{\cdot}9596\,q_2 - 0{\cdot}0591\,q_3
\end{aligned}$$

$$\begin{aligned}
-51{\cdot}8 = {} & \delta\epsilon - 0{\cdot}2616\,\delta x_1 - 0{\cdot}8631\,\delta x_2 - 0{\cdot}9436\,h_1 + 0{\cdot}7809\,h_2 \\
& - 19{\cdot}59\,\delta n + 0{\cdot}9652\,\delta y_1 - 0{\cdot}5050\,\delta y_2 - 0{\cdot}3310\,k_1 + 0{\cdot}6247\,k_2 \\
& - 0{\cdot}5301\,h_3 - 0{\cdot}0731\,p_1 + 0{\cdot}3991\,p_2 - 0{\cdot}6801\,p_3 \\
& - 0{\cdot}8479\,k_3 - 0{\cdot}9973\,q_1 - 0{\cdot}9169\,q_2 + 0{\cdot}7331\,q_3
\end{aligned}$$

$$\begin{aligned}
-43{\cdot}2 = {} & \delta\epsilon - 0{\cdot}4741\,\delta x_1 - 0{\cdot}5505\,\delta x_2 - 0{\cdot}8861\,h_1 + 0{\cdot}5704\,h_2 \\
& - 18{\cdot}58\,\delta n + 0{\cdot}8805\,\delta y_1 - 0{\cdot}8348\,\delta y_2 - 0{\cdot}4634\,k_1 + 0{\cdot}8213\,k_2 \\
& - 0{\cdot}1248\,h_3 + 0{\cdot}0115\,p_1 + 0{\cdot}4532\,p_2 - 0{\cdot}8147\,p_3 \\
& - 0{\cdot}9922\,k_3 - 0{\cdot}9999\,q_1 - 0{\cdot}8914\,q_2 + 0{\cdot}5798\,q_3
\end{aligned}$$

$$\begin{aligned}
-50{\cdot}1 = {} & \delta\epsilon - 0{\cdot}6430\,\delta x_1 - 0{\cdot}1731\,\delta x_2 - 0{\cdot}8191\,h_1 + 0{\cdot}3420\,h_2 \\
& - 17{\cdot}68\,\delta n + 0{\cdot}7659\,\delta y_1 - 0{\cdot}9849\,\delta y_2 - 0{\cdot}5736\,k_1 + 0{\cdot}9397\,k_2 \\
& + 0{\cdot}2588\,h_3 + 0{\cdot}0871\,p_1 + 0{\cdot}5001\,p_2 - 0{\cdot}9063\,p_3 \\
& - 0{\cdot}9659\,k_3 - 0{\cdot}9962\,q_1 - 0{\cdot}8660\,q_2 + 0{\cdot}4225\,q_3
\end{aligned}$$

$$\begin{aligned}
-37{\cdot}8 = {} & \delta\epsilon - 0{\cdot}9492\,\delta x_1 + 0{\cdot}8021\,\delta x_2 - 0{\cdot}5743\,h_1 - 0{\cdot}3404\,h_2 \\
& - 15{\cdot}25\,\delta n + 0{\cdot}3145\,\delta y_1 - 0{\cdot}5972\,\delta y_2 - 0{\cdot}8186\,k_1 + 0{\cdot}9403\,k_2 \\
& + 0{\cdot}9652\,h_3 + 0{\cdot}2872\,p_1 + 0{\cdot}6192\,p_2 - 0{\cdot}9984\,p_3 \\
& - 0{\cdot}2613\,k_3 - 0{\cdot}9579\,q_1 - 0{\cdot}7852\,q_2 - 0{\cdot}0560\,q_3
\end{aligned}$$

$$\begin{aligned}
-20''\cdot 5 = \quad & \delta\epsilon - 0\cdot 9985\,\delta x_1 + 0\cdot 9942\,\delta x_2 - 0\cdot 3671\ h_1 - 0\cdot 7304\ h_2 \\
& - 13\cdot 60\,\delta n - 0\cdot 0538\,\delta y_1 + 0\cdot 1074\,\delta y_2 - 0\cdot 9302\ k_1 + 0\cdot 6830\ k_2 \\
& + 0\cdot 9035\ h_3 + 0\cdot 4164\ p_1 + 0\cdot 6928\ p_2 - 0\cdot 9251\ p_3 \\
& + 0\cdot 4286\ k_3 - 0\cdot 9092\ q_1 - 0\cdot 7212\ q_2 - 0\cdot 3796\ q_3
\end{aligned}$$

$$\begin{aligned}
-\ 2\cdot 4 = \quad & \delta\epsilon - 0\cdot 9633\,\delta x_1 + 0\cdot 8560\,\delta x_2 - 0\cdot 2363\ h_1 - 0\cdot 8883\ h_2 \\
& - 12\cdot 64\,\delta n - 0\cdot 2684\,\delta y_1 + 0\cdot 5170\,\delta y_2 - 0\cdot 9717\ k_1 + 0\cdot 4593\ k_2 \\
& + 0\cdot 6562\ h_3 + 0\cdot 4882\ p_1 + 0\cdot 7327\ p_2 - 0\cdot 8345\ p_3 \\
& + 0\cdot 7546\ k_3 - 0\cdot 8727\ q_1 - 0\cdot 6806\ q_2 - 0\cdot 5511\ q_3.
\end{aligned}$$

41. The equation for finding p_3 may be formed as before, by multiplying the above equations taken in order by

$$-0\cdot 8,\quad -0\cdot 6,\quad +1\cdot 0,\quad +1\cdot 0,\quad +0\cdot 9,\quad +0\cdot 6,\quad +0\cdot 4,\quad +0\cdot 3,$$

beginning with the second; and the equation for q_3 by multiplying the same equations by

$$1\cdot 0,\quad 1\cdot 0,\quad 0\cdot 5,\quad 0\cdot 4,\quad 0\cdot 3,\quad 0\cdot 2,\quad 0\cdot 1,\quad 0\cdot 1.$$

Thus we obtain

$$\begin{aligned}
-279''\cdot 64 = \quad & 2\cdot 80\,\delta\epsilon - 3\cdot 3742\,\delta x_1 + 0\cdot 0265\,\delta x_2 - 2\cdot 9237\ h_1 + 2\cdot 2232\ h_2 \\
& - 27\cdot 82\,\delta n + 3\cdot 7593\,\delta y_1 - 1\cdot 0986\,\delta y_2 - 3\cdot 8471\ k_1 + 3\cdot 6706\ k_2 \\
& + 0\cdot 1281\ h_3 + 1\cdot 7522\ p_1 + 2\cdot 6081\ p_2 - 4\cdot 9033\ p_3 \\
& - 1\cdot 2295\ k_3 - 3\cdot 4661\ q_1 - 2\cdot 2221\ q_2 + 1\cdot 5785\ q_3
\end{aligned}$$

$$\begin{aligned}
83\cdot 56 = \quad & 3\cdot 60\,\delta\epsilon + 0\cdot 2714\,\delta x_1 - 0\cdot 9567\,\delta x_2 - 1\cdot 5602\ h_1 - 1\cdot 3924\ h_2 \\
& - 91\cdot 84\,\delta n - 0\cdot 5116\,\delta y_1 - 2\cdot 7976\,\delta y_2 + 1\cdot 0937\ k_1 + 0\cdot 6112\ k_2 \\
& + 1\cdot 0027\ h_3 - 1\cdot 6385\ p_1 + 0\cdot 1802\ p_2 + 0\cdot 6529\ p_3 \\
& - 2\cdot 7879\ k_3 - 2\cdot 4746\ q_1 - 3\cdot 2736\ q_2 + 0\cdot 3113\ q_3.
\end{aligned}$$

42. Eliminate $\delta\epsilon$ and δn by means of (ϵ) and (n) of Articles 35 and 36, and these equations become

$$\begin{aligned}
-361''\cdot 72 = & - 4\cdot 1831\,\delta x_1 + 0\cdot 6179\,\delta x_2 - 4\cdot 7839\ h_1 + 2\cdot 0969\ h_2 \\
& + 7\cdot 1533\,\delta y_1 - 1\cdot 5388\,\delta y_2 - 0\cdot 6909\ k_1 + 6\cdot 4242\ k_2 \\
& + 0\cdot 7410\ h_3 - 0\cdot 7000\ p_1 - 0\cdot 0153\ p_2 - 6\cdot 0258\ p_3 \\
& - 1\cdot 2508\ k_3 - 1\cdot 3068\ q_1 - 0\cdot 6369\ q_2 + 5\cdot 0525\ q_3
\end{aligned}$$

$$\begin{aligned}-146''.69 = &- 0.7686\,\delta x_1 - 0.1963\,\delta x_2 - 3.9519\,h_1 - 1.5548\,h_2\\ &+10.6926\,\delta y_1 - 4.2508\,\delta y_2 + 11.5128\,k_1 + 9.7013\,k_2\\ &+1.7907\,h_3 - 4.7913\,p_1 - 3.1927\,p_2 - 0.7902\,p_3\\ &-2.8583\,k_3 + 4.6536\,q_1 + 1.9595\,q_2 + 11.7796\,q_3.\end{aligned}$$

43. Substituting for δx_2, δy_2, their values in terms of δx_1, δy_1, we find

$$-4.1831\,\delta x_1 + 7.1533\,\delta y_1 + 0.6179\,\delta x_2 - 1.5388\,\delta y_2 = -4.1647\,\delta x_1 + 7.1473\,\delta y_1$$

$$-0.7686\,\delta x_1 + 10.6926\,\delta y_1 - 0.1963\,\delta x_2 - 4.2508\,\delta y_2 = -0.7319\,\delta x_1 + 10.6591\,\delta y_1.$$

Hence, if to the equations just found we add

$$+0.60808\,(x) - 5.5942\,(y)$$

and

$$+0.07306\,(x) - 8.3110\,(y) \text{ respectively,}$$

δx_1 and δy_1 will be eliminated, and we shall obtain the following equations:—

$$\begin{aligned}(3)\quad -476''.84 = &-2.7630\,h_1 + 6.9793\,h_2 + 4.6473\,h_3\\ &-2.8290\,k_1 - 5.1777\,k_2 - 20.2242\,k_3\\ &+0.0698\,p_1 + 0.3785\,p_2 - 2.5884\,p_3\\ &-1.7748\,q_1 - 0.8036\,q_2 - 0.2693\,q_3\\ (4)\quad -486.03 = &-3.7091\,h_1 - 0.9682\,h_2 + 2.2600\,h_3\\ &+8.3364\,k_1 - 7.5348\,k_2 - 31.0457\,k_3\\ &-4.6988\,p_1 - 3.1454\,p_2 - 0.3772\,p_3\\ &+3.9584\,q_1 + 1.7118\,q_2 + 3.8734\,q_3.\end{aligned}$$

44. Eliminate the left-hand members from equations (2), (3), and (4), of Articles 39 and 43, by means of equation (1), and we have

$$\begin{aligned}0 = &\;0.4200\,h_1 - 0.4114\,h_2 - 4.2014\,h_3 + 0.1980\,p_1 + 0.1069\,p_2 + 0.4236\,p_3\\ &-0.4964\,k_1 + 2.3306\,k_2 + 23.3213\,k_3 - 0.1567\,q_1 - 0.0409\,q_2 - 0.4531\,q_3\end{aligned}$$

$$\begin{aligned}0 = &-1.0507\,h_1 + 2.6465\,h_2 - 21.8182\,h_3 + 0.9482\,p_1 + 0.8614\,p_2 - 1.4023\,p_3\\ &-2.7471\,k_1 - 4.7334\,k_2 - 19.4976\,k_3 - 1.7569\,q_1 - 0.7972\,q_2 - 0.0655\,q_3\end{aligned}$$

$$\begin{aligned}0 = &-1.9638\,h_1 - 5.3845\,h_2 - 24.7155\,h_3 - 3.8034\,p_1 - 2.6532\,p_2 + 0.8317\,p_3\\ &+8.4199\,k_1 - 7.0819\,k_2 - 30.3051\,k_3 + 3.9767\,q_1 + 1.7183\,q_2 + 4.0811\,q_3.\end{aligned}$$

45. If, as before, we put $\epsilon - \epsilon' = \theta$, and $\epsilon - \varpi = \beta$, it may be seen that

$$\frac{h_1}{m'} = -42''\!\cdot\!33 \sin\theta \qquad \frac{h_2}{m'} = 76''\!\cdot\!55 \sin 2\theta$$

$$\frac{k_1}{m'} = -42\cdot33 \cos\theta \qquad \frac{k_2}{m'} = 76\cdot55 \cos 2\theta$$

$$\frac{h_3}{m'} = 7\cdot25 \sin 3\theta + 0\cdot007460 \frac{p_3}{m'} + 0\cdot008974 \frac{q_3}{m'}$$

$$\frac{k_3}{m'} = 7\cdot25 \cos 3\theta - 0\cdot008974 \frac{p_3}{m'} + 0\cdot007460 \frac{q_3}{m'}$$

$$\frac{p_1}{m'} = 0\cdot20 \sin(\theta - \beta) - 0\cdot074738 \left\{\frac{p_3}{m'} \cos 2\theta - \frac{q_3}{m'} \sin 2\theta\right\}$$

$$\frac{q_1}{m'} = -\ 0\cdot20 \cos(\theta - \beta) + 0\cdot074738 \left\{\frac{p_3}{m'} \sin 2\theta + \frac{q_3}{m'} \cos 2\theta\right\}$$

$$\frac{p_2}{m'} = 32\cdot91 \sin(2\theta - \beta) + 0\cdot259765 \left\{\frac{p_3}{m'} \cos\theta - \frac{q_3}{m'} \sin\theta\right\}$$

$$\frac{q_2}{m'} = 32\cdot91 \cos(2\theta - \beta) + 0\cdot259765 \left\{\frac{p_3}{m'} \sin\theta + \frac{q_3}{m'} \cos\theta\right\}.$$

46. Substituting these expressions in the above equations, and putting for β its value, $50^\circ\, 15'\, 8$, we obtain

$$\begin{aligned}0 = &-(1\cdot24872)\sin\theta + (1\cdot32231)\cos\theta - (1\cdot48110)\sin 2\theta + (2\cdot24265)\cos 2\theta \\ &-(1\cdot48373)\sin 3\theta + (2\cdot22809)\cos 3\theta + (9\cdot26254)\frac{p_3}{m'} - (9\cdot50079)\frac{q_3}{m'} \\ &+(8\cdot44376)\left\{\frac{p_3}{m'}\cos\theta - \frac{q_3}{m'}\sin\theta\right\} - (8\cdot02630)\left\{\frac{p_3}{m'}\sin\theta + \frac{q_3}{m'}\cos\theta\right\} \\ &-(8\cdot17031)\left\{\frac{p_3}{m'}\cos 2\theta - \frac{q_3}{m'}\sin 2\theta\right\} - (8\cdot06861)\left\{\frac{p_3}{m'}\sin 2\theta + \frac{q_3}{m'}\cos 2\theta\right\}.\end{aligned}$$

$$\begin{aligned}0 = &\ (1\cdot65190)\sin\theta + (2\cdot06584)\cos\theta + (2\cdot30220)\sin 2\theta - (2\cdot60306)\cos 2\theta \\ &-(2\cdot19916)\sin 3\theta - (2\cdot15032)\cos 3\theta - (0\cdot14305)\frac{p_3}{m'} - (9\cdot60933)\frac{q_3}{m'} \\ &+(9\cdot34981)\left\{\frac{p_3}{m'}\cos\theta - \frac{q_3}{m'}\sin\theta\right\} - (9\cdot31615)\left\{\frac{p_3}{m'}\sin\theta + \frac{q_3}{m'}\cos\theta\right\} \\ &-(8\cdot85046)\left\{\frac{p_3}{m'}\cos 2\theta - \frac{q_3}{m'}\sin 2\theta\right\} - (9\cdot11828)\left\{\frac{p_3}{m'}\sin 2\theta + \frac{q_3}{m'}\cos 2\theta\right\}.\end{aligned}$$

$$0 = (1{\cdot}91407)\sin\theta - (2{\cdot}55189)\cos\theta - (2{\cdot}62790)\sin 2\theta - (2{\cdot}64230)\cos 2\theta$$

$$- (2{\cdot}25331)\sin 3\theta - (2{\cdot}34185)\cos 3\theta + (9{\cdot}96344)\frac{p_3}{m'} + (0{\cdot}56029)\frac{q_3}{m'}$$

$$- (9{\cdot}83835)\left\{\frac{p_3}{m'}\cos\theta - \frac{q_3}{m'}\sin\theta\right\} + (9{\cdot}64968)\left\{\frac{p_3}{m'}\sin\theta + \frac{q_3}{m'}\cos\theta\right\}$$

$$+ (9{\cdot}45371)\left\{\frac{p_3}{m'}\cos 2\theta - \frac{q_3}{m'}\sin 2\theta\right\} + (9{\cdot}47306)\left\{\frac{p_3}{m'}\sin 2\theta + \frac{q_3}{m'}\cos 2\theta\right\},$$

where the numbers enclosed within parentheses denote the logarithms of the corresponding coefficients, as before.

47. From these equations we find, by the same method as before,

$$\theta = -46^\circ\ 55' \qquad \frac{p_3}{m'} = 138''{\cdot}92 \qquad \frac{q_3}{m'} = -109''{\cdot}83$$

Hence, since $\epsilon = 217^\circ\ 55'$, $\epsilon' = 264^\circ\ 50'$, the mean longitude of the disturbing planet at the epoch 1810·328. The sidereal motion in 36 synodic periods of *Uranus* $= 57^\circ\ 42'$, Precession $= 30'$. $\therefore$ mean longitude at the time 1846·762, or October 6, 1846, $= 323^\circ\ 2'$.

Also, the expressions for $\frac{p_3}{m'}$ and $\frac{q_3}{m'}$ are

$$\frac{p_3}{m'} = 33''{\cdot}93\sin(3\theta - \beta) - 63''{\cdot}41\, e'\sin(3\theta - \beta')$$

$$\frac{q_3}{m'} = 33{\cdot}93\cos(3\theta - \beta) - 63{\cdot}41\, e'\cos(3\theta - \beta')\,;$$

where $\epsilon - \varpi' = \beta'$.

Equating these to the values given above, we find $e' = 2{\cdot}4123$, $\beta' = 279^\circ\ 14'$, and $\therefore\ \varpi' = 298^\circ\ 41'$. Hence longitude of the perihelion in 1846 $= 299^\circ\ 11'$.

Lastly, substituting the values just obtained in equation (1) of Article 39, we find $m' = 0{\cdot}75017$.

48. Hence the values of the mass and elements of the orbit of the disturbing planet, resulting from the second hypothesis as to the mean distance, are the following:—

$$\frac{a}{a'} = 0{\cdot}515$$

Mean longitude of the Planet, October 6, 1846...	323° 2′
Longitude of the Perihelion	299 11
Eccentricity of the Orbit................................	0·120615
Mass (that of the Sun being 1)	0·00015003.

49. From the values of m', θ, $\frac{p_3}{m'}$, and $\frac{q_3}{m'}$, found above, the values of the quantities h, k, p, and q, corresponding to each hypothesis, are immediately determined. Thus we find,

1st Hypothesis. $\frac{a}{a'} = 0·5$		2nd Hypothesis. $\frac{a}{a'} = 0·515$	
$h_1 = \quad 23''·98$	$k_1 = - \quad 19''·07$	$h_1 = \quad 23''·19$	$k_1 = -21''·69$
$h_2 = - \quad 47·58$	$k_2 = - \quad 11·00$	$h_2 = - \quad 57·30$	$k_2 = - \quad 3·83$
$h_3 = - \quad 1·93$	$k_3 = - \quad 7·64$	$h_3 = - \quad 3·40$	$k_3 = - \quad 5·76$
$p_1 = \quad 9·93$	$q_1 = - \quad 8·31$	$p_1 = \quad 6·52$	$q_1 = - \quad 7·34$
$p_2 = - \quad 8·54$	$q_2 = - \quad 55·36$	$p_2 = - \quad 11·62$	$q_2 = -54·39$
$p_3 = \quad 224·90$	$q_3 = -171·63$	$p_3 = \quad 104·21$	$q_3 = -82·39$

50. And by substituting these values in the equations (ϵ), (n), (x), and (y), we obtain

1st Hypothesis. $\frac{a}{a'} = 0·5$		2nd Hypothesis. $\frac{a}{a'} = 0·515$	
$\delta\epsilon = - \quad 49''·77$	$\delta n = - \quad 0''·702$	$\delta\epsilon = -43''·23$	$\delta n = - \quad 0''·5417$
$\delta x_1 = -130·69$	$\delta y_1 = \quad 222·38$	$\delta x_1 = \quad 1·77$	$\delta y_1 = \quad 123·98$
$\delta x_2 = \quad 1·02$	$\delta y_2 = \quad 2·83$	$\delta x_2 = \quad 1·13$	$\delta y_2 = \quad 0·91$

and the corresponding corrections of the elliptic elements will be

$\frac{\delta a}{a} = 0·00000999$	$\frac{\delta a}{a} = 0·00000771$
$\delta e = \quad 20''·83$	$\delta e = 40''·31$
$e\delta\varpi = 127·27$	$e\delta\varpi = 47·10$

It will be seen that the corrections of the eccentricity and longitude of perihelion vary very rapidly with a change in the assumed mean distance.

51. If these quantities be substituted in the expressions before given, we obtain the following theoretical corrections of the mean longitude, each of these corrections being divided into two parts, of which the first is due to the changes in the elements of the orbit of *Uranus*, and the second to the action of the disturbing planet.

Hypothesis I.

Ancient Observations.

Year.			
1712	−288″·0	+365″·8	= +77″·8
1715	−283·1	+357·1	= +74·0
1750	+210·5	−260·7	= −50·2
1753	+218·1	−267·0	= −48·9
1756	+214·0	−260·0	= −46·0
1764	+154·0	−186·7	= −32·7
1769	+ 79·6	−100·7	= −21·1
1771	+ 27·6	− 41·8	= −14·2

Modern Observations.

Year.			
1780	−126″·12	+129″·27	= + 3″·15
1783	−180·28	+188·70	= + 8·42
1786	−227·66	+240·36	= +12·70
1789	−265·70	+281·63	= +15·93
1792	−292·25	+310·38	= +18·13
1795	−305·84	+325·27	= +19·43
1798	−305·67	+325·72	= +20·05
1801	−291·77	+312·05	= +20·28
1804	−264·95	+285·38	= +20·43
1807	−226·78	+247·51	= +20·73
1810	−179·43	+200·76	= +21·33
1813	−125″·59	+147″·72	= +22″·13
1816	− 68·21	+ 91·02	= +22·81
1819	− 10·40	+ 33·18	= +22·78
1822	+ 44·84	− 23·64	= +21·20
1825	+ 94·69	− 77·64	= +17·05
1828	+136·73	−127·48	= + 9·25
1831	+168·94	−172·17	= − 3·23
1834	+189·85	−211·04	= −21·19
1837	+198·51	−243·59	= −45·08
1840	+194·54	−269·36	= −74·82

Hypothesis II.

Ancient Observations.

Year.	
1712	$-133''{\cdot}7+211''{\cdot}9=+78''{\cdot}2$
1715	$-117{\cdot}7+191{\cdot}5=+73{\cdot}8$
1750	$+\ 85{\cdot}2-134{\cdot}4=-49{\cdot}2$
1753	$+\ 73{\cdot}8-122{\cdot}2=-48{\cdot}4$
1756	$+\ 59{\cdot}1-105{\cdot}2=-46{\cdot}1$
1764	$+\ 2{\cdot}7-\ 36{\cdot}4=-33{\cdot}7$
1769	$-\ 43{\cdot}1+\ 20{\cdot}8=-22{\cdot}3$
1771	$-\ 69{\cdot}9+\ 54{\cdot}7=-15{\cdot}2$

Modern Observations.

Year.		Year.	
1780	$-133''{\cdot}10+135''{\cdot}98=+\ 2''{\cdot}88$	1813	$-12''{\cdot}72+\ 34''{\cdot}91=+22''{\cdot}19$
1783	$-149{\cdot}47+157{\cdot}87=+\ 8{\cdot}40$	1816	$+13{\cdot}08+\ 9{\cdot}88=+22{\cdot}96$
1786	$-160{\cdot}15+172{\cdot}99=+12{\cdot}84$	1819	$+35{\cdot}71-\ 12{\cdot}74=+22{\cdot}97$
1789	$-164{\cdot}52+180{\cdot}64=+16{\cdot}12$	1822	$+54{\cdot}04-\ 32{\cdot}68=+21{\cdot}36$
1792	$-162{\cdot}30+180{\cdot}58=+18{\cdot}28$	1825	$+67{\cdot}18-\ 50{\cdot}08=+17{\cdot}10$
1795	$-153{\cdot}59+173{\cdot}07=+19{\cdot}48$	1828	$+74{\cdot}52-\ 65{\cdot}37=+\ 9{\cdot}15$
1798	$-138{\cdot}87+158{\cdot}86=+19{\cdot}99$	1831	$+75{\cdot}74-\ 79{\cdot}21=-\ 3{\cdot}47$
1801	$-118{\cdot}95+139{\cdot}08=+20{\cdot}13$	1834	$+70{\cdot}85-\ 92{\cdot}31=-21{\cdot}46$
1804	$-\ 94{\cdot}96+115{\cdot}21=+20{\cdot}25$	1837	$+60{\cdot}08-105{\cdot}25=-45{\cdot}17$
1807	$-\ 68{\cdot}25+\ 88{\cdot}85=+20{\cdot}60$	1840	$+43{\cdot}98-118{\cdot}38=-74{\cdot}40$
1810	$-\ 40{\cdot}33+\ 61{\cdot}61=+21{\cdot}28$		

52. Comparing these with the corrections of mean longitude derived from observation, we find the remaining differences to be the following:—

ANCIENT OBSERVATIONS.

Year.	Observation – Theory. Hypoth. I.	Hypoth. II.
	"	"
1712	+ 6·7	+ 6·3
1715	− 6·8	− 6·6
1750	− 1·6	− 2·6
1753	+ 5·7	+ 5·2
1756	− 4·1	− 4·0
1764	− 5·1	− 4·1
1769	+ 0·6	+ 1·8
1771	+11·8	+12·8

MODERN OBSERVATIONS.

Year.	Observation – Theory. Hypoth. I.	Hypoth. II.	Year.	Observation – Theory. Hypoth. I.	Hypoth. II.
	"	"		"	"
1780	+0·27	+0·54	1813	−0·94	−1·00
1783	−0·23	−0·21	1816	−0·31	−0·46
1786	−0·96	−1·10	1819	−2·00	−2·19
1789	+1·82	+1·63	1822	+0·30	+0·14
1792	−0·91	−1·06	1825	+1·92	+1·87
1795	+0·09	+0·04	1828	+2·25	+2·35
1798	−0·99	−0·93	1831	−1·06	−0·82
1801	−0·04	+0·11	1834	−1·44	−1·17
1804	+1·76	+1·94	1837	−1·62	−1·53
1807	−0·21	−0·08	1840	+1·73	+1·31
1810	+0·56	+0·61			

The largest difference in the above table, viz. that for 1771, is deduced from a single observation; whereas the difference immediately preceding it, which is deduced from the mean of several, is very small.

53. The results of the two theories agree very closely with each other, and with observation, till we come to the later years of the series; and it is to be observed, that the difference between the theories becomes sensible at precisely the point where they both shew symptoms of diverging from the observations, the errors of the second hypothesis, however, being less than those of the other.

Recent observations shew that the errors of the theory soon become very sensible, though decidedly less for the second hypothesis than for the first. The following are the differences of mean longitude, as deduced from theory and observation, for the oppositions of 1843, 1844, and 1845:—

Year.	Observation – Theory. Hypoth. I.	Hypoth. II.
	"	"
1843	+ 7·11	+ 5·77
1844	+ 8·79	+ 7·05
1845	+12·40	+10·18

For the observations of the last two years, I am indebted to the kindness of the Astronomer Royal. The three years nearly agree in shewing that the errors of the first hypothesis are to those of the second in the ratio of 5 to 4, from which I inferred, in a letter to the Astronomer Royal, dated September 2, 1846, that the assumption of $\frac{a}{a'} = \sin 35^\circ = 0{\cdot}574$, would probably satisfy all the observations very nearly.

54. The results which I have deduced from Professor Challis's observations of the planet, strongly confirm the inference that the mean distance should be considerably diminished. It is of course impossible to determine precisely, without actual calculation, the alteration in longitude which would be produced by such a diminution in the distance. By comparing the values of θ given by the two hypotheses, it may be seen, however, that if we took successively smaller and smaller values for the mean distance, the values found for the mean longitude in 1810 would probably go on diminishing, while at the same time the mean motion from 1810 to 1846 would rapidly increase, so that the corresponding values of the mean longitude at the present time would probably soon arrive at a minimum, and afterwards begin again to increase. This I believe to be the reason why the longitude found on the supposition of too large a value for the mean distance agrees so nearly with observation. In consequence of not making sufficient allowance for the increase in the mean motion, I hastily inferred, in my letter to the Astronomer Royal mentioned above, that the effect of a diminution in the mean distance would be to diminish the mean longitude.

55. I have already mentioned, that I thought it unsafe to employ Flamsteed's observation of 1690 in forming the equations of condition, as the interval between it and all the others is so large. The difference between it and the theory appears to be very considerable, and greater for the second hypothesis than for the first, the errors being $+44''{\cdot}5$ and $+50''{\cdot}0$ respectively. These errors would probably be increased by diminishing the mean distance. It would be desirable that Flamsteed's manuscripts should be examined with reference to this point.

56. The corrections of the tabular radius vector of *Uranus* may be easily deduced from those of the mean longitude by means of the following formula:

$$\frac{\delta r}{r} = \frac{1}{r}\frac{dr}{d\epsilon}\delta\zeta - \frac{1}{2}\frac{d\,\delta\zeta}{n\,dt} + \frac{1}{4}\frac{\delta a}{a} - \frac{1}{2}\frac{e\delta e}{1-e^2} - \frac{1}{6}m'a^2\frac{dA_0}{da}$$

$$+\frac{m'}{2}\;\Sigma C_i \cos i\left\{nt - n't + \epsilon - \epsilon'\right\}$$

$$+ m'e\Sigma D_i \cos\;\left\{i\left(nt - n't + \epsilon - \epsilon'\right) - nt - \epsilon + \varpi\right\}$$

$$+ m'e'\Sigma E_i \cos\;\left\{i\left(nt - n't + \epsilon - \epsilon'\right) - nt - \epsilon + \varpi'\right\}$$

where $\delta\zeta$ denotes the whole correction of the mean longitude at the time t,

$$\frac{1}{r}\frac{dr}{d\epsilon} = e\sin\left\{nt + \epsilon - \varpi\right\} + \frac{3e^2}{2}\sin 2\left\{nt + \epsilon - \varpi\right\} \text{ nearly,}$$

$$C_i = \frac{1}{2}\frac{n}{n - {}_{,}u}\,aA_i$$

$$D_i = -\frac{1}{4}\frac{in}{i(n-n') - n}\left\{2iaA_i + a^2\frac{dA_i}{da}\right\}$$

$$E_i = \;\frac{1}{4}\frac{(i-1)\,n}{i(n-n') - n}\left\{(2i-1)\,aA_{i-1} + a^2\frac{dA_{i-1}}{da}\right\}$$

i assuming all integral values positive and negative not including zero.

57. By substituting in this formula the values of m', δa, δe, &c., already obtained, and putting $a = 19{\cdot}191$, we find the following results corresponding to the two assumed values of the mean distance.

Hypothesis I.

$$\frac{a}{r}\delta r = \frac{a}{r}\frac{dr}{d\epsilon}\delta\zeta - \frac{a}{2}\frac{d\delta\zeta}{ndt} - 0{\cdot}000089$$

$$\begin{aligned}
&+ 0{\cdot}000069 \cos\;\left\{nt - n't + \epsilon - \epsilon'\right\}\\
&+ 0{\cdot}000259 \cos 2\left\{nt - n't + \epsilon - \epsilon'\right\}\\
&+ 0{\cdot}000109 \cos 3\left\{nt - n't + \epsilon - \epsilon'\right\}\\
&+ 0{\cdot}000016 \cos\;\left\{n't + \epsilon' - \varpi\right\}\\
&- 0{\cdot}000168 \cos\;\left\{nt - 2n't + \epsilon - 2\epsilon' + \varpi\right\}\\
&+ 0{\cdot}000078 \cos\;\left\{nt - 2n't + \epsilon - 2\epsilon' + \varpi'\right\}\\
&- 0{\cdot}000049 \cos\;\left\{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi\right\}\\
&+ 0{\cdot}000209 \cos\;\left\{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi'\right\}.
\end{aligned}$$

HYPOTHESIS II.

$$\frac{a}{r}\delta r = \frac{a}{r}\frac{dr}{d\epsilon}\delta\zeta - \frac{a}{2}\frac{d\delta\zeta}{ndt} - 0{\cdot}000144$$
$$+ 0{\cdot}000073 \cos\ \{nt - n't + \epsilon - \epsilon'\}$$
$$+ 0{\cdot}000266 \cos 2\{nt - n't + \epsilon - \epsilon'\}$$
$$+ 0{\cdot}000115 \cos 3\{nt - n't + \epsilon - \epsilon'\}$$
$$+ 0{\cdot}000016 \cos\ \{n't + \epsilon' - \varpi\}$$
$$- 0{\cdot}000188 \cos\ \{nt - 2n't + \epsilon - 2\epsilon' + \varpi\}$$
$$+ 0{\cdot}000068 \cos\ \{nt - 2n't + \epsilon - 2\epsilon' + \varpi'\}$$
$$- 0{\cdot}000053 \cos\ \{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi\}$$
$$+ 0{\cdot}000165 \cos\ \{2nt - 3n't + 2\epsilon - 3\epsilon' + \varpi'\}.$$

58. The values of $\delta\zeta$ and $\frac{d\delta\zeta}{dt}$ for some recent years are the following:—

HYPOTHESIS I.

Year.	$\delta\zeta$	$\frac{d\delta\zeta}{dt}$
1834	$-\ 21''{\cdot}19$	$-20''{\cdot}93$
1840	$-\ 74{\cdot}82$	$-32{\cdot}34$
1846	$-148{\cdot}65$	$-39{\cdot}94$

HYPOTHESIS II.

1834	$-\ 21''{\cdot}46$	$-20''{\cdot}85$
1840	$-\ 74{\cdot}40$	$-31{\cdot}62$
1846	$-145{\cdot}91$	$-38{\cdot}30$

Hence, by means of the above formulæ, we find the corrections of the tabular radius vector to be

Year.	Hypothesis I.	Hypothesis II.
1834	$+0{\cdot}00505$	$+0{\cdot}00492$
1840	$+0{\cdot}00722$	$+0{\cdot}00696$
1846	$+0{\cdot}00868$	$+0{\cdot}00825$

59. By far the most important part of these corrections arises from the term $-\frac{1}{2}r\frac{d\delta\zeta}{ndt}$, and may therefore be immediately deduced from a comparison of the observed angular motion of *Uranus* with that given by the tables. In fact, the corrections given by this term alone for the epochs above mentioned are

Year.	Hypothesis I.	Hypothesis II.
1834	+0·00447	+0·00445
1840	+0·00694	+0·00678
1846	+0·00853	+0·00818

which, as we see, differ very little from the complete values just found. The correction for 1834 very nearly agrees with that which Mr Airy has deduced from observation in the *Astronomische Nachrichten* (No. 349). The corrections for subsequent years are rather larger than those given by the Greenwich Observations, the results of the second hypothesis, as in the case of the longitude, being nearer the truth than those of the first.

60. I made some attempts, by discussing the observations of latitude, to find approximate values of the longitude of the node and inclination of the orbit of the disturbing planet, but the results were not satisfactory. The perturbations of the latitude are, in fact, exceedingly small, and during the comparatively short period of three-fourths of a revolution are nearly confounded with the effects of a constant alteration in the inclination and the position of the node of *Uranus*, so that very small errors in the observations may entirely vitiate the result.

61. The perturbations of *Saturn* produced by the new planet, though small, will still be sensible, and it would be interesting to enquire whether, if they were taken into account, the values of the masses of *Jupiter* and *Uranus* found from their action on *Saturn* would be more consistent with those determined by other means than they appear to be at present. The reduction of the Greenwich planetary observations renders such an inquiry comparatively easy, and it is to be hoped that English astronomers will not be the last to avail themselves of the treasures of observation thus laid open to the world.

THE SEARCH FOR THE PLANET NEPTUNE BY PROFESSOR CHALLIS.

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CAMBRIDGE OBSERVATORY,
October 21, 1846.

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My more immediate purpose in writing to you at present, is to give some account of observations which I undertook this summer in search of the recently-discovered planet. Mr Adams, a young Cambridge mathematician, had for a long time turned his attention to the perturbations of *Uranus*, and in the autumn of last year communicated to me and to Mr Airy, the Astronomer Royal, values which he had obtained of the heliocentric longitude, mass, eccentricity of orbit, and longitude of perihelion of a supposed disturbing planet, revolving at a mean distance from the Sun about double that of *Uranus*. These results were deduced entirely from a consideration of perturbations of *Uranus* not otherwise accounted for. M. Le Verrier, by an investigation published in June last, obtained almost precisely the same heliocentric longitude which Mr Adams had arrived at. This coincidence from two independent sources very naturally inspired confidence in the theoretical deductions, and accordingly Mr Airy shortly after suggested to me the employing of the Northumberland telescope of this Observatory in a systematic search after the planet. I commenced observing July 29.

Unfortunately I was not then aware of the publication of hour XXI of the Berlin star-maps, and consequently had to proceed on the principle of comparison of observations made at intervals. On July 30 I recorded the approximate places of stars in a zone 9′ in breadth, in such a manner as to be sure that none brighter than the 11th magnitude escaped me, which a peculiar arrangement in the construction of the Northumberland Equatorial enabled me to do. On August 4 I took the places of the brighter stars in a zone 80′ broad, and among these recorded a place of the planet. My next observations were on August 12, on which day I met with a star of the 8th magnitude in the zone which I had taken on July 30, which did not then contain this star. This again was the planet. So exactly had theory indicated the proper place for making the search, that in four days only of observing I had recorded two positions of the planet. Also according to the principle of search I had adopted, the observations of two of those days (July 30 and August 12) were sufficient to discover it. My time, however, was so occupied with comet reductions, and so little expectation had I of discovering the planet by a brief search, that I was only just preparing to map the places of the stars to see what success I had had, when the announcement of the discovery reached me. My observations after August 12 were purposely made early in Right Ascension for the sake of being able to carry them on during a longer portion of the year. Accordingly I did not again meet with the planet till September 29, on which day I saw for the first time the results of M. Le Verrier's last investigations. By these I was induced to return again to the theoretical position of the planet, and to endeavour to detect it by the appearance of a disk. In fact on the night of September 29, out of a very large number of stars whose approximate places I recorded, I fixed upon one which appeared to me to have a disk, and which proved to be the planet. On October 1 I had intelligence of Dr Galle's discovery.

The foregoing account, while it shews that I cannot lay claim to any discovery, may perhaps be regarded with some degree of interest. In particular, the places which I have obtained for the planet on August 4 and August 12, though they cannot pretend to great accuracy, for the present possess a value which they will lose when accurate observations have been continued for a longer period. I have, therefore, thought it worth while to send them to you, and to describe in detail the manner in which they have been deduced, that an opportunity may be given of judging of the degree of confidence they deserve.

My observations were all made with the large Northumberland Refractor, and with a magnifying power of 170. On August 4, the Hour Circle being fixed, the telescope was moved in declination, and the transits were all taken at the same part of the field, at the toothed edge of the comb of a micrometer eye-piece. Differences of declination were measured by means of a graduated sector-arc, which was read off by a microscope-micrometer, one revolution of which is 10″. The stars were accurately bisected by a fixed wire equatorially adjusted, but to gain time the micrometer was read off to integral revolutions, and by estimation to a fourth part of a revolution. The error of reading off in this way could hardly be more than 3″, and the error of comparison with a single star might possibly amount to 6″. On August 12, the telescope was absolutely fixed, and the zone, which was 9′ in breadth, was limited by the field of view. The transits were taken at the toothed edge of the comb carefully adjusted, and the differences of declination were measured by revolutions of the eye-piece micrometer, read off in integral revolutions, and by estimation to a fourth part of a revolution, by means of the teeth of the comb. Occasionally, as it happened in the instance of the planet, the tenth part of a revolution was estimated. The value of one revolution of this micrometer is 17″, and I should therefore estimate the error of comparison with a single star, so far as it depended on error of reading off, to be at most 8″. I now give the places of the planet resulting from a comparison with every known star that was taken in the same series on each of the two days.

August 4

Star of Comparison, and authority for its place.			Right Asc. of Planet.	Decl. of Planet.
			h. m. s.	° ′ ″
50 Capricorni	British Association Catalogue		21 58 14·13	−12 57 18·4*
	Bessel Z. 127	$21^h\,37^m\,13^s$	15·21	21·7*
British Association Catalogue 7599			14·89	32·0
38 Aquarii B. A. C. 7722		h. m. s.	14·86	41·9*
Bessel Z. 127 and Z. 129		21 59 10	14·48	33·6
——	127 and Z. 129	22 5 6	14·80	34·9
——	127	21 45 34	14·69	28·4
——	127	21 34 54	14·18	35·1
——	127	21 32 30	14·94	30·2
——	127	21 48 50	14·77	33·6

* There can be little doubt that there is an error of 10″ in these from error in the number of micrometer revolutions.

August 12

Star of Comparison.			h. m. s.	R. A. of Planet. h. m. s.	Decl. of Planet. ° ′ ″
Bessel Z.	127		22 0 51......	21 57 26·14......	−13 1 55·2
——	127 and 129		22 5 6......	25·98......	64·0
——	127 and 129		22 8 15......	26·27......	59·3
——	127 and 129		22 10 53......	26·05......	61·5
——	127 and 129		22 11 18......	26·10......	61·9
——	127		22·18 20......	25·99......	62·7
——	127		22 19 26......	26·32......	60·9
——	127		22 24 45......	26·35......	54·4
——	127 and 129		22 32 7......	26·21......	57·8
——	129		22 27 31......	25·99......	65·2
——	127		22 36 51......	25·84......	62·7
75 Aquarii B. A. C. 7976			....	26·34......	57·1

Not knowing whether Bessel's place of 50 Capricorni or that of B. A. C. is preferable, I have adopted the mean of the two. The following are the places of the planet given by the means of the above determinations.

	Greenwich mean time. h. m. s.	R. A. h. m. s.	Decl. ° ′ ″
August 4	13 36 25	21 58 14·70	−12 57 32·2
12	13 3 26	21 57 26·13	−13 2 0·2

in which the errors of R. A. are probably not greater than those incident to results depending on single transits, and the errors of declination, according to the estimate already given, may amount to 3 or 4 seconds.

From these places, compared with recent observations extending to October 13, Mr Adams has obtained the following results:—

Distance of the planet from the Sun	30·05
Inclination of the orbit	1° 45′
Longitude of the descending node	309·43
Heliocentric Longitude, August 4	326·39

The present distance from the Sun is therefore about a tenth less than theory had predicted. Guided by these results I have been seeking for previous accidental observations of the planet, but without success. The position at the date of the *Histoire Céleste* is now too near the Sun.

DETERMINATION OF THE ORBIT OF THE PLANET NEPTUNE (PROFESSOR CHALLIS).

[From the *Astronomische Nachrichten.* No. 596 (1847). Pp. 309—314.]

In conformity with a wish expressed by the Vice-Chancellor and the Observatory Syndicate at their ordinary terminal meeting, held on March 15, I propose in this Report to carry on, for the information of members of the Senate, the account of proceedings in the Observatory relative to the new planet, a first Report of which was made on December 12 of last year. The theoretical grounds on which a search for the planet was instituted, the manner in which the search was conducted, and the degree of success that attended it, were stated in the former Report, which brought the history of proceedings down to the date at which the planet was discovered. I have now to give an account of the subsequent observations both of its position in the heavens, and of its physical appearance, and to state the results respecting the orbit which have been deduced from the observations by calculation.

A regular series of observations of the planet was commenced on October 3, 1846, and continued at all available opportunities, partly with the meridian instruments, and partly with the Northumberland Equatorial, to December 4, soon after which the planet became too faint to observe on the meridian on account of daylight. The observations were subsequently carried on with the Equatorial to January 15. The series was much interrupted by cloudy weather, particularly in the months of December and January. On the whole I have obtained 28 positions of the planet with the meridian instruments, and 25 positions with the Northumberland Equatorial by means of 92 differential observations of Right Ascension and as many of North Polar Distance. The Equatorial measures were all referred to the same star, No. 7648 of the British Association Catalogue, the exact place of which was determined by 16 observations with the Transit, and 8 observations with the Mural Circle. I have reason to think that the positions obtained with the equatorial are entitled to very nearly the same weight as those obtained on the meridian. All the above observations I have completely reduced, and have placed the results at the disposal of Mr Adams for deducing elements of the planet's orbit.

On January 12, I had for the first time a distinct impression that the planet was surrounded by a ring. The appearance noticed was such as would be presented by a ring like that of *Saturn*, situated with its plane very oblique to the direction of vision. I felt convinced that the observed elongation could not be attributed to atmospheric refraction, or to any irregular action on the pencils of light, because when the object was seen most steadily I distinctly perceived a symmetrical form. My assistant, Mr Morgan, being requested to pay particular attention to the appearance of the planet, gave the same direction of the axis of elongation as that in which it appeared to me. I saw the ring again on the evening of January 14. In my note-book I remark, "The ring is very apparent with a power of 215, in a field considerably illumined by lamp-light. Its brightness seems equal to that of the planet itself." On that evening, Mr Morgan, at my request, made a drawing of the form, which on comparison coincided very closely with a drawing made independently by myself. The ratio of the diameter of the ring to that of the planet, as measured from the drawings, is about that of 3 to 2. The angle made by the axis of the ring with a parallel of declination, in the south-preceding or north-following quarter, I estimated at 60°. By a measurement taken with the position-circle on January 15, under very unfavourable circumstances, this angle was found to be 65°. I am unable to account entirely for my not having noticed the ring at an earlier period of the observations. It may, however, be said that an appearance like this, which it is difficult to recognize except in a good state of the atmosphere, might for a long time escape detection, if not expressly and repeatedly looked for. To force itself on the attention, it would require to be seen under extremely favourable circumstances. Previous to the observations in January, the planet had been hid for more than three weeks by clouds. The evenings of January 12 and 14 were particularly good, and the planet was at first looked at in strong twilight. Under very similar circumstances I have twice seen with the Northumberland telescope the second division of Saturn's Ring.

I communicated to Mr Lassell of Liverpool, who was the first to suspect the existence of a Ring, my observations upon it, accompanied with a drawing; and I have received from him in return a drawing of the appearance presented in his twenty-feet reflector, closely resembling mine both as to the form and the position of the Ring. Mr Lassell writes, "I cannot refuse to consider that your observation puts beyond reasonable doubt the reality of mine." In this conclusion I concur, and accordingly in communications to the Royal Astronomical Society and to Schumacher's *Astronomische Nachrichten*, containing my reduced observations, I have ventured to express my conviction of the existence of a Ring.

By micrometer measures taken with the Northumberland telescope, I find the apparent diameter of the body of the planet to be very nearly 3″.

The above account includes all the observations on the planet I could obtain before its disappearance in the solar rays. By the kindness of Mr Adams I am able to add some particulars respecting its orbit, which he has derived by calculation from the reduced places with which I furnished him. As was stated in the former Report, Mr Adams calculated first approximations to the elements, by employing the places I obtained on August 4 and August 12 in the course of searching for the planet, with observations since the discovery extending to October 13. For the sake of comparison with the second approximations, I now give the first results.

Heliocentric Longitude	326° 39′	Aug. 4, 1846
Longitude of the Descending Node	309 43	
Inclination of the Orbit	1 45	
Distance of the Planet from the Sun	30·05.	

In calculating the following second approximations Mr Adams used the mean of the two places of August as a single place, and of the others he selected nine which seemed to be the best determined, and which were separated by convenient intervals. All the results are calculated for the epoch of 1846, August 8·0, mean time at Greenwich.

	°	′	″
Heliocentric Longitude of the Planet referred to the mean Equinox of 1847·0	326	41	12·3
Heliocentric motion in Longitude in 100 days......		36	5·52
Heliocentric Latitude South		30	34·4
Change of Heliocentric Latitude in 100 days		1	4·44
Longitude of the Descending Node	310	3	44·0
Inclination of the Orbit	1	46	49·1
Distance of the Planet from the Sun			30·008
Half the Latus Rectum of the Orbit			30·228.

The first position on which the above results depend, that of August 4, was obtained 16 days before the planet was in opposition, and the last position, that of January 15, 32 days before it was in conjunction. The great variation of the planet's elongation from the Sun in this interval is favourable to the correctness of the above determinations, which, although they cannot pretend to extreme accuracy on account of the short period over which the observations extend, are yet entitled to considerable weight. Mr Adams has in fact calculated the probable errors of the above results by supposing each observation of Right Ascension or of North Polar Distance to be liable to an error of 3″, and he finds that there is little probability of their receiving any great amount of correction by taking account of future observations. It may be remarked that the first and second approximations do not differ by any large quantities. Hence it may be inferred that the places of August are deserving of confidence, and that, on account of the extension given to the period of observation by including those places, this second approximation to the elements is more accurate than it would have been if it depended solely on observations made since the discovery of the planet.

The calculations give 59′ 8″ for the planet's heliocentric motion from August 4 to January 15. This is so small an arc that it is not possible to deduce with any degree of certainty those elements the determination of which depends on change of the heliocentric distance. Mr Adams has, however, discussed the observations with this object in view, and has obtained certain limiting results, which, as possessing considerable interest, I here subjoin.

The eccentricity of the orbit cannot exceed 0·18. The most probable value is 0·06, which differs but little from the eccentricities of the orbits of *Jupiter*, *Saturn*, and *Uranus*.

The most probable longitude of perihelion is 49° 58′, and the probable true anomaly 276° 43′, according to which the planet is near the extremity of the latus rectum and is descending towards perihelion. These results are extremely uncertain.

The mean distance is 30·35, with a probable error of 0·25; and the corresponding sidereal period is 167 years, with a probable error of about 2 years. It is remarkable that the periodic time is very nearly double that of *Uranus*; so that these two bodies will offer an instance of mutual perturbations of large amount, differing in character from those of the older planets, but analogous to the mutual perturbations of the first and second, and second and third satellites of *Jupiter*.

According to Bode's law of the planetary distances, the mean distance of the new planet would be nearly 38. The actual mean distance differs so much from this, that we are compelled to conclude that this singular law fails in this instance.

Since the apparent diameter of the new planet is to that of *Uranus* nearly in the ratio of 3 to 4, according to the foregoing determination of the distance its bulk is to that of *Uranus* in the ratio of 8 to 5.

The above is the sum of the results derivable from the first series of observations. For further and more exact information we must wait till the planet emerges from the solar rays. Before concluding this Report, I am desirous of saying a few words respecting the name of the planet. I recently had the satisfaction of receiving from M. Struve the copy of a communication read by him at the general annual meeting of the Imperial Academy of Sciences of St Petersburg on December 29, in which he states the reasons that have induced himself and the other Poulkova astronomers to adhere to the name of *Neptune*, which name was first proposed by the French Board of Longitude, shortly after the discovery of the planet. These reasons are thus briefly expressed in a note addressed to me personally: "The Poulkova astronomers have resolved to maintain the name of *Neptune*, in the opinion that the name of Leverrier would be against the accepted analogy, and against historical truth, as it cannot be denied that Mr Adams has been the first theoretical discoverer of that body, though not so happy as to effect a direct result of his indications." M. Struve's communication has been published in this country by the Astronomer Royal, who has expressed his assent to the reasons therein contained, and his determination to adopt the name of *Neptune*. Professor Gauss and Professor Encke have also, as I understand, adopted this name. I have only to add that it is my intention (and I am permitted to say, the intention of Mr Adams also) to follow the example set by these eminent astronomers.

OBSERVATIONS OF THE PLANET NEPTUNE, BY PROFESSOR CHALLIS.

[From the *Monthly Notices of the Royal Astronomical Society.* Vol. VII. (1847.)]

CAMBRIDGE.

In the Meridian.

			Greenwich M. T. h. m. s.	R. A. h. m. s.	N. P. D. ° ′ ″
1846	Oct.	8	8 43 27	21 52 13·29	103 29 43·4
		10	8 35 29	21 52 6·42	103 30 18·7
		13	8 23 21	21 51 56·90	103 31 8·7
		15	8 15 34	21 51 51·05	103 31 37·5
		16	8 11 35	21 51 48·43	103 31 53·9
		17	8 7 37	21 51 45·89	103 32 6·4
		19	7 59 40	21 51 40·98	103 32 31·2
		20	7 55 42	21 51 38·76	103 32 41·6
		23	7 43 48	21 51 32·60	103 33 7·1
		30	7 16 7	21 51 22·86	103 33 58·9
	Nov.	1	7 8 13	21 51 21·16	103 34 9·7
		4	6 56 25	21 51 19·91	103 34 14·3
		11	6 28 54	21 51 20·78	103 34 6·1
		16	6 9 19	21 51 25·63	103 33 38·7
		18	6 1 30	21 51 28·43	103 33 23·8
		19	5 57 36	21 51 30·42	103 33 13·4
		20	5 53 43	... ...	103 33 3·3
		21	5 49 48	21 51 33·71	103 32 52·8
		22	5 45 54	21 51 35·40	103 32 41·7
		24	5 38 6	21 51 40·03	103 32 17·1
		26	5 30 19	21 51 44·91	103 31 52·7
		28	5 22 33	21 51 50·25	103 31 22·6
		30	5 14 47	21 51 56·30	103 30 50·6
	Dec.	1	5 10 55	21 51 59·58	103 30 33·4
		3	5 3 9	21 52 6·11	103 29 56·6
		4	4 59 17	21 52 9·38	103 29 39·6

With the Northumberland Equatorial.

			Greenwich M. T. h. m. s.	R. A. h. m. s.	N. P. D. ° ′ ″	No. of Measures.
1846	Oct.	3	8 2 58	21 52 32·58	103 28 2·5	7
			10 22 45	21 52 32·22	103 28 4·2	6
		5	10 57 13	21 52 24·24	103 28 47·2	6
		8	10 49 27	21 52 13·14	103 29 44·5	6
		13	7 29 46	21 51 57·08	103 31 5·7	6

		Greenwich M. T.	R. A.	N. P. D.	No. of Measures.
		h. m. s.	h. m. s.	° ′ ″	
	Oct. 17	7 56 21	21 51 45·92	103 32 4·7	9
	30	6 30 17	21 51 22·84	103 34 1·5	8
	Nov. 2	8 42 56	21 51 20·97	103 34 13·1	6
	3	9 52 17	21 51 20·20	103 34 10·9	6
	16	7 4 42	21 51 25·77	103 33 35·0	5
	18	7 26 54	21 51 28·68	103 33 23·7	3
	19	6 50 47	21 51 30·21	103 33 14·5	6
	26	6 0 16	21 51 44·83	103 31 49·8	4
	Dec. 11	6 50 37	21 52 38·64	103 27 6·6	6
	12	7 12 35	21 52 43·13	103 26 40·2	6
	13	6 5 49	21 52 47·60	103 26 15·1	6
	14	7 42 37	21 52 52·59	103 25 50·4	4
	15	5 50 57	21 52 57·02	103 25 27·6	3
	18	4 52 3	21 53 12·19	103 24 5·3	4
1847	Jan. 11	5 35 20	21 55 46·97	103 10 27·4	6
	12	5 37 43	21 55 54·87	103 9 45·9	6
	14	5 45 48	21 56 9·82	103 8 27·3	6
	15	5 49 9	21 56 18·04	103 7 42·4	6

The star of reference throughout is No. 7648 of the British Association Catalogue, the assumed mean place of which, January 1, 1846, determined by 16 transit and 8 circle observations, is

$$\text{R. A.} = 21^{\text{h}}\ 50^{\text{m}}\ 5^{\text{s}}{\cdot}91, \qquad \text{N. P. D.} = 103^\circ\ 23'\ 55''{\cdot}56.$$

I found the apparent diameter of the planet by micrometer measures taken October 3 to be 3″·07. I have been able with the Northumberland telescope to verify Mr Lassell's suspicion of a ring. I first received the impression of a ring on January 12. Two independent drawings, made by myself and my assistant, Mr Morgan, gave the same representation of its appearance and position. The ring is very little open. Its diameter makes an angle in the south preceding quadrant of 66° with the parallel of declination, according to a measurement (not very satisfactorily taken) on January 15. The ratio of the diameter of the ring to that of the planet is by estimation that of 3 to 2. I am unable to account for my not having noticed the ring earlier.

3.

CORRECTED ELEMENTS OF NEPTUNE.

[From the *Monthly Notices of the Royal Astronomical Society* (1847). Vol. VII.]

THE following results respecting the orbit of the recently discovered planet *Neptune* may, perhaps, not be uninteresting to the Society. They are deduced from the early Cambridge observations of August 4 and August 12, combined with nine later ones made at the same observatory, those being generally selected where the planet was observed with the equatorial and meridian instruments on the same day. To each element found I have annexed the probable error to which it is subject, in order that it may be judged what reliance may be placed upon the value obtained. It will be seen that some tolerably definite information respecting the orbit is already afforded by the observations, though they are, of course, insufficient to determine, even roughly, all the elements.

Epoch 1846, Aug. 8·0, G. M. T.

True Long. of the Planet, M. Eq. 1847·0	$326^\circ\ 41'\ 12''\cdot3 \pm\ 2''\cdot55$
Motion in Longitude in 100 days	$36'\ 5''\cdot52 \pm\ 2''\cdot82$
Distance of Planet from the Sun	$30\cdot008 \pm 0\cdot0312$
Change of distance from the Sun in 100 days ...	$-0\cdot01947 \pm 0\cdot0365$
Heliocentric Latitude, South	$30'\ 34''\cdot35 \pm\ 2''\cdot24$
Increase of Heliocentric Latitude in 100 days......	$1'\ 4''\cdot44 \pm\ 2''\cdot05$

Hence we find,

Inclination of the Orbit	$1^\circ\ 46'\ 49''\cdot 1 \pm\ 3'\ 7''$
Longitude of Descending Node	$310^\circ\ \ 3'\ 44''\cdot 0 \pm 30'\ 37''$
Semi-latus Rectum	$30\cdot 228 \pm 0\cdot 0922$.

Also, if e be the eccentricity of the orbit and α the true anomaly,

$$e \cos \alpha = \quad 0\cdot 00733 \pm 0\cdot 00235$$

$$e \sin \alpha = -0\cdot 06223 \pm 0\cdot 1167.$$

Hence the most probable values of the eccentricity and longitude of the perihelion appear to be,

$$\text{Eccentricity} \ldots\ldots\ldots\ldots\ldots = 0\cdot 06266.$$

$$\text{Longitude of Perihelion} = \ 49^\circ\ 58'.$$

These latter are merely given as the results of the calculation, the magnitude of the probable error of $e \sin \alpha$ shewing that no weight is to be attached to them. It may be seen, however, that the eccentricity cannot be large.

The most probable value of the mean distance $= 30\cdot 35$, with a probable uncertainty of about $0\cdot 25$: the corresponding periodic time $= 167\cdot 2$ sidereal years, which is very nearly double that of *Uranus*. Hence the mutual disturbances of these two planets will present some remarkable peculiarities analogous to those of the first and second, and of the second and third satellites of *Jupiter*.

The probable errors given above have been found by considering the probable error of each observation to be $3''$, the mean of the observations on August 4 and August 12 (which, however, agree very well with each other) being taken as a single observation.

This estimation appears to be quite high enough, as the remaining differences between theory and observation only exceed that amount in two instances.

NOTE. *Extract of a Letter from Professor Schumacher.*

"I have received to-day a very interesting letter from M. Le Verrier. The star observed by Lalande on May 10, 1795, is undoubtedly the planet (*Neptune*). On consulting the original MSS. it appears that he observed the planet on May 10, *and also on May* 8; but in printing the *Histoire Céleste*, these two observations, supposed to be of the same fixed star, were found discordant. Hence the observation of May 8 *was not printed at all*, and to that of May 10 were affixed the two points, signifying doubt, *which are not* in the MSS. The MSS. observations stand thus:"*—

			Middle Wire.	Third Wire.	Zenith Distance.
			h. m. s.	h. m. s.	° ′ ″
May	8	7·8	14 11 24	....	59 54 40
		Planet	11 36·5	....	60 8 17
	10	Planet	11 23·5	....	60 7 19
		7·8	....	14 11 50·5	59 54 40.

Observations of Neptune since its Reappearance.

CAMBRIDGE. Northumberland Equatorial. (Prof. Challis.)

			Greenwich M. T.	R. A.	N. P. D.
			h. m. s.	h. m. s.	° ′ ″
1847	May	6	15 13 49	22 9 53·40	101 55 53·1
		11	15 24 11	21 10 9·03	101 54 38·0

"*Neptune* was compared with a star in Bessel's Zones 127, 129, R. A. = $22^h\,15^m\,11^s$, and Bessel's place was employed. On May 6th, the observation was difficult from twilight and unfavourable atmosphere."

			Greenwich M. T.	R. A.	N. P. D.
			h. m. s.	h. m. s.	° ′ ″
1847	May	26	14 58 36	22 10 36·72	101 52 33·5
	June	1	14 5 4	22 10 39·86	101 52 27·0

"The planet was compared on May 26 three times with B. A. C. 7740 and twice with a star in Bessel's zones 127 and 129, R. A. = $22^h\,15^m\,11^s$. On June 1 it was compared five times with the former star and four times with the latter. The places of the stars are taken from the British Association Catalogue and from Bessel."

* The mean places of the star for 1800, by Schumacher's Tables, are

R. A.	h. m. s. 14 12 0·83	N. P. D.	101° 8′ 19″·4
	11 59·81		8 17·8

There is probably an error of 1^s in one of the observed R. Ascensions.

4.

NEW ELEMENTS OF NEPTUNE.

[From the *Monthly Notices of the Royal Astronomical Society* (May, 1847), Vol. VII.]

THE following elements of *Neptune* have been obtained by taking into account Professor Challis's Observations made since the reappearance. * * * The elements are now sufficiently correct to enable me to approximate to the perturbations of *Neptune* by the action of *Uranus*, in order to compare more accurately the ancient observations of 1795 with those . . . . made recently. I have used the old observations, supposing the elements not to have changed. I hope immediately to set about a new solution of the perturbations of *Uranus*, starting with a very approximate value of the mean distance. * * * I do not think with Professor Pierce, that the near commensurability of the mean motions will interfere seriously with the results obtained by the treatment of perturbations; but it will be interesting to see how nearly the real elements can be obtained by means of the perturbations.

Elements of the Orbit of Neptune.

Mean Longitude, Jan. 1, 1847, G. M. T.	328° 13′ 54″·5	
Longitude of Perihelion (on the Orbit)	11 13 41·5	M. Eq. 1847·0
—— Ascending Node	130 5 39·0	
Inclination to Ecliptic	1 47 1·5	
Mean Daily Motion	21·3774	
Semi-axis Major	30·2026	
Eccentricity of Orbit	0·0083835	

5.

EPHEMERIS OF NEPTUNE AND MERIDIAN OBSERVATIONS.

[From the *Astronomische Nachrichten.* XXVI. (1847). No. 604, pp. 51, 52.]

Communicated by Rev. R. Sheepshanks.

Ephemeris of Neptune for Mean Midnight Greenwich.

			R. A.	N. P. D.
			h. m. s.	° ′ ″
1847	April	30	22 9 30·79	101 57 48·1
	May	10	22 10 5·75	101 54 52·5
		20	22 10 28·63	101 53 4·4
		30	22 10 39·07	101 52 25·4
	June	9	22 10 37·10	101 52 55·3
		19	22 10 22·91	101 54 32·4
		29	22 9 57·22	101 57 12·3
	July	9	22 9 21·08	102 0 48·9
		19	22 8 35·72	102 5 14·4
		29	22 7 42·91	102 10 18·6
	Aug.	8	22 6 44·57	102 15 50·3

This ephemeris is deduced from the Elements of Neptune last communicated to the Royal Astronomical Society.

Professor Challis' observations give the following equations for the difference between Observation and Ephemeris.

Observation – Ephemeris.

	R. A.	N. P. D.
	s.	″
May 26	+0·18	+1·5
June 1	+0·21	+1·0

I am hard at work on the perturbations of *Uranus*, in order to obtain a new theoretical determination of the place.... The general values of the perturbations are enormous, far exceeding anything else of the same kind in the system of the primary planets. A comparison of the numerical expressions for the perturbations which I have now obtained with those, which I used before, would justify some scepticism as to former conclusions. But we shall soon see how this great apparent difference affects the result.

From the *Astronomische Nachrichten*, No. 616, pp. 241—244.

Ephemeris of Neptune for Greenwich Mean Midnight.

1847	R. A. h. m. s.	N. P. D. ° ′ ″	1847	R. A. h. m. s.	N. P. D. ° ′ ″
Sept. 12	22 3 9·27	102 35 45·8	Sept. 28	22 1 44·76	102 43 23·7
13	3 3·53	36 17·2	29	1 40·50	43 48·7
14	2 57·83	36 48·2	30	1 35·52	44 13·3
15	2 52·19	37 19·0	Oct. 1	1 31·03	44 37·4
16	2 46·61	37 49·4	2	1 26·62	45 1·0
*17	2 41·08	38 19·4	3	1 22·30	45 24·1
18	2 35·62	38 49·1	4	1 18·08	45 46·6
19	2 30·21	39 18·4	5	1 13·95	46 8·6
20	2 24·87	39 47·3	6	1 9·92	46 30·1
21	2 19·61	40 15·8	*7	1 5·99	46 51·0
22	2 14·41	40 44·0	8	1 2·16	47 11·4
23	2 9·27	41 11·7	9	0 58·42	47 31·1
24	2 4·22	41 38·9	10	0 54·79	47 50·4
25	1 59·24	42 5·8	11	0 51·27	48 9·0
26	1 54·34	42 32·2	12	0 47·85	48 27·1
*27	1 49·51	42 58·2	13	0 44·54	48 44·5

1847	R. A. h. m. s.	N. P. D. ° ′ ″	1847	R. A. h. m. s.	N. P. D. ° ′ ″
Oct. 14	22 0 41·33	102 49 1·3	13	22 0 1·09	102 52 21·1
15	0 38·24	49 17·6	14	0 1·74	52 17·0
16	0 35·26	49 33·2	15	0 2·52	52 12·2
*17	0 32·39	49 48·2	*16	0 3·43	52 6·6
18	0 29·63	50 2·6	17	0 4·47	52 0·3
19	0 27·00	50 16·3	18	0 5·65	51 53·3
20	0 24·47	50 29·5	19	0 6·96	51 45·7
21	0 22·07	50 41·9	20	0 8·41	51 37·2
22	0 19·78	50 53·7	21	0 9·99	51 28·1
23	0 17·61	51 4·8	22	0 11·69	51 18·3
24	0 15·56	51 15·3	23	0 13·53	51 7·8
25	0 13·64	51 25·1	24	0 15·49	50 56·5
26	0 11·83	51 34·3	25	0 17·60	50 44·6
*27	0 10·15	51 42·8	*26	0 19·83	50 32·0
28	0 8·59	51 50·6	27	0 22·19	50 18·7
29	0 7·15	51 57·8	28	0 24·67	50 4·7
30	0 5·85	52 4·2	29	0 27·30	49 50·0
31	0 4·66	52 10·0	30	0 30·05	49 34·6
Nov. 1	0 3·61	52 15·1	Dec. 1	0 32·92	49 18·5
2	0 2·68	52 19·5	2	0 35·92	49 1·7
3	0 1·88	52 23·2	3	0 39·05	48 44·3
4	0 1·21	52 26·2	4	0 42·31	48 26·2
5	0 0·67	52 28·4	5	0 45·69	48 7·4
*6	0 0·26	52 30·0	*6	0 49·20	47 47·9
7	21 59 59·98	52 30·9	7	0 52·83	47 27·8
8	59 59·84	52 31·0	8	0 56·59	47 7·0
9	59 59·82	52 30·4	9	1 0·47	46 45·5
10	59 59·94	52 29·2	10	1 4·47	46 23·4
11	22 0 0·19	52 27·2	11	1 8·59	46 0·6
12	0 0·57	52 24·5			

Meridian Observations of Neptune made at the Cambridge Observatory by Professor Challis, and compared with the Ephemeris.

	Greenwich M. T.	Observed R. A.	Obs. R. A. – Cal. R. A.	Observed N. P. D.	Obs. N. P. D. – Cal. N. P. D.
1847	h. m. s.	h. m. s.		° ′ ″	″
July 22	14 7 9·4	22 8 19·90	–0·23	102 6 43·3	–1·4
26	13 51 4·7	7 58·81	0·20	8 46·5	+0·2
27	13 47 3·3	7 53·27	0·30		
29	13 39 0·5	7 42·28	0·25	10 22·6	+1·8
30	13 34 59·1	7 36·72	0·20	10 50·6	–2·2
Aug. 3	13 18 52·4	7 13·63	0·37	13 2·1	–1·5
7	13 2 45·3	6 50·11	0·22	15 16·6	–1·1
9	12 54 41·1	6 37·71	0·57	16 27·0	+1·2
10	12 50 39·3	6 31·80	0·41	16 58·0	–2·0
11	12 46 37·5	6 25·88	0·22	17 34·7	+0·2
13	12 38 33·4	6 13·51	0·29	18 39·6	–4·0
14	12 34 31·2	6 7·28	0·35		
20	12 10 18·5	5 29·87	0·38	22 48·0	+0·4
21	12 6 16·2	5 23·50	0·48	23 23·8	+0·3
23	11 58 11·8	5 10·84	0·62	24 33·9	+1·6
24	11 54 9·8	5 4·70	0·49	25 9·2	+2·0
27	11 42 3·4	4 45·97	0·46	26 51·6	+0·2
31	11 25 54·9	4 21·07	0·52	29 7·2	–1·6
Sept. 1	11 21 52·9	4 14·97	0·45	29 42·9	+0·1
2	11 17 50·8	4 8·77	0·50	30 20·3	+3·6
4	11 9 46·9	3 56·59	0·46	31 26·6	+2·7
8	10 53 39·3	3 32·58	0·44	33 36·3	+0·4
9	10 49 37·4	3 26·57	0·54	34 9·4	+1·1
16	10 21 25·9	2 46·29	0·70		
17	10 17 24·7	2 41·04	0·43	38 18·4	+1·1

The observations of Aug. 9, 13 and Sept. 6 were somewhat uncertain on account of clouds. The N. P. D. has been corrected for parallax.

6.

THE MASS OF URANUS.

[From the *Monthly Notices of the Royal Astronomical Society.* Vol. IX. (1849.)]

THE mass of *Uranus* is a very important element in the determination of the orbit of *Neptune.* Two values of this mass have been given, differing widely from each other. Bouvard, from the action of *Uranus* on *Saturn,* found the mass to be $\frac{1}{17918}$, that of the sun being $=1$; while more recently, from observations of the satellites, Lamont has obtained the value $\frac{1}{24605}$. In order to throw light on this subject, Mr Lassell was kind enough to make for me the observations of the satellites of *Uranus,* which are given in the *Monthly Notice* for March last.

These I have carefully reduced, and the value of the mass which I have found from the observations of the fourth satellite (which are more to be depended on for this purpose than those of the second) is $\frac{1}{20897}$, which is almost exactly a mean between the results of Bouvard and Lamont. In obtaining this result, I have rejected the first day's observations, which are discordant both for the second and fourth satellites.

I have also reduced all Sir Wm. Herschel's measures of distance of the satellites given in his paper in the *Phil. Trans.,* 1815, and the value of the mass obtained from the observations of the fourth satellite is $\frac{1}{21165}$, which agrees very closely with that found from Mr Lassell's observations. Although, therefore, more numerous observations will be requisite in order to obtain a mass which may be used with confidence in the theory of *Neptune,* I have no doubt that the value $\frac{1}{21000}$ is much nearer the truth than either of those which have been previously given, and I shall accordingly employ it in my subsequent calculations respecting the orbit of *Neptune.*

The most probable values of the periods of the second and fourth satellites, given by the combination of the observations of Sir Wm. Herschel, Sir J. Herschel, Lamont, and Mr Lassell, are $8^{d}\cdot7058435$ and $13^{d}\cdot463139$ respectively; but the remaining errors of the epochs are greater than can with probability be ascribed to mere errors of observation, and seem to indicate the existence of considerable perturbations.

7.

APPENDIX ON THE DISCOVERY OF NEPTUNE.

[From *Liouville's Journal de Mathématiques*, New Series, Tome II. (1876).]

BESSEL a inséré au no. 48 des *Astronomische Nachrichten*, t. II., p. 441, une Lettre qui est accompagnée d'une note explicative se rapportant à ses Tables d'Uranus et émanant de Bouvard lui-même.

Il résulte évidemment des remarques I, II, III de M. Le Verrier, aux pages 92—94 de son Mémoire sur les perturbations d'Uranus, qu'il n'avait pas connaissance de ces Lettres de Bessel et Bouvard; car elles auraient fait disparaître la plupart des doutes qu'il y exprime relativement aux Tables de ce dernier. Il aurait vu, par exemple, que la correction $2\delta e$, qu'il suppose pouvoir s'élever à 100 secondes sexagésimales, n'était réellement que d'environ 10 secondes centésimales. Au haut de la page 90 de son Mémoire, M. Le Verrier remarque, avec beaucoup de justesse, qu'une erreur dans l'inégalité d'une longue période n'a pas d'importance pour l'objet en vue; mais il aurait dû aussi remarquer qu'une erreur dans une inégalité, dont la période était presque égale à celle d'Uranus, serait pareillement presque insignifiante, puisque l'effet de cette erreur, durant le temps pendant lequel Uranus a été observé, serait, à peu de chose près, représenté par une correction constante appliquée à l'excentricité et à la longitude du périhélie, comme je l'ai dit à la fin du no. 7 de mon Mémoire.

J'attache une très-grande importance à la remarque faite au no. 9, relativement à l'avantage d'employer la correction de la longitude moyenne au lieu de celle de la longitude vraie. M. Hansen a fortement insisté sur ce point dans sa *Théorie de la Lune* et dans ses autres ouvrages.

Par suite de cela, les termes qui sont nécessairement omis dans une première approximation sont plus faibles que si l'on avait employé les perturbations de la longitude vraie.

Je vais maintenant faire un petit nombre de remarques, en réponse aux objections de M. le professeur Pierce, contre la légitimité du procédé suivi, tant par M. Le Verrier que par moi-même, pour la solution de notre problème. Le professeur Pierce prétend que la période de notre planète hypothétique diffère si considérablement de celle de Neptune, que l'on pourrait indiquer quelques périodes intermédiaires, lesquelles seraient exactement commensurables avec la période d'Uranus, et qu'il y aurait une solution de continuité dans les perturbations d'Uranus, causée par deux planètes hypothétiques, dont l'une aurait une plus grande période et l'autre une période plus petite que la période commensurable dont il vient d'être question. De plus, la période de Neptune lui-même est, à très-peu de chose près, double de celle d'Uranus, et cette circonstance donne naissance à des perturbations réciproques très-considérables, d'un caractère tout à fait différent de celles qui seraient causées par nos planètes hypothétiques.

Peu de mots, à mon avis, suffiront pour aplanir cette difficulté. Il est vrai que, si nous voulions représenter les perturbations d'Uranus causées par une planète supérieure, pendant deux ou plusieurs périodes synodiques, cela ne pourrait se faire qu'en adoptant une période approximativement vraie pour la planète perturbatrice; mais le cas est différent lorsque, comme ici, nous n'avons à représenter que les perturbations produites durant une fraction d'une période synodique.

Dans ce cas, si nous prenions pour quantités inconnues, non les corrections applicables aux éléments moyens de l'orbite d'Uranus, mais celles qui seraient applicables aux éléments adoptés pour l'époque de 1810, par exemple, alors toutes les considérations relatives à une commensurabilité approximative dans les deux périodes, deviendraient étrangères à la question, et les perturbations pour l'intervalle limité requis pourraient être représentées approximativement, pourvu que les forces perturbatrices de la planète réelle et de la planète présumée fussent approximativement les mêmes en grandeur et en direction, durant le temps où ces forces perturbatrices agiraient avec la plus grande intensité, c'est-à-dire lorsque les planètes ne seraient pas fort éloignées de leur conjonction. Sir John Herschel a montré dans ses *Outlines of Astronomy* que ces conditions sont remplies d'une manière satisfaisante par les planètes hypothétiques de M. Le Verrier et de moi-même, quand leur action est comparée à celle de Neptune.

On ne devait attacher aucune valeur à la forte excentricité ni à la longitude de l'apside de l'orbite de la planète présumée, si ce n'est en tant qu'elles fournissaient les moyens d'approcher de plus près de la distance actuelle et du mouvement angulaire du corps perturbateur, dans l'intervalle où l'action perturbatrice se faisait le plus sentir.

Ainsi donc, de la circonstance que le périhélie de la planète présumée sortit du premier calcul, non loin de la ligne de conjonction, on aurait pu raisonnablement conclure, ce qu'a donné en effet le second calcul, que l'hypothèse d'une plus faible valeur de la distance moyenne conduirait à une valeur plus faible de l'excentricité.

On fera bien aussi de remarquer que les grands changements dans les valeurs de δe et $e\delta\varpi$, qui se trouvent dans le no. 50, résultant de la transition de ma première à ma seconde hypothèse, sont des changements dans les valeurs des éléments moyens de l'orbite d'Uranus, lesquels sont grandement affectés par l'inégalité de la longitude moyenne avec les coefficients p_3 et q_3, dont la période ne diffère pas beaucoup de celle d'Uranus, particulièrement pour le cas de la première hypothèse. On verra que δx_1+p_3 et δy_1+q_3 varient bien moins en passant d'une hypothèse à l'autre que δx_1 et δy_1. Nous avons donc:

Première hypothèse.	Seconde hypothèse.
$\delta x_1+p_3=94{,}''21$	$\delta x_1+p_3=105{,}''98$
$\delta y_1+q_3=50{,}75$	$\delta y_1+q_3=\ \ 41{,}59$

Et les corrections des éléments adoptés, à l'époque de 1810, seront approximativement déduites de ces quantités, absolument comme δe et $e\delta\varpi$ ont été formés de δx_1 et δy_1.

L'observation de Flamsteed, en 1690, remonte à une époque trop éloignée pour qu'elle puisse être bien représentée par les formules dont les résultats s'accordent assez bien avec ceux des observations plus récentes.

Ma seconde hypothèse a donné une erreur plus forte que la première. C'est donc probablement pour avoir eu trop de confiance dans la possibilité d'appliquer ses formules à cette observation ancienne, que M. Le Verrier s'est trouvé amené à fixer une limite inférieure à la distance moyenne de sa planète perturbatrice, laquelle ne concorde pas avec la distance moyenne de Neptune, telle qu'elle a été observée.

8.

ELEMENTS OF THE COMET OF FAYE.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. VI. (1844).]

THE observations used were made with the Northumberland telescope of the Cambridge Observatory; and the deduced places are as follows:

		Greenwich Mean Solar Time.	Apparent R. A. of Comet.	Apparent N. P. D. of Comet.
		h. m. s.	h. m. s.	° ′ ″
1843	Nov. 29	11 12 23	5 21 37·5	84 24 55
	Dec. 8	9 59 18	5 17 28·7	85 47 53
	16	11 55 45	5 13 33·0	86 35 55

At first I computed the orbit by the method of Olbers, on the supposition of its being a parabola, but found that the middle observation was so badly represented, that this hypothesis could not be correct. I then proceeded to determine the elements without making any hypothesis as to the conic section, and the resulting elements are as follows:

Perihelion passage, 1843, October $26^{d}\cdot33$ Greenwich mean time.

Longitude of Perihelion on the Orbit......	54° 27′·8	From the equinox of Dec. 5
Longitude of ascending Node	207 38·0	
Inclination to the Ecliptic	10 48·9	
Perihelion Distance.............................	1·687	
Semi-axis Major	3·444	
Eccentricity	0·510	
Periodic Time	6·39	Sidereal years.

Motion direct.

I would suggest that the comet may not have been moving long in its present orbit, and that, as in the case of the comet of 1770, we are indebted to the action of *Jupiter* for its present apparition. In fact, supposing the above elements to be correct, the aphelion distance is very nearly equal to the distance of *Jupiter* from the Sun: also the time of the comet's being in aphelion was $1843\cdot8 - 3\cdot2 = 1840\cdot6$, at which time its heliocentric longitude was $234°\cdot5$ nearly, and the longitude of *Jupiter* was $231°\cdot5$; and, therefore, since the inclination to the plane of *Jupiter's* orbit is also small, the comet must have been very near *Jupiter* when in aphelion, and must have suffered very great perturbations, which may have materially changed the nature of its orbit.

9.

THE ORBIT OF THE NEW COMET.

[From the *Times*, October 15, 1844.]

HAVING obtained some results of an interesting nature respecting the new comet, I am induced to communicate them to the world through the medium of your widely-spread journal. My first investigations were founded on three observations made by Prof. Challis with the Northumberland equatorial on the 15th, 20th and 25th of September, and the orbit found from them appeared to be an ellipse of moderate eccentricity and short period. To test the accuracy of this result, Prof. Challis kindly favoured me with some more recent observations, which were made on the meridian, and therefore entitled to more confidence. Availing myself of the extension thus given to the arc described by the comet, I have re-calculated the orbit from the observations on the 15th and 25th of September and the 5th of October. The following are the results which I have obtained:

Perihelion passage, Sept. 2·4159 mean time at Greenwich.

Longitude of perihelion of the orbit...	342° 28′ 25″	From the mean equinox of Sept. 25
Longitude of ascending node............	63 47 7	
Inclination to the Ecliptic..............	2 56 13	
Log. ($\frac{1}{2}$ axis major)........................	0·500660	
Eccentricity..........................	=sin 38°40′22″	
Longitude perihelion distance	0·074841	
Period in sidereal years.................	5·636	

Motion direct.

These elements compared with observations give the following errors:—

Date	Error in Long.	Error in S. Lat.
Sept. 15	0°	0°
Sept. 25	+1·0	+ 3·5
Oct. 2	+6·1	−28·9 (merid. obs.)
Oct. 5	+0·0	0·0 (merid. obs.)

Though the period found may require considerable correction, I think there can be no doubt that the orbit is really elliptic. If this be the case, it is a remarkable fact that this is the second comet whose periodicity has been discovered during the present year.

10.

THE RELATIVE POSITION OF THE TWO HEADS OF BIELA'S COMET.

[Communicated to the *Royal Astronomical Society* (March 14, 1846).]

THE diagram shows the relative position of the two heads of Biela's Comet on Jan. 26·5, Feb. 11·5 and Feb. 27·5 mean Greenwich time, projected on a plane parallel to the equator. The rectangular coordinates of the smaller head, referred to the larger as origin, are as follows

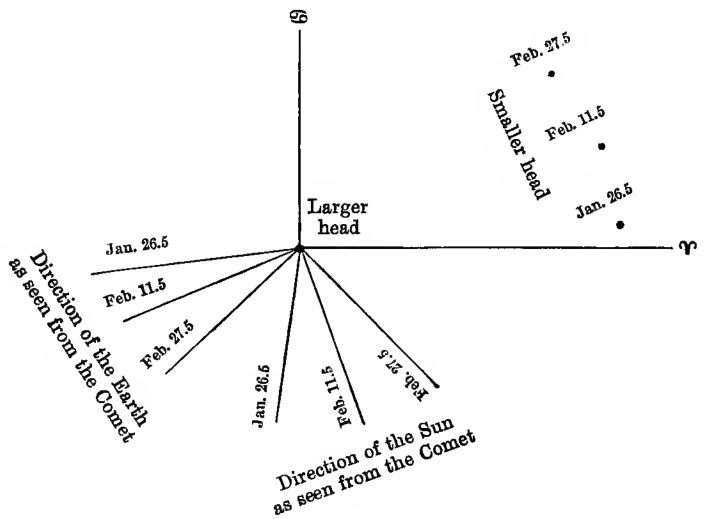
	x	y	z
Jan. 26·5	504·06	25·74	85·06
Feb. 11·5	481·99	154·95	107·12
27·5	404·65	270·21	118·26

The unit of measure is a line subtending an angle of 1″ at the mean distance of the Earth from the Sun; the plane parallel to the equator is the plane of xy; and the axis of x is a line drawn in the direction of the first point of *Aries*.

The relative velocities on Feb. 11·5, in the directions of the axes are as follows

$$\frac{dx}{dt} = -3{\cdot}2647, \quad \frac{dy}{dt} = 8{\cdot}1047, \quad \frac{dz}{dt} = 1{\cdot}1415\,;$$

the linear unit being the same as before, and the unit of time a mean solar day.



From these results it will be easy to deduce the differences of the elements of the orbits of the two heads. According to my calculations the periodic time of the smaller head is 8·48 days longer than the periodic time of the larger.

11.

ON THE APPLICATION OF GRAPHICAL METHODS TO THE SOLUTION OF CERTAIN ASTRONOMICAL PROBLEMS, AND IN PARTICULAR TO THE DETERMINATION OF THE PERTURBATIONS OF PLANETS AND COMETS.

[From the *Report of the British Association* (1849).]

AFTER briefly pointing out the advantages of graphical methods, the author proceeded to give some instances of their practical application. It was shewn that the solutions of the transcendental equation which expresses the relation between the mean and eccentric anomalies in an elliptic orbit is obtained in the most simple manner by the intersection of a straight line with the curve of sines. Attention was directed to Mr Waterston's graphical method of finding the distance of a comet from the Earth, and an analogous method was given for determining the distance of a planet, on the supposition that the orbit is a circle in the plane of the ecliptic.

The author then passed on to the more immediate object of his communication, the graphical treatment of the problem of perturbations of planets and comets. He first shewed how to obtain geometrical representations of the disturbing forces, and then gave simple constructions for determining the changes produced by these forces in each of the elements of the orbit, in a given small interval of time. Having obtained the total changes of the elements in any number of such intervals, it was shewn in the last place how to find their effect on the longitude, radius vector and latitude of the disturbed body, and thus to effect the complete solution of the problem of perturbations without calculation.

12.

ELEMENTS OF COMET II. 1854.

[From the *Monthly Notices of the Royal Astronomical Society,* Vol. XIV. (1854).]

PROBABLY you will have plenty of elements of the comet which is now starring it, nevertheless I may mention the following, which I deduced from Professor Challis's observations on March 30, April 1, 3. A comparison of these elements with an observation on April 7, gave an error of only 10″ in longitude, and nothing in latitude, so that they are probably not far from the truth.

Perihelion Passage, March 24·01221, G. M. T.

Longitude of Perihelion..............	213° 51′ 32″
Longitude of the Ascending Node	315 29 52
Inclination	82 34 28
Log. Perihelion Distance	9·4426170

Motion retrograde.

13.

OBSERVATIONS OF COMET II. 1861.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. XXII. (1862) and *Astronomische Nachrichten*, LVII. (1862).]

	G. M. S. T. 1861	Observed R. A.	Parallax × Δ.	Observed N. P. D.	Parallax × Δ.
d.	h. m. s.	h. m. s.		° ′ ″	
June 30	11 6 7·4	6 40 14·94	+0·127	43 25 37·1	−8·343
	11 19 51·1	6 40 40·50	+0·099	43 19 35·0	−8·391
July 2	10 41 46·6	8 30 28·47	+0·541	...	...
	10 57 47·4	...	...	27 36 40·5	−6·571
3	9 57 55·6	9 39 53·92	+0·822	24 10 11·5	−4·128
	11 4 52·4	9 43 15·42	+0·726	24 4 38·6	−5·330
5	10 29 33·1	11 44 52·88	+0·868	23 38 32·8	−2·301
8	9 54 51·8	13 17 34·82	+0·624	27 54 50·1	−0·573
	10 53 7·9	13 18 22·15	+0·706	27 58 30·1	−1·627
9	11 4 31·9	13 35 4·31	+0·673	29 23 45·7	−1·767
	11 56 10·5	13 35 36·11	+0·709	29 26 47·9	−2·768
10	11 7 1·7	13 47 55·32	+0·641	30 40 45·8	−1·800
13	11 22 27·6	14 13 11·05	+0·588	33 47 49·3	−2·208
23	10 32 49·0	14 46 40·09	+0·469	39 26 26·8	−2·151
26	10 32 17·3	14 51 52·27	+0·466	40 27 2·2	−2·369
27	10 33 40·2	14 53 23·97	+0·469	40 44 52·1	−2·456
31	10 26 32·3	14 58 53·10	+0·462	41 47 58·5	−2·630
Aug. 1	10 35 45·8	15 0 8·70	+0·473	42 2 10·6	−2·835
2	10 32 1·3	15 1 21·77	+0·469	42 15 28·0	−2·850
6	10 6 13·3	15 5 58·48	+0·448	43 3 51·5	−2·718
8	11 36 5·3	15 8 15·87	+0·508	43 26 28·1	−4·273
13	10 50 16·2	15 13 38·37	+0·489	44 14 56·1	−3·837
14	10 5 46·4	15 14 40·53	+0·459	44 23 31·9	−3·202
15	10 17 49·8	15 15 45·60	+0·470	44 32 18·2	−3·444

	G. M. S. T. 1861	Observed R. A.	Parallax × Δ.	Observed N. P. D.	Parallax × Δ.
d.	h. m. s.	h. m. s.		° ′ ″	
Aug. 16	10 9 30·0	15 16 49·44	+0·404	44 40 41·4	−3·375
19	10 29 28·4	15 20 3·57	+0·480	45 4 48·1	−3·860
20	10 17 53·8	15 21 7·81	+0·474	45 12 18·1	−3·735
21	9 22 2·6	15 22 10·59	+0·429	45 19 26·2	−2·960
23	9 45 40·3	15 24 22·39	+0·454	45 33 50·1	−3·414
24	9 31 49·0	15 25 27·54	+0·443	45 40 36·9	−3·265
27	10 12 6·3	15 28 50·08	+0·475	46 0 31·3	−4·034
28	10 12 2·4	15 29 57·85	+0·476	46 6 45·7	−4·088
30	9 22 4·7	15 32 10·76	+0·445	46 18 33·4	−3·437
Sept. 3	9 57 56·4	15 36 50·93	+0·472	46 40 58·0	−4·183
6	8 47 24·9	15 40 21·31	+0·427	46 55 58·1	−3·285
7	9 17 36·4	15 41 35·06	+0·453	47 0 49·0	−3·770
9	8 45 16·6	15 43 59·40	+0·431	47 10 1·5	−3·394
10	9 44 31·5	15 45 16·09	+0·470	47 14 35·8	−4·323
11	9 19 54·9	15 46 29·07	+0·460	47 18 47·9	−3·996
12	10 24 59·8	15 47 46·77	+0·477	47 23 9·1	−5·047
13	10 37 13·6	15 49 3·41	+0·475	47 27 11·9	−5·284
14	9 45 6·2	15 50 16·27	+0·472	47 30 53·2	−4·521
23	10 12 4·0	16 2 1·87	+0·470	47 59 51·8	−5·342
Oct. 9	9 30 36·3	16 24 25·11	+0·467	48 25 8·7	−5·358
11	8 57 45·5	16 27 20·63	+0·468	48 25 45·7	−4·935
12	10 38 55·7	16 28 56·26	+0·429	48 25 51·1	−3·473
14	9 55 27·1	16 31 52·46	+0·452	48 26 0·1	−5·918
15	9 14 45·4	16 33 20·43	+0·466	48 25 40·7	−5·342
16	8 32 50·6	16 34 47·90	+0·467	48 25 22·8	−4·740
23	8 21 25·1	16 45 33·56	+0·469	48 18 53·2	−4·813
28	7 35 56·1	16 53 23·50	+0·460	48 10 7·4	−4·290
Nov. 1	7 36 39·0	16 59 49·13	+0·465	48 0 34·7	−4·426
2	8 57 36·6	17 1 31·62	+0·464	47 57 36·8	−5·698
	9 1 53·1	17 1 32·41	+0·462	47 57 37·9	−5·761
5	8 3 14·1	17 6 21·53	+0·473	47 48 43·1	−4·959
6	7 52 51·5	17 7 59·25	+0·473	47 45 28·2	−4·830
7	8 2 21·5	17 9 38·79	+0·474	47 41 56·1	−5·008
9	8 43 42·4	17 13 1·20	+0·466	47 34 15·6	−5·721
11	8 38 55·1	17 16 20·95	+0·465	47 26 31·6	−5·689
20	6 53 55·6	17 31 30·01	+0·476	46 43 49·4	−4·315
23	7 49 1·8	17 36 44·66	+0·482	46 26 28·5	−5·327

	G. M. S. T. 1861	Observed R. A.	Parallax × Δ.	Observed N. P. D.	Parallax × Δ.
d.	h. m. s.	h. m. s.		° ′ ″	
Nov. 27	6 55 5·0	17 43 39·16	+0·485	46 2 18·8	−4·507
28	6 56 14·3	17 45 24·37	+0·486	45 55 48·9	−4·554
30	7 58 3·9	17 48 59·05	+0·479	45 42 56·9	−5·575
Dec. 3	7 27 26·7	17 54 16·40	+0·490	45 21 10·9	−5·193
4	7 55 49·1	17 56 5·36	+0·480	45 13 45·6	−5·653
5	8 7 48·6	17 57 53·10	+0·472	45 6 8·5	−5·882

The foregoing values were deduced as follows:—

	R. A. Comet − Star.	No. of Comp.	N. P. D. Comet − Star.	No. of Comp.	Star.
d.	m. s.		′ ″		
June 30	− 6 1·85	1	− 7 37·2	1	*a*
	−11 50·63	1	−31 27·0	1	*b*
July 2	{−11 26·54 {− 8 10·07	3	...	...	{*c* {*d*
			{+ 5 23·9 {− 2 54·2	2	{*c* {*d*
3	−28 2·06	1	− 1 49·4	1	*e*
	+ 3 46·57	3	+18 55·2	3	*f*
5	− 4 25·66	6	−20 23·8	6	*g*
8	−27 41·47	1	+ 5 53·8	1	*h*
	+ 3 2.19	5	+ 6 13·7	5	*i*
9	−29 25·99	1	−36 22·5	1	*k*
	− 3 33·03	2	+17 31·6	2	*l*
10	+ 2 11·99	6	− 5 28·7	6	*m*
13	− 6 13·30	4	+ 0 24·9	4	*n*
23	− 5 8·51	4	−21 37·8	4	*o*
26	− 5 2·31	7	+11 44·6	7	*p*
27	+ 5 42·19	4	− 0 32·1	4	*q*
31	− 0 21·65	11	− 0 9·5	11	*r*
Aug. 1	+ 0 53·98	6	+14 2·6	6	*r*
2	+ 5 25·25	6	+ 5 11·4	6	*s*
6	+ 1 54·96	8	+ 4 32·4	8	*t*
8	− 5 18·81	3	−25 49·5	3	*u*
13	− 5 47·78	6	+ 0 51·3	6	*v*
14	− 4 45·59	2	+ 9 27·1	2	*v*
15	+ 1 48·85	8	+ 3 47·8	8	*w*
16	+ 2 52·72	6	+12 10·9	6	*w*

		R. A. Comet – Star.	No. of Comp.	N. P. D. Comet – Star.	No. of Comp.	Star.
	d.	m. s.		′ ″		
Aug.	19	− 1 12·06	8	− 7 43·3	8	*x*
	20	+ 0 59·93	8	+ 3 45·8	8	*y*
	21	+ 0 55·02	8	+ 6 54·8	8	*x*
	23	+ 2 42·01	8	+ 3 30·7	8	*z*
	24	+ 3 47·18	8	+10 17·3	8	*z*
	27	− 4 51·96	6	+ 4 7·9	6	*a a*
	28	− 3 44·16	6	+10 22·2	6	*a a*
	30	+ 1 44·81	8	− 3 36·4	8	*b b*
Sept.	3	+ 2 3·92	6	−10 3·6	6	*c c*
	6	− 1 9·31	6	− 9 53·5	6	*d d*
	7	− 4 41·08	6	− 0 3·6	6	*e e*
	9	− 2 16·69	6	+ 9 8·7	6	*e e*
	10	− 2 37·66	6	+ 5 12·8	6	*ff*
	11	− 1 24·66	6	+ 9 24·8	6	*ff*
	12	− 0 6·93	8	+13 45·9	8	*ff*
	13	+ 1 9·73	6	+17 48·5	6	*ff*
	14	+ 2 22·62	6	+21 29·7	6	*ff*
	23	− 2 36·92	4	−32 34·3	4	*g g*
Oct.	9	+ 1 46·87	8	− 1 15·5	8	*h h*
	11	− 4 43·97	6	+ 6 7·8	6	*i i*
	12	− 3 8·32	2	+ 6 13·0	2	*i i*
	14	− 5 44·14	4	− 6 20·8	5	*k k*
	15	− 4 16·15	6	− 6 40·4	6	*k k*
	16	− 2 48·66	8	− 6 58·4	8	*k k*
	23	+ 1 0·70	6	+13 15·6	6	*l l*
	28	− 2 20·76	8	−10 55·2	8	*m m*
Nov.	1	+ 0 57·26	8	+ 6 18·9	8	*n n*
	2	− 0 34·14	7	− 0 6·2	7	*o o*
		+ 2 40·55	2	+ 3 21·9	2	*n n*
	5	− 2 47·38	8	+ 4 58·1	8	*p p*
	6	− 1 9·64	7	+ 1 42·9	7	*p p*
	7	+ 0 29·91	9	− 1 49·5	9	*p p*
	9	+ 2 39·71	4	− 3 46·3	3	*q q*
	11	− 0 48·10	9	− 9 12·8	9	*r r*
	20	+ 1 1·34	8	+13 17·5	8	*s s*
	23	+ 0 19·00	6	− 0 59·9	5	*t t*
	27	− 0 15·83	8	+11 34·8	6	*u u*

		R. A. Comet – Star.	No. of Comp.	N. P. D. Comet – Star.	No. of Comp.	Star.
	d.	m. s.		′ ″		
Nov.	28	+ 1 29·36	8	+ 5 4·7	6	*u u*
	30	– 0 51·62	8	– 5 34·2	6	*v v*
Dec.	3	– 1 9·17	8	+ 4 43·9	6	*w w*
	4	+ 0 39·79	8	– 2 41·7	6	*w w*
	5	+ 2 27·53	8	– 10 19·1	6	*w w*

The determinations of N. P. D. from July 2 to July 9, inclusive, are liable to some uncertainty, in consequence of the defective state of the clamp by which the declination-rod was attached to the polar frame. The determinations of R. A., however, are trustworthy.

The R. A. and N. P. D. for July 2 are obtained by taking a mean between the results of the comparisons with (*c*) and (*d*).

It is probable that in the observation of Nov. 30 the recorded micrometer-reading was too great by 5 revolutions, and that the N. P. D. should consequently be diminished by $5^r = 43''\cdot 2$.

Assumed Mean Places of the Stars of Comparison for 1861·0.

Star.	R. A. 1861·0.	N. P. D. 1861·0.	Authority.	
	h. m. s.	° ′ ″		
a	6 46 15·02	43 33 14·32	Johnson	1841
b	6 52 29·36	43 51 2·00	Arg.	7473
c	8 41 53·41	27 31 18·42	Johnson	2212
d	8 38 36·47	27 39 33·30	Arg.	9299
e	10 7 54·05	24 12 1·81	Johnson	2464
f	9 39 26·98	23 45 44·11	,,	2396
g	11 49 16·42	23 58 58·28	Arg.	12183-84
h	13 45 13·86	27 48 58·73	Johnson	3103
i	13 15 17·63	27 52 18·45	Arg.	13563
k	14 4 27·81	30 0 10·61	Johnson	3147
l	13 39 6·75	29 9 18·43	,,	3084
m	13 45 40·92	30 46 16·61	,,	3104
n	14 19 21·88	33 47 26·83	Arg.	14545
o	14 51 46·18	39 48 7·45	Johnson	3293
p	14 56 52·21	40 15 20·75	Arg.	15039
q	14 47 39·45	40 45 27·01	,,	14924-5 and 6
r	14 59 12·46	41 48 11·30	Johnson	3318
s	14 55 54·28	42 10 19·81	,,	3306

Star.	R. A. 1861·0. h. m. s.	N. P. D. 1861·0. ° ′ ″	Authority.
t	15 4 1·33	42 59 22·55	Arg. 15138, 39 &
u	15 13 32·48	43 52 21·45	„ 15266
v	15 19 24·05	44 14 8·99	„ 15347
w	15 13 54·71	44 28 34·20	„ 15272
x	15 21 13·64	45 12 35·38	Johnson 3385
y	15 20 5·93	45 8 36·32	Arg. 15355
z	15 21 38·49	45 30 23·41	Johnson 3387
a a	15 33 40·19	45 56 27·99	„ 3423
b b	15 30 24·17	46 22 13·89	„ 3413
c c	15 34 45·30	46 51 5·65	„ 3431
d d	15 41 28·95	47 5 56·02	„ 3448
e e	15 46 14·48	47 0 57·43	„ 3462
f f	15 47 52·16	47 9 27·67	„ 3464
g g	16 4 37·39	48 32 30·79	H. C. 29530
h h	16 22 37·14	48 26 28·41	„ 30042
i i	16 32 3·53	48 19 43·04	Eq. Comparison.
k k	16 37 35·56	48 32 26·10	H. C. 30489
l l	16 44 32·00	48 5 41·91	„ 30687
m m	16 55 43·43	48 21 7·36	„ 31031
n n	16 58 51·13	47 54 20·14	B. Z. 426 $16^h\ 57^m\ 41^s$
o o	17 2 5·02	47 57 47·50	Eq. Comparison.
p p	17 9 8·21	47 43 49·86	H. C. 31417
q q	17 10 20·85	47 38 5·93	„ 31456
r r	17 17 8·42	47 35 49·01	„ 31697
s s	17 30 28·18	46 30 36·13	„ 32154 and 5
t t	17 36 25·19	46 27 32·75	Johnson 3741
u u	17 43 54·58	45 50 48·38	„ 3763
v v	17 49 50·27	45 48 35·65	B. Z. 478. $17^h\ 47^m\ 53^s$
w w	17 55 25·32	45 16 31·59	Eq. Comparison.

The place assumed for the star (*i i*) is derived from equatorial comparisons made on Oct. 15 with H. C. 30489. The place of (*o o*) is derived from equatorial comparisons made on Nov. 20 with B. Z. 426. $16^d\ 57^m\ 41$, and the place of (*w w*) from equatorial comparisons with Johnson 3795 made on Feb. 20, 1862.

The observations up to July 13 were made by Professor Challis, and the subsequent ones by Mr Bowden, the senior Assistant at this Observatory.

14.

ON THE ORBIT OF γ VIRGINIS.

[From *Ædes Hartwellianæ*, Letter to Admiral Smythe, June, 1851.]

I HAVE great pleasure in sending you the results which I have obtained respecting the orbit of γ *Virginis*, and I feel the more indebted to you for having called my attention to the subject, inasmuch as the problem of determining the orbits of double stars is one with which I had previously only a theoretical acquaintance. The orbit, given by Sir John Herschel in the Results of his Cape Observations, was taken as the basis of the calculations, and equations of condition for the correction of the elements were formed by comparing certain selected angles of position deduced from observation with the values calculated by means of Sir John Herschel's elements.

The positions employed are those given by Bradley's observation in 1718, Sir William Herschel's observations in 1781 and 1803, a normal position for 1825 deduced from the observations of 1822, 1825, and 1828, one for 1833 from the observations of 1832, 1833, and 1834, another for 1839 from the observations of 1838, 1839, and 1840, and, lastly, a normal position for 1848 from the observations of 1846, 1847, 1848, 1849, and 1850. The number of these positions being greater by one than that absolutely necessary for the determination of the elements, I at first omitted the equation of condition for 1718 and solved the remaining ones in such a manner as to shew the effect which would be produced in each of the elements by a small given change in any one of the observed angles of position. The result proved that the elements would be greatly affected by small errors in the observed positions for 1781 and 1803, and I therefore called in the observation of 1718 to the rescue, and solved the equations anew, supposing the positions for 1825, 1833, 1839, and 1848 to be correct, and distributing the errors among the other three, according to the rules supplied by the method of least squares, giving double weight to the observations of 1781 and 1803.

The following are the resulting elements:—

Inclination of the orbit to the plane of projection	25° 27′
Position of the node	34 45
Distance of perihelion from the node	284 53
Angle of eccentricity	61 36
Eccentricity	0·87964
Perihelion passage	1836·34
Period	174·137 yrs.

The following table shews the differences between the observed positions and those calculated from the above elements:

Epoch.	Observed position.	Calculated position.	Differences.
1718·22	150° 52′	151° 3′	−11′
1781·89	130 44	130 29	+15
1803·20	120 15	120 43	−28
1825·32	97 46	97 43	+ 3
1833·27	61 16	61 11	+ 5
1839·36	215 51	216 2	−11
1848·37	180 6	180 6	0.

A better agreement could scarcely be desired. The observations made about the time of perihelion passage are liable to great errors in consequence of the excessive closeness of the stars, and therefore I did not take them into account in forming the equations of condition.

Sir John Herschel was obliged to admit large differences between these observations and the results of his theory, and these differences are considerably increased by using my elements. I am inclined to think that these observations cannot be satisfied without materially increasing the errors on both sides of the perihelion passage.

My elements agree very well with the latest observations which have come to my knowledge, as is shewn by the following comparison:

Observer.	Epoch.	Observed position.	Calculated position.	Differences.
Lord Wrottesley,	1851·172	175° 55′	175° 52′	+ 3′
Mr Dawes,	1851·217	176 35	175 49	+46
Mr Fletcher,	1851·401	175 58	175 34	+24

15.

ON THE TOTAL ECLIPSE OF THE SUN, 28 JULY 1851, AS SEEN AT FREDERIKSVAERN.

Latitude, 58° 59′ 33″·9 N. Longitude, $40^{m}\ 15^{s}\cdot 5$ East.

[From the *Memoirs of the Royal Astronomical Society.* Vol. XXI. (1852).]

THE approach of the total eclipse of July 28, 1851, produced in me a strong desire to witness so rare and striking a phenomenon. Not that I had much hope of being able to add anything of scientific importance to the accounts of the many experienced astronomers who were preparing to observe it; for I was not unaware of the difficulty which one not much accustomed to astronomical observation would have in preserving the requisite coolness and command of the attention amid circumstances so novel, where the points of interest are so numerous, and the time allowed for observation is so short. Certainly my experience has now shewn that I did not exaggerate these difficulties; but I have at least the satisfaction of having formed a far more vivid idea of the phenomenon than I could have obtained from any description; and I think that if I should ever have another opportunity of observing a total eclipse, I should be prepared to give a much better account of it than I can of the present.

I left Hull, by steamer, on the evening of Saturday, July 19, together with a large party of astronomers bound on the same errand with myself. In the afternoon of Tuesday the 22nd, we arrived at Christiania, where I landed with several other passengers, the remainder of the party going on to Göttenburg. We had no trouble in getting our instruments on shore;

the Norwegian Government having, in the most liberal and enlightened spirit, ordered the custom-house officers to allow them to pass without examination. This favour, I afterwards found, we owed to the kind offices of Professor Hansteen, whose acquaintance, as well as that of several other eminent Professors of the University, I had the happiness of making during my short stay at Christiania.

On Thursday the 24th, in company with my friend Mr Liveing, of St John's College, Cambridge, I proceeded by steamer to Frederiksværn, the point selected for making the observation, as being one easily accessible, and situated almost exactly on the central line of the path of the Moon's shadow. Here is one of the royal dockyards, containing a small observatory for giving time to the shipping. The officers of the dockyard shewed us much attention, and were anxious to render us every assistance in preparing for the observation. To Lieutenant Riis, in particular, we are under the deepest obligations. On Friday the 25th we inspected the Observatory, and examined the neighbourhood with the view of selecting a favourable spot for the observation. It rained heavily during the whole of Saturday, so that our prospects were not very encouraging, but on Sunday the weather improved, and on the morning of the eventful day, Monday the 28th, the sky was bright and clear, with the exception of a few light clouds, which, however, became more numerous as the day advanced, and at length overspread the heavens, as fresh vapour was brought up by the wind, which blew quite a gale from the south-west. I had intended to observe the eclipse from the summit of a rocky island lying just off the dockyard, and commanding an extensive prospect over the sea, though the view on the land side is cut off by a lofty ridge of rocks rising behind the town. The violence of the wind, however, made it necessary to choose some sheltered position for the instrument, and I fixed upon one in an angle within the ramparts of the dockyard. The telescope which I employed was one of Dollond's, which was kindly lent me by the Master and Fellows of St John's College. The aperture of the object-glass is $2\frac{3}{4}$ inches, and its focal length 42 inches. The astronomical eye-pieces belonging to the instrument giving too small a field of view, I employed a terrestrial eye-piece, with a magnifying power of about 20. The field was limited by a diaphragm having small teeth of different sizes arranged at intervals of 45° around its circumference, in order to enable me to estimate the position and magnitude of any small object that might be seen.

As the eastern limb of the Moon advanced over the Sun, I observed that it appeared uneven in several places, and two mountains were particularly noticed on the edge, about 5° apart and near the eastern extremity of the Moon's horizontal diameter. The cusps, too, as they were approaching each other, occasionally appeared to be somewhat blunted. I could see no trace of the Moon's limb extending beyond the Sun's disc. As the crescent became very narrow, it seemed to be in a state of violent agitation, and at last, just before the totality, it broke up into several parts. These, however, were not like the "beads" described by Mr Baily, but were quite irregular, being evidently occasioned by the inequalities on the Moon's limb. As the totality approached, the gloom rapidly increased; still, enough light remained up to the moment of total obscuration to render the change which then took place very marked and startling. For a few moments I felt somewhat confused, and did not immediately remove the dark glass. I then applied my eye to the finder, and saw the corona surrounding the dark body of the Moon. The light of the corona was pale, not sensibly coloured, and gradually faded away in receding from the Moon's edge. Its average breadth was perhaps about a third of the Moon's diameter, but it extended considerably farther in some directions than in others, its boundary being very irregular. It did not appear to consist of rays, and there was no marked annularity of structure, so that I could not decide whether it was concentric with the Sun or the Moon.

I now quitted the telescope and looked first at the Moon and then around on the sky. The appearance of the corona, shining with a cold unearthly light, made an impression on my mind which can never be effaced, and an involuntary feeling of loneliness and disquietude came upon me. I had previously ascertained the position of the principal stars and planets, but none of them could be seen on account of the clouds. I did not notice any peculiarity in the colours of surrounding objects. The light remaining was only just sufficient to enable me to read off the face of a box chronometer which I had with me. A party of haymakers, who had been laughing and chatting merrily at their work during the early part of the eclipse, were now seated on the ground, in a group near the telescope, watching what was taking place with the greatest interest, and preserving a profound silence.

About forty or fifty seconds after the commencement of the totality, I returned to the telescope, and cast my eye round the disc of the Moon. The light of the corona did not seem to be uniformly diffused round it, there being a patch brighter than the rest near the point where the Sun's last rays had disappeared. At the point nearly opposite, or about 105° from the upper point of the Moon, measured towards the west, I noticed a rosy-coloured prominence, about one minute in altitude. The upper or northern boundary of this was well defined, and had nearly the form of a quadrantal arc of a circle meeting the Moon's limb perpendicularly, the concavity being turned downwards; the southern boundary was also somewhat concave downwards, but the illumination near it was less, and diminished gradually, so that it was difficult to ascertain its exact form. The appearance was somewhat like the enlightened portion of a hemispherical mountain standing on the Moon's limb and illuminated on its northern side, whilst more than half the hemisphere on the opposite side was invisible. After watching this for a short time, I observed that its altitude was gradually increasing, and my attention became in consequence entirely engrossed by it. The southern boundary of this prominence soon became better defined than at first, while the northern boundary remained perfectly even and well defined throughout. The altitude continued to increase till the moment of the Sun's reappearance, when it amounted to nearly three minutes. The form of the prominence now resembled that of a sickle, and it projected nearly perpendicularly from the Moon's limb, the part nearest the Moon being nearly straight, but the curvature gradually increasing in approaching the point, which was sharp and turned downwards. The breadth at the base was, perhaps, two-thirds of a minute. There was no sensible, or at any rate, no marked change of form in the several parts after they had once been seen, but only a gradual lengthening by additions at the base, of such a kind as would have been occasioned by the motion of the Moon if the prominence had really belonged to the Sun[1]. My impression, however, is, that the increase of length was greater than can be accounted for by the Moon's motion, and that it proceeded more rapidly towards the end of the totality than at first, but I cannot feel certain on this point. A little before the end of the totality, the corona seemed to become brighter in the neighbourhood of the prominence, which was close to the point

[1] "While the Sun is totally covered by the Moon, the latter appears surrounded by a luminous ring, with rays proceeding from it, something in the manner of the glory which is placed by painters round the heads of saints. The most extraordinary appearances however were certain rosy-coloured flame-like projections from the limb of the Moon, one, which I noticed particularly, was very large. This was at the point of the limb at which the Sun reappeared, and it appeared gradually to lengthen out as the Sun's limb was approaching the Moon's, as if it had really been connected with the Sun and moved with it....... If these rosy flames really belong to the Sun, they must be of enormous magnitude, the one I noticed could not have been less than 50,000 miles in length." *From Letter written Aug.* 9, 1851.

where the Sun was about to reappear. On account of the clouds, I felt no inconvenience in observing the reappearance without the intervention of a dark glass. As the first ray of the Sun appeared the corona vanished, and at the same moment the prominence seemed suddenly to contract and change its form, the point of it disappearing and the remaining part becoming detached from the limb of the Moon. In about a second more the whole had vanished. I did not notice any interruption to the continuity of the Sun's limb in its reappearance, like that with which I had been struck when it disappeared, the Moon's western limb being apparently much more regular than the eastern.

The clouds now grew rapidly thicker, and completely hid the Sun from view before the end of the eclipse.

At the small observatory the eclipse was observed by Lieutenants Smith and Hjorth, two officers of the Norwegian Royal Navy, and also by the well-known French traveller, M. D'Abbadie. Lieut. Smith, who was specially charged by Professor Hansteen with the determination of the time, found the following results:

	h. m. s.	
Beginning of the Eclipse ...	2 41 40·3	Mean Time at the Observatory.
Beginning of the Totality ...	3 44 52·3	,, ,, ,,
End of Totality	3 48 17·8	,, ,, ,,

The end of the eclipse could not be observed.

According to Professor Hansteen, the longitude of the Observatory is $2^m 39^s{\cdot}3$ west of Christiania, or $40^m 15^s{\cdot}5$ east of Greenwich, and its latitude $58° 59' 33''.9$ north.

Lieut. Hjorth compares the appearance of the prominence to that of the flame of a candle acted on by the blowpipe.

Besides this prominence, which was the only one seen by me, Lieut. Hjorth observed two much smaller ones to spring up a little before the end of the totality, on the same side of the Moon as the former, one being above and the other below it.

Mr Liveing, who observed the eclipse from the same spot with myself, has kindly communicated the following observations, taken with the naked eye.

"The first appearance I noted was the formation of a halo round the Sun soon after the eclipse commenced; light clouds were at the same time flitting across the sky. When the totality approached, the passage of the shadow was not so rapid but that I could see the clouds to the north-west grow dark before the last direct beam of the Sun was extinguished. And at the reappearance of the Sun it was still more remarkable; the clouds to the north-west lightened up, making it much lighter where I stood; and I had time to exclaim that the Sun was going to appear, and to turn my eyes towards him, an appreciable interval before he actually shewed himself. The first appearance was a single point of light, like a very bright star, increasing in size, of course, very rapidly.

"I did not observe that the landscape was peculiarly livid; it had a cold appearance, but much such as it often has after sunset; and the only clear part of the sky, towards the south-east horizon, had quite an orange hue, also such as is not unusual after sunset; and it remained nearly the same colour the whole time of darkness.

"I looked for colour in the corona, but could see none; neither did it appear to me divided by a dark ring, or to be regular or well-defined on the outside; in four points it certainly appeared to project to a greater distance than at the intermediate points, and these four points were at unequal intervals; but I did not watch it long enough to observe how far this might be due to the clouds which covered it, and which had now become much thicker than at first. As I did not expect to be able to observe it, I had no means of exactly measuring the intensity of the light; but I could not distinguish the features of people about four yards from me; and a candle at about the same distance threw a well-defined shadow.

"A crow was the only animal near me; it seemed quite bewildered, croaking and flying backwards and forwards near the ground in an uncertain manner."

I have also been favoured with the following interesting account by another friend, who observed the eclipse in company with several other persons, from an elevated point about thirty-three miles west of Christiania, which commands an extensive view of the surrounding country.

"We observed the eclipse from the Skuderud Sæters, about nine miles north-east of Fossum, and nearly on the same parallel as Christiania. We had smoked glasses, and also a small telescope smoked. The eclipse appeared to begin about $2^h 45^m$. As the shadow increased the change in the appearance of the country was most curious. The light became pale; our shadows were sharply cut, as by moonlight, but the light was more yellow. A deep gray twilight seemed to come on. Perhaps two minutes before the totality a dark, thick shade appeared over the west and north-west mountains, which drew nearer, till, when the eclipse became total, it entirely surrounded us, though it was paler or less dense towards the east. But on the instant that we were in complete shade, a bright orange streak of light appeared on the horizon to the north-west, spreading west and south. The corona was orange. Bright, pale, and very irregular yellow rays streamed round like the glories round the heads of saints. Many stars were visible, but *Venus* was the only planet pointed out to me. The totality lasted $2^m 50^s$ to the best of our reckoning; but before the Sun reappeared the clouds thickened rapidly, and afterwards we only caught stray glimpses. For a minute after the totality was passed the dark shade lingered over the south and south-east.

"The following remarks are numbered with reference to the *Suggestions* drawn up by a Committee of the British Association.

"16. We noticed no variation of colour in the sky.

"18. The corona appeared to be formed instantaneously all round; equally broad; not divided into rings.

"22. The corona cast no shadow. I read the word 'Observation' at three yards, the remainder of the title at two, the interior print at the usual distance in my hand. I read the same at the same distances at $10^h 30^m$ the following evening, the book facing west; and at six, four, and two yards distance by sunlight.

"24. The outline of all the mountains was perfectly distinct."

I cannot close this account without expressing my sense of the kind hospitality which I met with during a subsequent tour of six weeks in Norway. To Mr Crowe, Her Majesty's Consul-general at Christiania, whose kindness is so well known to all English travellers in that country, I feel particularly bound to return my warmest thanks.

16.

ON AN IMPORTANT ERROR IN BOUVARD'S TABLES OF SATURN.

[From the *Memoirs of the Royal Astronomical Society* (1849), Vol. XVII., and *Monthly Notices of the Royal Astronomical Society* (1847), Vol. VII.].

HAVING lately entered upon a comparison of the theory of *Saturn* with the Greenwich observations, I was immediately struck with the magnitude of the tabular errors in heliocentric latitude, and the more so, since the whole perturbation in latitude is so small, that it could not be imagined that these errors arose from any imperfection in the theory. In order to examine the nature of the errors, I treated them by the method of curves, taking the times of observation as abscissæ, and the corresponding tabular errors as ordinates. After eliminating, by a graphical process, the effects of a change in the node and inclination, a well-defined inequality became apparent, the period of which was nearly twice that of *Saturn*. One of the principal terms of the perturbation in latitude (viz. that depending on the mean longitude of *Jupiter* minus twice that of *Saturn*) having nearly the same period, I was next led to examine whether this term had been correctly tabulated by Bouvard. The formula in the introduction appeared to be accurate; but on inspecting the Table XLII., which professes to be constructed by means of this formula, I was surprised to find that there was not the smallest correspondence between the numbers given by the formula and those contained in the table, the latter following the simple progression of sines, while the formula contained two terms. The origin of this mistake is rather curious. Bouvard's formula for the terms in question is

$$9''\cdot67 \sin\{\phi - 2\phi' - 60^\circ\cdot29\} + 28''\cdot19 \sin\{2\phi - 4\phi' + 66^\circ\cdot12\}$$

but in tabulating the last term he appears to have taken the simple argument $\phi - 2\phi'$ instead of $2\phi - 4\phi'$, so that the two parts may be united

into a single term, $$25''\cdot 85 \sin\{\phi - 2\phi' + 43^\circ\cdot 88\}$$
which I find very closely to represent Bouvard's Table XLII.

After correcting the above error, and making a proper alteration in the inclination and place of the node, the remaining errors of latitude are in general very small. I subjoin a correct table, to be used instead of Bouvard's. The constant added being 36″·0 instead of 26″·0, it will be necessary to subtract 10″·0 from the final result.

TABLE XLII.

Argument III. de la Longitude.

Argument.	Equation.	Argument.	Equation.	Argument.	Equation.	Argument.	Equation.
	″		″		″		″
0	52·4	2500	17·4	5000	68·1	7500	6·1
100	54·4	2600	16·2	5100	69·4	7600	4·0
200	56·0	2700	15·5	5200	70·2	7700	2·3
300	57·2	2800	15·2	5300	70·5	7800	1·1
400	58·0	2900	15·2	5400	70·4	7900	0·4
500	58·3	3000	15·7	5500	69·8	8000	0·1
600	58·3	3100	16·6	5600	68·7	8100	0·4
700	57·8	3200	17·9	5700	67·2	8200	1·0
800	56·9	3300	19·6	5800	65·3	8300	2·2
900	55·7	3400	21·7	5900	62·9	8400	3·7
1000	54·1	3500	24·1	6000	60·1	8500	5·7
1100	52·2	3600	26·7	6100	57·1	8600	8·0
1200	50·0	3700	29·7	6200	53·7	8700	10·7
1300	47·5	3800	32·8	6300	50·0	8800	13·7
1400	44·9	3900	36·2	6400	46·2	8900	16·8
1500	42·1	4000	39·6	6500	42·1	9000	20·2
1600	39·2	4100	43·1	6600	38·0	9100	23·7
1700	36·2	4200	46·5	6700	33·9	9200	27·3
1800	33·3	4300	50·0	6800	29·8	9300	31·0
1900	30·4	4400	53·3	6900	25·7	9400	34·5
2000	27·7	4500	56·5	7000	21·8	9500	38·0
2100	25·1	4600	59·4	7100	18·1	9600	41·4
2200	22·8	4700	62·1	7200	14·6	9700	44·6
2300	20·6	4800	64·5	7300	11·4	9800	47·5
2400	18·8	4900	66·5	7400	8·5	9900	50·1
2500	17·4	5000	68·1	7500	6.1	10000	52·4

Constante ajoutée 36″·0.

17.

ON NEW TABLES OF THE MOON'S PARALLAX.

[From the *Monthly Notices of the Royal Astronomical Society* (1853), Vol. XIII., and *Nautical Almanac for* 1856.]

THE importance of an accurate knowledge of the Moon's Parallax is very evident. No observation of the Moon's place can be compared with the Tables, or turned to any practical use, without undergoing a preliminary reduction of which the amount of the Parallax is the most important element. Now the same theory by which the angular motion of the Moon round the Earth is determined gives likewise the form of the orbit, and therefore the proportion between the Parallaxes at different times; hence, as the theory is sufficiently perfect to represent the place of the Moon within 10″, it cannot be doubted that it would be competent to give the variations of the Parallax within a small fraction of a second, provided the mean Parallax were known. To determine this, however, by theory, it is necessary to know, in addition to the elements furnished by observations of the Moon's motion, the ratio of the Moon's mass to that of the Earth. Hence, conversely, if the mean value of the Parallax be deduced from corresponding observations of the Moon's declination, made at distant points on the Earth's surface, one means is afforded of finding the ratio of the masses.

The most recent determination of the Parallax by means of observations of this kind is contained in a paper by Mr Henderson in the tenth volume of the *Memoirs of the Royal Astronomical Society*, and is founded on his own observations made at the Cape of Good Hope, combined with corresponding observations at Greenwich and Cambridge. In this paper Mr Henderson compares the Parallaxes deduced from observation with those calculated by means of the Tables both of Burckhardt and Damoiseau. It is remarkable that he finds a difference of $1''\cdot3$ in the value of the mean Parallax, according as one set of Tables or the other is employed in the comparison, and not knowing which value to prefer, he adopts the mean of the two for his final result.

If we consider, however, that the only part of this process which depends on the Tables consists in the reduction of the actual Parallaxes at the times of observation to the mean value, it is plain that so large a difference in the mean of thirty-four observations can only arise from intolerable errors in the periodic terms of Parallax given by one of the two sets of Tables.

The Parallax in Damoiseau's Tables is given at once in the form in which it is furnished by theory, but that in Burckhardt's Tables is adapted to his peculiar form of the arguments, and requires transformation in order to be compared with the former. When this was done, I found that several of the minor equations of Parallax deduced from Burckhardt differed completely from their theoretical values given by Damoiseau.

On further inquiry, I discovered that the difference between Burckhardt's equations of Parallax and those of Bürg and Damoiseau had been long since remarked by Clausen in a comparative analysis of the three sets of Lunar Tables given in the seventeenth volume of the *Astronomische Nachrichten*, but no notice appears to have been taken of this remark.

With regard to the Parallax, Burckhardt professes to have followed the theory of Laplace, but this agrees very closely with that of Damoiseau, so that errors have evidently been committed by him in the transformation of Laplace's formula.

These appear to have originated in the following manner:

In the formation of Burckhardt's Arguments of Evection and Variation, the *mean* longitude of the Sun is employed. Now four of the errors in the coefficients of the minor equations may be accounted for, by supposing him to have erroneously employed the *true* instead of the *mean* longitude of the Sun in forming the above-mentioned arguments. In another of these equations, the coefficient is taken with a wrong sign, and in another a wrong argument is employed.

A strange fatality seems to have attended all Burckhardt's calculations respecting the Moon's Parallax. In the *Connaissance des Temps* for the year XV of the Republic, he gives a comparison between the values furnished by Mayer's and Laplace's theories, and he concludes that the error of the former may sometimes amount to 7″.

But this difference is caused almost wholly by an error in his own transformation of Laplace's expression. In the formation of Mayer's Arguments of Evection and Variation, the *true* longitude of the Sun is employed, but Burckhardt appears to have inadvertently used the *mean* longitude instead of it, an error which is the exact converse of the one above noticed with respect to his own Tables.

After examining Burckhardt's Table of Parallax, I was naturally led to scrutinize more closely the results of the theories of Damoiseau, Plana, and Pontécoulant, with respect to the same subject. Although the differences between these were very trifling when compared with the errors of Burckhardt, still they were greater than we had a right to expect, considering the close agreement which existed with respect to the equations of longitude. In the theories of Damoiseau and Plana, the expression for the projection of the Moon's radius vector on the Ecliptic in terms of her true longitude is required in order to find the relation between that longitude and *the time*, and therefore no pains have been spared to obtain it with accuracy; but in the subsequent operations and transformations necessary in order to deduce the expression for the Parallax in terms of the time, the same care has not been employed. In Pontécoulant's theory the time is taken as the independent variable, and consequently the analytical expression for the Parallax in the form required is obtained immediately, and is developed to as great an extent as the corresponding expression for the longitude, yet in the conversion of his formula into numbers he neglects all the terms beyond the fifth order, so that several of the resulting coefficients are sensibly in error.

I have endeavoured to supply these defects and omissions.

In the seventeenth volume of the *Astronomische Nachrichten*, M. Hansen gives the expression which he has obtained for the logarithm of the sine of the horizontal Parallax, by means of his new method of treating the Lunar Theory. I have transformed this expression with the care which its great value deserves, so as to compare it with the results of the former theories.

The agreement thus found between the several theories is most satisfactory, the difference of the separate values of each coefficient and the general mean rarely amounting to a hundredth of a second. There are only two instances in which this amount is much exceeded. One of these relates to the constant of Parallax, the value of which, given by M. Hansen's method, is $0''\cdot06$ less than the corresponding value found from the same fundamental data by the other methods, and the second relates to the term whose argument in Damoiseau's notation is $t+z$, the coefficient being $0''\cdot146$ according to Damoiseau and Plana, $0''\cdot140$ according to Pontécoulant, and $0''\cdot181$ according to Hansen.

The values of the constant of Parallax which I have deduced from the theories of Damoiseau, Plana, and Pontécoulant agree perfectly with one another, and from the particular examination which I have given to this subject, I am induced to place considerable reliance on the result. It is possible that M. Hansen's definitive value of the constant may differ slightly from that which he has given in the paper above referred to.

From the value of the constant of Nutation found by M. Peters, it follows that the ratio of the Moon's mass to that of the Earth is as 1 to 81·5 nearly. Employing this ratio, together with the dimensions of the Earth according to Bessel, and the length of the seconds' pendulum in latitude $35\frac{1}{4}°$, deduced from Mr Baily's Report on Foster's Pendulum experiments, I find the value of the constant of Parallax to be $3422''\cdot325$.

Now Henderson, in the paper cited above, has found the value of the constant, by comparison with Damoiseau's Tables, to be $3422''\cdot46$.

It should, however, be remarked that what the Table calls the Parallax is more strictly the *sine* of the Parallax converted into seconds of arc. In Henderson's calculations he has taken the tabular quantity to denote the Parallax itself, so that the value found must be diminished by $0''\cdot15$ in order to obtain the constant of the *sine* of the Parallax. Thus the value deduced in this manner is $3422''\cdot31$, a result admirably agreeing with that just derived from theory.

I have carefully transformed the expression for the Parallax given by theory, so as to make it depend on Burckhardt's Arguments of Longitude, and from the resulting formula Mr Farley has calculated the Tables which are appended to this paper. Constants are added to the several equations so as to render them always positive.

The Minor Equations of Equatorial Horizontal Parallax are comprised in Table I.

Table II. contains the Equation depending on the Argument of Evection;

Table III. that depending on the Argument of Variation; and

Table IV. that depending on the Argument of Anomaly.

The formulæ employed in their construction are the following, in which

E denotes Burckhardt's argument of Evection;

V that of Variation; and

A that of Anomaly;

and the Arguments of the Minor Equations are denoted by their numbers as in Burckhardt.

$$
\begin{array}{rl}
0''\cdot34 - & 0''\cdot34 \cos(\text{Arg. } 1) \\
1\cdot73 + & 1\cdot73 \cos(\text{Arg. } 2) \\
1\cdot46 + & 1\cdot46 \cos(\text{Arg. } 4) \\
0\cdot87 + & 0\cdot87 \cos(\text{Arg. } 5) \\
0\cdot71 - & 0\cdot71 \cos(\text{Arg. } 6) \\
0\cdot11 - & 0\cdot11 \cos(\text{Arg. } 7) \\
0\cdot62 - & 0\cdot62 \cos(\text{Arg. } 8) \\
1\cdot81 - & 0\cdot05 \cos(\text{Arg. } 9) + 1''\cdot81 \cos 2(\text{Arg. } 9) \\
0\cdot21 - & 0\cdot21 \cos(\text{Arg. } 12) \\
0\cdot16 - & 0\cdot16 \cos(\text{Arg. } 13) \\
0\cdot14 + & 0\cdot14 \cos(\text{Arg. } 16) \\
0\cdot12 + & 0\cdot12 \cos(\text{Arg. } 23) \\
0\cdot10 + & 0\cdot10 \cos(\text{Arg. } 25) \\
36\cdot81 + & 37\cdot22 \cos E + \ 0''\cdot41 \cos 2E \\
26\cdot18 - & 0\cdot94 \cos V + 26''\cdot34 \cos 2V + 0''\cdot16 \cos 4V \\
55'\ 50\cdot92 + & 187\cdot14 \cos A + 10''\cdot27 \cos 2A + 0''\cdot64 \cos 3A + 0''\cdot04 \cos 4A
\end{array}
$$

In this formula, a few terms have been neglected, the largest of the coefficïents of which does not exceed $0''\cdot08$.

The sum of the constants in this formula is $3422''\cdot29$, slightly differing from what is called the constant of Parallax, in consequence of the change in the form of developement.

For the sake of comparison I will here give the formula on which Burckhardt's own Tables are constructed, which is as follows:

$$
\begin{array}{r l}
0''\cdot4 - & 0''\cdot4 \cos(\text{Arg. } 1) \\
0\cdot8 + & 0\cdot8 \cos(\text{Arg. } 2) \\
0\cdot3 + & 0\cdot3 \cos(\text{Arg. } 4) \\
0\cdot8 + & 0\cdot8 \cos(\text{Arg. } 5) \\
1\cdot1 + & 0\cdot8 \cos(\text{Arg. } 6) \\
0\cdot6 - & 0\cdot6 \cos(\text{Arg. } 8) \\
1\cdot8 + & 1\cdot8 \cos 2\,(\text{Arg. } 9) \\
0\cdot7 + & 0\cdot7 \cos(\text{Arg. } 12) \\
1\cdot0 + & 1\cdot0 \cos(\text{Arg. } 13) \\
43\cdot0 + & 37\cdot4 \cos E + \;0''\cdot4 \cos 2E \\
30\cdot0 - & 1\cdot0 \cos V + 26''\cdot3 \cos 2V + 0''\cdot3 \cos 3V \\
55'\;40\cdot0 + & 187\cdot0 \cos A + 10''\cdot2 \cos 2A + 0''\cdot3 \cos 3A
\end{array}
$$

The sum of the constants in this formula is 3420″·5.

The errors of the coefficients of Equations 2 and 12 arise from the mistake respecting the formation of the Argument of Variation before explained, and those of the coefficients of Equations 4 and 13 from the similar mistake respecting the Argument of Evection.

Equation 6 is taken with a wrong sign, and in the Variation Equation $3V$ appears to be wrongly substituted for $4V$, though I find that the corresponding term, when reduced to Burckhardt's form, has a smaller coefficient.

In consequence of the way in which most of these errors originate, their amount will be generally greatest in March and September, and least about the beginning of January and July, when the Sun's mean and true places coincide.

The total error of Burckhardt's Tables may amount to nearly 6″, independently of the change in the value of the constant.

Looking at the accuracy of modern observations, it is easy to imagine to what an extent the value of comparisons between observed and tabular places may be diminished by their being liable to an error of this kind.

In determining differences of longitude by means of occultations, it is plain that the results may be considerably affected by such an error in the Parallax. It has often been remarked that differences of longitude obtained by means of different occultations are not so consistent with each other as might be expected from the precise character of the observation, and I have no doubt that a great part of the discrepancy is to be attributed to the use of an erroneous Parallax.

Mr Maclear's observations at the Cape, combined with European observations, would doubtless furnish most valuable materials for a new determination of the constant of Parallax, care being of course taken to employ correct Tables in the reductions; and such a work would be a useful contribution to Astronomy.

In order to facilitate these and similar objects, Mr Stratford has calculated the Parallaxes from my Tables for each Greenwich mean noon in the years 1840—1855, and has thus obtained the corrections to be applied to the corresponding quantities given in the *Nautical Almanac.*

These corrections are embodied in Tables which are appended to the present paper. Subsequently to 1855, the Moon's Parallax given in the *Nautical Almanac* is calculated from my Tables.

TABLE I. Of the Moon's Equatorial Horizontal Parallax.

Arg.	Argument:—Arg^s 1, 2, 4, &c. from calculations of the Moon's Place by Burckhardt.													Arg.
	1	2	4	5	6	7	8	9	12	13	16	23	25	
	″	″	″	″	″	″	″	″	″	″	″	″	″	
000	0·00	3·46	2·92	1·74	0·00	0·00	0·00	3·57	0·00	0·00	0·28	0·24	0·20	1000
010	0·00	3·46	2·92	1·74	0·00	0·00	0·00	3·56	0·00	0·00	0·28	0·24	0·20	990
020	0·00	3·45	2·91	1·73	0·01	0·00	0·01	3·51	0·00	0·00	0·28	0·24	0·20	980
030	0·01	3·43	2·89	1·72	0·01	0·00	0·01	3·44	0·00	0·00	0·28	0·24	0·20	970
040	0·01	3·41	2·87	1·71	0·02	0·00	0·02	3·35	0·01	0·01	0·27	0·24	0·20	960
050	0·02	3·38	2·85	1·70	0·03	0·00	0·03	3·23	0·01	0·01	0·27	0·23	0·20	950
060	0·02	3·34	2·82	1·68	0·05	0·01	0·05	3·08	0·02	0·01	0·27	0·23	0·19	940
070	0·03	3·30	2·78	1·66	0·07	0·01	0·06	2·92	0·02	0·02	0·27	0·23	0·19	930
080	0·04	3·25	2·74	1·63	0·09	0·01	0·08	2·74	0·03	0·02	0·26	0·23	0·19	920
090	0·05	3·19	2·69	1·60	0·11	0·02	0·10	2·54	0·03	0·03	0·26	0·22	0·18	910
100	0·06	3·13	2·64	1·57	0·13	0·02	0·12	2·33	0·04	0·03	0·25	0·22	0·18	900
110	0·08	3·06	2·58	1·53	0·16	0·03	0·14	2·11	0·05	0·04	0·25	0·21	0·18	890
120	0·09	2·99	2·52	1·50	0·19	0·03	0·17	1·89	0·06	0·04	0·24	0·21	0·17	880
130	0·11	2·91	2·46	1·46	0·22	0·03	0·20	1·66	0·07	0·05	0·24	0·20	0·17	870
140	0·12	2·83	2·39	1·42	0·26	0·04	0·23	1·44	0·08	0·06	0·23	0·20	0·16	860
150	0·14	2·75	2·32	1·38	0·29	0·04	0·26	1·22	0·09	0·07	0·22	0·19	0·16	850
160	0·16	2·66	2·24	1·34	0·33	0·05	0·29	1·01	0·10	0·07	0·21	0·18	0·16	840
170	0·18	2·56	2·16	1·29	0·37	0·06	0·32	0·82	0·11	0·08	0·21	0·18	0·15	830
180	0·20	2·47	2·08	1·24	0·41	0·06	0·36	0·63	0·12	0·09	0·20	0·17	0·14	820
190	0·22	2·37	2·00	1·19	0·45	0·07	0·39	0·47	0·13	0·10	0·19	0·16	0·14	810
200	0·24	2·27	1·91	1·14	0·49	0·07	0·43	0·33	0·14	0·11	0·18	0·16	0·13	800
210	0·26	2·16	1·82	1·09	0·53	0·08	0·47	0·21	0·16	0·12	0·17	0·15	0·13	790
220	0·28	2·05	1·73	1·03	0·58	0·08	0·50	0·12	0·17	0·13	0·17	0·14	0·12	780
230	0·30	1·95	1·64	0·98	0·62	0·09	0·54	0·05	0·18	0·14	0·16	0·14	0·11	770
240	0·32	1·84	1·55	0·92	0·67	0·10	0·58	0·01	0·20	0·15	0·15	0·13	0·11	760
250	0·34	1·73	1·46	0·87	0·71	0·11	0·62	0·00	0·21	0·16	0·14	0·12	0·10	750
260	0·36	1·62	1·37	0·82	0·75	0·12	0·66	0·02	0·22	0·17	0·13	0·11	0·09	740
270	0·38	1·51	1·28	0·76	0·80	0·12	0·70	0·06	0·24	0·18	0·12	0·11	0·09	730
280	0·40	1·41	1·19	0·71	0·84	0·13	0·74	0·14	0·25	0·19	0·11	0·10	0·08	720
290	0·42	1·30	1·10	0·65	0·89	0·14	0·77	0·24	0·26	0·20	0·10	0·09	0·07	710
300	0·45	1·19	1·01	0·60	0·93	0·14	0·81	0·36	0·28	0·21	0·10	0·08	0·07	700
310	0·47	1·09	0·92	0·55	0·97	0·15	0·85	0·51	0·29	0·22	0·09	0·08	0·06	690
320	0·48	0·99	0·84	0·50	1·01	0·16	0·88	0·68	0·30	0·23	0·08	0·07	0·06	680
330	0·50	0·90	0·76	0·45	1·05	0·16	0·92	0·87	0·31	0·24	0·07	0·06	0·05	670
340	0·52	0·80	0·68	0·41	1·09	0·17	0·95	1·07	0·32	0·25	0·07	0·06	0·05	660
350	0·54	0·71	0·60	0·36	1·13	0·18	0·98	1·28	0·33	0·25	0·06	0·05	0·04	650
360	0·56	0·63	0·53	0·32	1·16	0·18	1·01	1·50	0·34	0·26	0·05	0·04	0·04	640
370	0·57	0·55	0·46	0·27	1·19	0·19	1·04	1·73	0·35	0·27	0·04	0·04	0·03	630
380	0·59	0·47	0·40	0·24	1·23	0·19	1·07	1·96	0·36	0·28	0·04	0·03	0·03	620
390	0·60	0·40	0·34	0·20	1·26	0·19	1·10	2·19	0·37	0·28	0·03	0·03	0·02	610
400	0·62	0·33	0·28	0·17	1·29	0·20	1·12	2·41	0·38	0·29	0·03	0·02	0·02	600
410	0·63	0·27	0·23	0·14	1·31	0·20	1·14	2·62	0·39	0·29	0·02	0·02	0·02	590
420	0·64	0·21	0·18	0·11	1·33	0·21	1·16	2·82	0·39	0·30	0·02	0·01	0·01	580
430	0·65	0·16	0·14	0·08	1·35	0·21	1·18	3·01	0·40	0·31	0·01	0·01	0·01	570
440	0·66	0·12	0·10	0·06	1·37	0·21	1·20	3·18	0·40	0·31	0·01	0·01	0·01	560
450	0·66	0·08	0·07	0·04	1·39	0·21	1·21	3·32	0·41	0·31	0·01	0·01	0·00	550
460	0·67	0·05	0·05	0·03	1·40	0·22	1·22	3·44	0·41	0·31	0·00	0·00	0·00	540
470	0·67	0·03	0·03	0·02	1·41	0·22	1·23	3·54	0·42	0·32	0·00	0·00	0·00	530
480	0·68	0·01	0·01	0·01	1·41	0·22	1·23	3·61	0·42	0·32	0·00	0·00	0·00	520
490	0·68	0·00	0·00	0·00	1·42	0·22	1·24	3·65	0·42	0·32	0·00	0·00	0·00	510
500	0·68	0·00	0·00	0·00	1·42	0·22	1·24	3·67	0·42	0·32	0·00	0·00	0·00	500
	1	2	4	5	6	7	8	9	12	13	16	23	25	

To be substituted for Burckhardt's Table XXVIII.

TABLE II. Of the Moon's Equatorial Horizontal Parallax.

Argument:—The Argument of Evection from calculations of the Moon's Place by Burckhardt.

	0s		Is		IIs		IIIs		IVs		Vs		
°	′ ″	diff. ″	′ ″	diff. ″	′ ″	diff. ″	′ ″	diff. ″	′ ″	diff. ″	′ ″	diff. ″	°
0	1 14·44		1 9·25		0 55·22		0 36·40		0 18·00		0 4·78		30
1	1 14·43	0·01	1 8·91	0·34	0 54·64	0·58	0 35·75	0·65	0 17·45	0·55	0 4·47	0·31	29
2	1 14·42	0·01	1 8·56	0·35	0 54·05	0·59	0 35·10	0·65	0 16·91	0·54	0 4·17	0·30	28
3	1 14·39	0·03	1 8·19	0·37	0 53·47	0·58	0 34·45	0·65	0 16·37	0·54	0 3·89	0·28	27
4	1 14·35	0·04	1 7·82	0·37	0 52·88	0·59	0 33·81	0·64	0 15·84	0·53	0 3·61	0·28	26
5	1 14·29	0·06	1 7·44	0·38	0 52·28	0·60	0 33·16	0·65	0 15·32	0·52	0 3·34	0·27	25
6	1 14·23	0·06	1 7·05	0·39	0 51·67	0·61	0 32·52	0·64	0 14·81	0·51	0 3·08	0·26	24
7	1 14·15	0·08	1 6·65	0·40	0 51·07	0·60	0 31·88	0·64	0 14·30	0·51	0 2·83	0·25	23
8	1 14·06	0·09	1 6·24	0·41	0 50·46	0·61	0 31·24	0·64	0 13·80	0·50	0 2·59	0·24	22
9	1 13·96	0·10	1 5·82	0·42	0 49·84	0·62	0 30·60	0·64	0 13·30	0·50	0 2·37	0·22	21
10	1 13·85	0·11	1 5·39	0·43	0 49·22	0·62	0 29·96	0·64	0 12·81	0·49	0 2·15	0·22	20
11	1 13·73	0·12	1 4·96	0·43	0 48·60	0·62	0 29·33	0·63	0 12·33	0·48	0 1·94	0·21	19
12	1 13·59	0·14	1 4·51	0·45	0 47·98	0·62	0 28·70	0·63	0 11·86	0·47	0 1·74	0·20	18
13	1 13·44	0·15	1 4·06	0·45	0 47·35	0·63	0 28·07	0·63	0 11·40	0·46	0 1·56	0·18	17
14	1 13·28	0·16	1 3·60	0·46	0 46·72	0·63	0 27·44	0·63	0 10·94	0·46	0 1·38	0·18	16
15	1 13·12	0·16	1 3·13	0·47	0 46·09	0·63	0 26·82	0·62	0 10·49	0·45	0 1·21	0·17	15
16	1 12·94	0·18	1 2·65	0·48	0 45·45	0·64	0 26·20	0·62	0 10·05	0·44	0 1·05	0·16	14
17	1 12·74	0·20	1 2·17	0·48	0 44·81	0·64	0 25·59	0·61	0 9·62	0·43	0 0·91	0·14	13
18	1 12·54	0·20	1 1·67	0·50	0 44·17	0·64	0 24·98	0·61	0 9·19	0·43	0 0·78	0·13	12
19	1 12·33	0·21	1 1·17	0·50	0 43·53	0·64	0 24·37	0·61	0 8·78	0·41	0 0·65	0·13	11
20	1 12·10	0·23	1 0·66	0·51	0 42·89	0·64	0 23·77	0·60	0 8·37	0·41	0 0·54	0·11	10
21	1 11·86	0·24	1 0·15	0·51	0 42·24	0·65	0 23·17	0·60	0 7·97	0·40	0 0·44	0·10	9
22	1 11·61	0·25	0 59·63	0·52	0 41·60	0·64	0 22·57	0·60	0 7·58	0·39	0 0·35	0·09	8
23	1 11·36	0·25	0 59·10	0·53	0 40·95	0·65	0 21·98	0·59	0 7·20	0·38	0 0·27	0·08	7
24	1 11·09	0·27	0 58·56	0·54	0 40·30	0·65	0 21·40	0·58	0 6·82	0·38	0 0·20	0·07	6
25	1 10·81	0·28	0 58·02	0·54	0 39·65	0·65	0 20·82	0·58	0 6·46	0·36	0 0·14	0·06	5
26	1 10·52	0·29	0 57·47	0·55	0 39·00	0·65	0 20·24	0·58	0 6·11	0·35	0 0·09	0·05	4
27	1 10·22	0·30	0 56·92	0·55	0 38·35	0·65	0 19·67	0·57	0 5·76	0·35	0 0·05	0·04	3
28	1 9·90	0·32	0 56·36	0·56	0 37·70	0·65	0 19·11	0·56	0 5·43	0·33	0 0·02	0·03	2
29	1 9·58	0·32	0 55·79	0·57	0 37·05	0·65	0 18·55	0·56	0 5·10	0·33	0 0·00	0·02	1
30	1 9·25	0·33	0 55·22	0·57	0 36·40	0·65	0 18·00	0·55	0 4·78	0·32	0 0·00	0·00	0
	XIs		Xs		IXs		VIIIs		VIIs		VIs		

To be substituted for Burckhardt's Table XXIX.

TABLE III. Of the Moon's Equatorial Horizontal Parallax.

Argument:—The Argument of Variation from calculations of the Moon's Place by Burckhardt.

°	0ˢ ′ ″	diff. ″	Iˢ ′ ″	diff. ″	IIˢ ″	diff. ″	IIIˢ ′ ″	diff. ″	IVˢ ′ ″	diff. ″	Vˢ ′ ″	diff. ″	°
0	0 51·74	0·02	0 38·46	0·81	0 12·46	0·76	0 0·00	0·03	0 13·40	0·81	0 40·08	0·81	30
1	0 51·72	0·05	0 37·65	0·82	0 11·70	0·75	0 0·03	0·06	0 14·21	0·82	0 40·89	0·79	29
2	0 51·67	0·08	0 36·83	0·83	0 10·95	0·73	0 0·09	0·10	0 15·03	0·84	0 41·68	0·77	28
3	0 51·59	0·11	0 36·00	0·85	0 10·22	0·71	0 0·19	0·13	0 15·87	0·85	0 42·45	0·75	27
4	0 51·48	0·15	0 35·15	0·85	0 9·51	0·69	0 0·32	0·15	0 16·72	0·87	0 43·20	0·74	26
5	0 51·33	0·18	0 34·30	0·87	0 8·82	0·66	0 0·47	0·19	0 17·59	0·87	0 43·94	0·71	25
6	0 51·15	0·20	0 33·43	0·88	0 8·16	0·65	0 0·66	0·22	0 18·46	0·89	0 44·65	0·69	24
7	0 50·95	0·24	0 32·55	0·88	0 7·51	0·62	0 0·88	0·25	0 19·35	0·89	0 45·34	0·66	23
8	0 50·71	0·28	0 31·67	0·89	0 6·89	0·60	0 1·13	0·28	0 20·24	0·91	0 46·00	0·65	22
9	0 50·43	0·30	0 30·78	0·90	0 6·29	0·58	0 1·41	0·30	0 21·15	0·91	0 46·65	0·62	21
10	0 50·13	0·33	0 29·88	0·90	0 5·71	0·55	0 1·71	0·34	0 22·06	0·92	0 47·27	0·59	20
11	0 49·80	0·37	0 28·98	0·90	0 5·16	0·53	0 2·05	0·37	0 22·98	0·92	0 47·86	0·57	19
12	0 49·43	0·39	0 28·08	0·91	0 4·63	0·50	0 2·42	0·40	0 23·90	0·93	0 48·43	0·54	18
13	0 49·04	0·43	0 27·17	0·91	0 4·13	0·48	0 2·82	0·42	0 24·83	0·92	0 48·97	0·52	17
14	0 48·61	0·45	0 26·26	0·91	0 3·65	0·44	0 3·24	0·45	0 25·75	0·93	0 49·49	0·49	16
15	0 48·16	0·48	0 25·35	0·90	0 3·21	0·42	0 3·69	0·48	0 26·68	0·94	0 49·98	0·46	15
16	0 47·68	0·50	0 24·45	0·91	0 2·79	0·39	0 4·17	0·51	0 27·62	0·93	0 50·44	0·43	14
17	0 47·18	0·53	0 23·54	0·90	0 2·40	0·37	0 4·68	0·53	0 28·55	0·93	0 50·87	0·40	13
18	0 46·65	0·56	0 22·64	0·90	0 2·03	0·34	0 5·21	0·56	0 29·48	0·92	0 51·27	0·37	12
19	0 46·09	0·59	0 21·74	0·89	0 1·69	0·30	0 5·77	0·58	0 30·40	0·92	0 51·64	0·34	11
20	0 45·50	0·61	0 20·85	0·88	0 1·39	0·28	0 6·35	0·61	0 31·32	0·92	0 51·98	0·31	10
21	0 44·89	0·63	0 19·97	0·88	0 1·11	0·24	0 6·96	0·63	0 32·24	0·92	0 52·29	0·27	9
22	0 44·26	0·65	0 19·09	0·87	0 0·87	0·22	0 7·59	0·65	0 33·16	0·90	0 52·56	0·25	8
23	0 43·61	0·68	0 18·22	0·86	0 0·65	0·19	0 8·24	0·68	0 34·06	0·89	0 52·81	0·22	7
24	0 42·93	0·70	0 17·36	0·85	0 0·46	0·15	0 8·92	0·70	0 34·95	0·89	0 53·03	0·18	6
25	0 42·23	0·72	0 16·51	0·84	0 0·31	0·13	0 9·62	0·72	0 35·84	0·87	0 53·21	0·15	5
26	0 41·51	0·73	0 15·67	0·82	0 0·18	0·09	0 10·34	0·74	0 36·71	0·86	0 53·36	0·11	4
27	0 40·78	0·76	0 14·85	0·81	0 0·09	0·06	0 11·08	0·75	0 37·57	0·85	0 53·47	0·08	3
28	0 40·02	0·77	0 14·04	0·80	0 0·03	0·03	0 11·83	0·78	0 38·42	0·84	0 53·55	0·05	2
29	0 39·25	0·79	0 13·24	0·78	0 0·00	0·00	0 12·61	0·79	0 39·26	0·82	0 53·60	0·02	1
30	0 38·46		0 12·46		0 0·00		0 13·40		0 40·08		0 53·62		0
	XIˢ		Xˢ		IXˢ		VIIIˢ		VIIˢ		VIˢ		

To be substituted for Burckhardt's Table XXX.

TABLE IV. Of the Moon's Equatorial Horizontal Parallax.

Argument:—The Argument of Anomaly from calculations of the Moon's Place by Burckhardt.

	0ˢ			Iˢ			IIˢ			IIIˢ			IVˢ			Vˢ			
°	′	″	diff. ″	′	″	diff. ″	′	″	diff. ″	′	″	diff. ″	′	″	diff. ″	′	″	diff. ″	°
0	59	9·01	0·03	58	38·10	2·00	57	18·70	3·15	55	40·69	3·23	54	12·84	2·51	53	13·97	1·33	30
1	59	8·98	0·11	58	36·10	2·06	57	15·55	3·17	55	37·46	3·21	54	10·33	2·47	53	12·64	1·29	29
2	59	8·87	0·18	58	34·04	2·12	57	12·38	3·18	55	34·25	3·20	54	7·86	2·44	53	11·35	1·25	28
3	59	8·69	0·25	58	31·92	2·17	57	9·20	3·20	55	31·05	3·18	54	5·42	2·40	53	10·10	1·20	27
4	59	8·44	0·33	58	29·75	2·22	57	6·00	3·22	55	27·87	3·17	54	3·02	2·36	53	8·90	1·16	26
5	59	8·11	0·39	58	27·53	2·27	57	2·78	3·23	55	24·70	3·15	54	0·66	2·33	53	7·74	1·11	25
6	59	7·72	0·46	58	25·26	2·32	56	59·55	3·24	55	21·55	3·13	53	58·33	2·30	53	6·63	1·07	24
7	59	7·26	0·53	58	22·94	2·36	56	56·31	3·26	55	18·42	3·12	53	56·03	2·26	53	5·56	1·02	23
8	59	6·73	0·60	58	20·58	2·42	56	53·05	3·26	55	15·30	3·10	53	53·77	2·22	53	4·54	0·98	22
9	59	6·13	0·68	58	18·16	2·46	56	49·79	3·28	55	12·20	3·08	53	51·55	2·19	53	3·56	0·94	21
10	59	5·45	0·74	58	15·70	2·50	56	46·51	3·28	55	9·12	3·06	53	49·36	2·14	53	2·62	0·89	20
11	59	4·71	0·81	58	13·20	2·55	56	43·23	3·29	55	6·06	3·03	53	47·22	2·11	53	1·73	0·85	19
12	59	3·90	0·88	58	10·65	2·59	56	39·94	3·30	55	3·03	3·01	53	45·11	2·08	53	0·88	0·80	18
13	59	3·02	0·95	58	8·06	2·63	56	36·64	3·31	55	0·02	2·99	53	43·03	2·03	53	0·08	0·75	17
14	59	2·07	1·02	58	5·43	2·67	56	33·33	3·30	54	57·03	2·97	53	41·00	1·99	52	59·33	0·71	16
15	59	1·05	1·08	58	2·76	2·72	56	30·03	3·31	54	54·06	2·94	53	39·01	1·96	52	58·62	0·67	15
16	58	59·97	1·15	58	0·04	2·75	56	26·72	3·31	54	51·12	2·92	53	37·05	1·92	52	57·95	0·62	14
17	58	58·82	1·22	57	57·29	2·78	56	23·41	3·31	54	48·20	2·89	53	35·13	1·87	52	57·33	0·57	13
18	58	57·60	1·29	57	54·51	2·82	56	20·10	3·31	54	45·31	2·86	53	33·26	1·84	52	56·76	0·53	12
19	58	56·31	1·34	57	51·69	2·85	56	16·79	3·31	54	42·45	2·84	53	31·42	1·79	52	56·23	0·48	11
20	58	54·97	1·41	57	48·84	2·89	56	13·48	3·31	54	39·61	2·81	53	29·63	1·75	52	55·75	0·44	10
21	58	53·56	1·48	57	45·95	2·92	56	10·17	3·30	54	36·80	2·78	53	27·88	1·72	52	55·31	0·39	9
22	58	52·08	1·53	57	43·03	2·95	56	6·87	3·30	54	34·02	2·76	53	26·16	1·67	52	54·92	0·34	8
23	58	50·55	1·60	57	40·08	2·98	56	3·57	3·30	54	31·26	2·72	53	24·49	1·63	52	54·58	0·30	7
24	58	48·95	1·66	57	37·10	3·00	56	0·27	3·28	54	28·54	2·70	53	22·86	1·59	52	54·28	0·25	6
25	58	47·29	1·72	57	34·10	3·03	55	56·99	3·28	54	25·84	2·66	53	21·27	1·54	52	54·03	0·21	5
26	58	45·57	1·78	57	31·07	3·06	55	53·71	3·27	54	23·18	2·64	53	19·73	1·51	52	53·82	0·16	4
27	58	43·79	1·84	57	28·01	3·08	55	50·44	3·26	54	20·54	2·60	53	18·22	1·46	52	53·66	0·12	3
28	58	41·95	1·89	57	24·93	3·11	55	47·18	3·25	54	17·94	2·57	53	16·76	1·42	52	53·54	0·07	2
29	58	40·06	1·96	57	21·82	3·12	55	43·93	3·24	54	15·37	2·53	53	15·34	1·37	52	53·47	0·02	1
30	58	38·10		57	18·70		55	40·69		54	12·84		53	13·97		52	53·45		0
	XIˢ			Xˢ			IXˢ			VIIIˢ			VIIˢ			VIˢ			

To be substituted for Burckhardt's Table XXXI.

TABLES

CONTAINING CORRECTIONS TO BE APPLIED TO THE VALUES OF THE MOON'S EQUATORIAL HORIZONTAL PARALLAX GIVEN IN THE NAUTICAL ALMANACS 1840—1855, IN ORDER TO MAKE THEM AGREE WITH THOSE CALCULATED FROM THE NEW TABLES.

Day of the Month	1840											
	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	″	″	″	″	″	″	″	″	″	″	″	″
1	+0·3	+0·9	+1·2	+2·9	+3·7	+3·8	+1·6	−0·3	−0·5	+0·9	+3·5	+4·3
2	0·3	1·4	2·1	4·3	4·9	3·9	1·2	0·7	0·3	1·5	4·1	4·7
3	0·3	2·3	3·2	5·8	6·0	3·8	1·2	0·6	−0·2	2·1	4·4	4·4
4	0·7	3·0	4·3	6·8	6·7	3·9	1·0	0·8	+0·3	2·5	4·5	4·2
5	1·0	3·3	5·3	7·6	6·7	3·5	0·6	0·8	0·5	2·7	4·4	4·0
6	1·2	3·8	6·0	7·6	6·3	2·9	+0·4	0·8	1·0	3·1	4·3	3·8
7	1·6	4·1	6·5	7·2	5·7	2·1	−0·1	0·6	1·2	3·0	3·9	3·9
8	2·0	4·1	6·7	6·0	4·4	1·0	0·5	−0·4	1·5	2·8	3·4	3·6
9	2·2	3·9	6·0	4·2	3·2	0·4	0·4	0·0	1·8	2·6	2·6	3·0
10	2·4	3·1	4·8	2·8	1·7	0·3	−0·3	+0·6	1·5	2·1	1·8	2·0
11	2·6	2·3	3·2	1·3	1·0	0·0	0·0	1·0	1·3	1·2	+0·6	+0·9
12	2·6	1·4	1·6	+0·4	0·4	0·2	+0·6	1·4	+0·7	+0·2	−0·4	0·0
13	2·4	0·6	+0·3	−0·2	0·0	0·7	1·2	1·3	−0·1	−0·7	1·1	−0·6
14	2·3	0·1	−0·4	0·2	0·3	1·2	1·7	1·1	0·5	1·2	1·2	0·8
15	2·1	0·3	0·7	0·3	0·4	1·4	1·8	0·8	0·6	1·2	1·0	−0·5
16	2·2	0·5	0·6	0·1	0·6	1·6	1·9	0·8	−0·1	−0·8	−0·6	+0·1
17	2·6	0·8	0·2	0·2	0·5	1·6	1·8	1·1	+0·8	+0·1	+0·1	0·3
18	2·7	0·9	0·0	0·2	0·6	1·5	1·9	1·7	2·0	0·8	0·6	0·5
19	2·6	0·9	0·2	0·5	0·3	1·7	2·1	2·7	3·2	1·6	1·3	0·8
20	2·4	0·7	0·4	0·5	0·3	1·9	2·7	3·9	3·9	2·4	1·4	0·9
21	1·9	0·4	0·6	0·8	0·4	2·1	3·5	4·8	4·3	2·7	1·4	1·0
22	1·6	0·3	0·7	0·9	0·4	2·5	4·2	5·3	4·3	2·8	1·7	1·3
23	1·2	0·1	0·9	0·8	0·5	2·8	5·0	5·3	4·0	2·9	1·5	1·4
24	1·2	+0·1	1·1	0·7	0·7	3·1	4·8	4·8	3·5	2·6	1·5	1·8
25	1·2	−0·1	1·1	0·7	0·9	3·0	4·8	4·0	2·9	2·4	1·6	2·1
26	1·0	0·0	1·0	0·4	0·8	3·1	4·0	3·4	2·4	1·9	1·6	2·7
27	0·7	+0·1	0·9	−0·2	1·1	2·9	3·4	2·6	1·6	1·5	2·0	3·0
28	0·7	0·2	0·5	+0·3	1·5	2·8	2·5	1·5	0·9	1·4	2·6	3·4
29	0·4	+0·4	−0·2	1·1	2·1	2·5	1·6	0·7	0·8	1·7	3·3	3·6
30	0·2		+0·4	+2·3	2·9	+2·3	0·6	+0·2	+0·7	2·2	+3·8	3·7
31	+0·6		+1·5		+3·5		+0·2	−0·2		+2·7		+3·5

Day of the Month	1841											
	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	″	″	″	″	″	″	″	″	″	″	″	″
1	+3·2	−0·3	−0·2	−3·0	−2·2	+0·3	+2·1	+3·4	+2·2	+0·2	−2·3	−1·7
2	2·5	0·6	1·4	3·7	2·0	0·9	2·8	3·3	1·8	−0·7	2·8	1·7
3	2·1	0·7	2·5	3·6	1·7	1·6	3·4	3·3	1·1	1·5	2·4	1·4
4	2·1	−0·1	3·1	3·0	0·8	2·1	3·4	2·9	0·6	1·8	1·8	−0·7
5	2·4	0·0	2·7	2·1	−0·2	2·4	3·3	2·5	0·5	1·4	−0·5	+0·3
6	2·4	+0·1	2·3	1·1	+0·6	2·5	3·2	2·1	0·6	−0·7	+0·9	1·2
7	2·4	0·4	1·6	−0·3	0·9	2·2	2·8	2·2	1·5	+0·9	2·0	2·0
8	2·0	0·4	−0·9	+0·4	1·4	2·2	2·8	2·2	2·6	2·2	2·7	2·5
9	1·2	0·6	0·0	1·0	1·6	2·2	2·6	2·7	3·5	3·4	3·4	3·3
10	0·7	0·8	+0·3	1·4	1·8	2·3	2·7	3·3	4·2	4·3	3·9	3·5
11	0·3	1·3	1·0	1·8	1·9	2·2	2·6	3·6	4·5	4·4	4·2	3·8
12	0·3	1·7	1·4	2·2	2·1	2·2	2·6	3·6	4·4	4·4	4·4	3·8
13	0·6	2·0	2·0	2·5	2·1	2·0	2·4	3·3	3·6	4·2	4·2	3·8
14	0·8	2·2	2·1	2·5	2·1	1·6	2·0	2·8	2·9	4·0	4·0	3·7
15	1·2	2·2	2·5	2·7	1·9	1·2	1·7	2·2	2·2	3·2	3·6	3·3
16	1·1	2·3	2·6	2·8	2·0	1·0	1·2	1·3	1·2	2·8	3·5	3·2
17	1·3	2·6	2·8	3·1	1·7	0·8	0·9	+0·3	0·7	2·4	3·6	3·4
18	1·3	2·9	3·0	3·4	2·0	1·0	+0·4	−0·7	0·2	2·4	3·6	3·3
19	1·4	3·5	3·5	3·8	2·1	1·0	−0·1	1·2	0·3	2·6	3·7	3·1
20	1·9	4·2	4·0	4·4	2·5	1·1	0·7	1·2	0·4	2·8	3·6	2·5
21	2·3	5·0	4·8	4·7	3·1	1·0	1·0	1·0	1·1	3·0	3·3	2·1
22	2·8	5·4	5·3	5·2	3·4	1·0	1·0	0·9	1·5	3·2	3·1	1·5
23	3·4	5·4	5·7	5·3	3·7	1·2	0·8	−0·2	1·7	3·1	2·7	0·9
24	3·8	5·2	5·9	5·3	3·5	0·8	0·5	+0·3	2·0	3·0	2·1	0·4
25	3·9	4·6	5·8	4·9	3·4	0·9	−0·1	0·8	2·2	2·8	1·3	0·1
26	3·9	3·7	5·3	4·3	2·8	0·8	+0·2	1·2	2·3	2·5	0·9	0·3
27	3·7	2·6	4·5	3·0	1·8	0·6	0·7	1·8	2·3	1·9	+0·5	0·6
28	3·2	+1·3	3·5	+1·4	0·7	0·7	1·3	2·2	2·0	1·3	−0·1	0·7
29	2·4		2·0	−0·1	+0·1	1·0	1·8	2·6	1·8	+0·6	0·8	0·7
30	1·6		+0·2	−1·4	−0·3	+1·7	2·4	2·7	+1·3	−0·2	−1·4	0·7
31	+0·7		−1·5		−0·2		+3·0	+2·7		−1·5		+0·6

1842

Day of the Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	″	″	″	″	″	″	′	″	″	″	″	″
1	+0·6	+3·5	+4·8	+5·2	+3·7	+1·4	+0·4	+1·5	+4·0	+6·3	+6·4	+4·7
2	0·8	3·9	5·2	4·9	3·3	1·2	0·4	1·4	3·9	6·0	5·3	3·9
3	1·5	4·0	5·4	4·7	3·0	0·7	+0·1	1·1	3·5	5·2	4·2	3·3
4	1·9	3·9	5·3	4·3	2·5	0·4	−0·1	0·9	2·7	4·3	3·3	2·6
5	2·5	3·9	5·1	3·7	1·9	0·3	0·3	0·8	2·2	3·0	2·4	1·9
6	2·8	3·4	4·6	3·1	1·8	0·3	0·4	0·5	1·2	1·8	1·7	1·5
7	3·0	3·2	4·1	2·8	1·7	0·2	0·4	0·2	0·5	1·1	1·5	1·4
8	3·1	3·0	3·5	2·9	1·9	0·5	−0·3	0·1	0·3	0·7	1·0	1·0
9	3·4	2·8	3·2	2·8	1·9	0·8	0·0	0·2	0·2	0·6	0·7	0·6
10	3·3	2·7	3·2	3·0	2·2	1·1	+0·5	0·5	0·3	0·5	0·5	+0·3
11	3·2	2·6	3·0	3·2	2·2	1·6	1·1	0·6	0·7	0·6	+0·2	0·0
12	3·0	2·6	3·0	3·0	2·6	2·3	1·6	1·3	1·1	0·5	−0·2	−0·4
13	2·9	2·2	3·0	2·9	2·7	2·6	2·1	1·7	1·3	0·4	0·4	0·4
14	2·6	1·7	2·7	2·5	2·6	2·9	2·4	2·3	1·4	0·2	0·5	0·4
15	2·5	1·3	2·2	1·8	2·4	2·7	2·6	2·4	1·4	+0·2	0·6	−0·2
16	2·2	+0·4	1·5	1·0	2·1	2·3	2·7	2·8	1·6	−0·1	0·6	+0·1
17	1·9	−0·8	+0·7	+0·1	1·4	1·9	3·0	2·9	1·5	0·1	0·5	0·5
18	1·3	1·7	−0·5	−0·9	+0·6	1·8	3·1	3·2	1·3	0·2	0·6	0·8
19	+0·6	2·6	1·5	1·8	−0·2	2·0	3·6	3·1	0·8	0·3	0·9	0·7
20	−0·4	3·1	2·7	2·4	0·4	2·5	3·9	2·9	+0·5	0·9	1·2	0·7
21	1·0	2·9	3·4	2·4	−0·3	2·9	4·1	2·1	−0·1	1·1	1·0	1·2
22	1·6	2·1	3·6	1·9	+0·5	3·4	3·7	1·6	0·5	1·1	−0·5	1·8
23	1·4	−0·7	3·2	−0·9	1·5	3·6	3·1	0·9	0·5	0·8	+0·5	2·6
24	−0·7	+0·8	2·1	+0·3	2·2	3·2	2·3	0·4	−0·2	−0·3	2·0	3·3
25	+0·3	2·1	−0·6	1·5	2·8	2·5	1·6	0·0	+0·5	+1·2	3·4	3·8
26	1·4	3·0	+0·7	2·5	3·0	1·9	0·8	0·4	1·7	2·8	4·5	3·8
27	2·1	3·5	1·9	3·2	2·9	1·3	0·5	0·7	3·2	4·4	5·4	3·5
28	2·5	+4·2	3·1	3·6	2·6	0·9	0·3	1·7	4·4	5·6	5·7	3·6
29	2·7		3·9	3·9	2·4	0·7	0·6	2·8	5·4	6·7	5·5	3·2
30	2·8		4·7	+3·9	2·0	+0·4	0·9	3·3	+6·2	7·0	+5·3	2·9
31	+3·0		+4·9		+1·8		+1·2	+3·9		+6·7		+2·5

1843

Day of the Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	″	″	″	″	″	″	″	″	″	″	″	″
1	+2·0	+0·2	−0·1	+0·8	+2·5	+3·4	+2·9	+2·2	+0·3	−1·1	−1·0	−0·4
2	1·6	0·0	0·0	1·0	2·6	3·6	3·5	2·4	0·8	0·6	0·6	−0·1
3	1·3	−0·2	0·0	1·0	2·6	3·6	3·6	2·5	1·2	−0·1	−0·2	+0·4
4	1·0	0·3	0·0	1·2	2·6	3·3	3·7	2·5	1·5	+0·5	+0·3	0·9
5	0·5	0·4	0·0	0·9	2·4	2·8	3·3	2·3	2·0	0·9	1·0	1·5
6	0·5	0·5	0·0	+0·3	1·6	2·4	2·9	2·3	2·4	1·3	1·2	2·1
7	+0·5	1·0	0·3	−0·3	0·8	1·7	2·5	2·5	2·6	1·3	1·3	2·4
8	0·0	1·2	0·8	0·9	+0·3	1·0	1·9	2·8	2·3	1·4	1·6	2·5
9	−0·5	1·1	1·1	1·3	−0·1	0·8	1·8	2·7	2·1	1·4	1·7	2·4
10	0·7	0·9	1·5	1·1	−0·1	1·0	2·0	2·3	1·5	1·3	1·9	2·4
11	0·5	−0·4	1·5	−0·5	+0·4	1·1	2·0	1·6	1·3	1·4	2·3	2·5
12	−0·5	+0·8	−1·0	+0·6	1·1	1·2	1·5	1·2	1·1	1·6	3·0	2·7
13	0·0	2·5	+0·1	2·2	1·6	1·0	1·0	0·6	1·4	2·4	3·7	3·1
14	+0·9	4·0	1·8	3·5	2·0	0·8	+0·5	0·3	1·8	3·3	4·6	3·5
15	2·0	5·5	3·7	4·2	2·4	0·4	0·0	0·3	2·9	4·5	5·3	3·8
16	2·8	6·2	5·4	4·5	2·5	0·2	−0·3	0·9	4·0	5·5	5·8	3·6
17	3·4	6·4	6·2	4·8	2·2	0·3	−0·3	1·5	4·9	6·3	6·1	3·4
18	3·7	6·4	6·5	4·8	2·3	+0·3	+0·1	2·2	5·5	6·7	5·8	2·9
19	4·1	6·3	6·5	4·7	2·3	−0·1	0·5	3·1	5·8	6·8	5·2	2·3
20	4·2	5·7	6·5	4·3	1·8	+0·1	0·8	3·6	5·5	6·2	4·3	1·4
21	4·5	5·2	6·2	4·0	1·4	0·3	1·4	3·6	4·9	5·4	3·1	0·7
22	4·4	4·3	5·5	3·0	1·1	0·6	1·7	3·4	3·9	4·4	1·7	+0·3
23	4·2	3·5	4·6	2·1	0·7	0·6	2·0	2·9	2·9	3·0	+0·3	−0·2
24	3·6	2·5	3·7	1·3	0·5	0·9	1·9	2·4	1·7	+1·4	−0·7	0·2
25	3·0	1·3	2·6	0·8	0·6	1·3	1·9	1·4	+0·4	−0·2	1·2	−0·2
26	2·3	0·8	1·6	0·6	1·0	1·7	1·5	0·8	−0·7	1·4	1·5	+0·1
27	1·8	0·4	0·8	0·8	1·4	2·1	1·2	+0·2	1·4	2·0	1·2	0·3
28	1·4	+0·1	0·3	1·1	2·0	2·3	1·2	−0·1	1·9	2·2	1·2	0·4
29	0·9		0·1	1·6	2·6	2·3	1·1	0·3	1·9	2·1	1·0	0·7
30	0·6		0·2	+2·0	3·1	+2·7	1·5	0·3	−1·6	1·8	−0·6	0·6
31	+0·4		+0·5		+3·2		+1·7	−0·1		−1·4		+0·9

Day of the Month	1844 Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	"	"	"	"	"	"	"	"	"	"	"	"
1	+1·3	+2·9	+2·1	+1·8	+1·1	+0·9	+0·4	+1·0	+2·4	+3·0	+3·0	+1·9
2	2·0	3·7	2·9	3·2	2·2	0·9	0·1	0·9	2·4	3·1	3·5	2·2
3	2·5	4·5	3·9	4·5	2·8	0·8	0·2	1·0	2·9	3·4	3·6	2·3
4	3·1	5·0	4·9	4·9	3·1	1·1	0·3	1·5	3·2	3·9	3·7	2·2
5	3·7	5·0	5·6	5·2	3·2	1·2	0·9	2·0	3·7	4·2	3·6	2·0
6	3·9	4·9	5·8	5·1	3·0	1·8	1·4	2·4	3·9	4·1	3·1	1·9
7	3·7	4·5	5·7	4·7	3·2	2·0	1·7	2·9	3·8	3·9	2·4	1·7
8	3·4	4·0	5·5	4·1	3·1	2·1	2·1	3·1	3·5	3·1	1·4	1·5
9	3·1	3·5	4·8	3·4	2·8	2·2	2·3	3·1	2·6	1·8	+0·8	1·3
10	2·7	2·7	3·8	2·5	2·4	2·4	2·5	3·1	1·7	+0·7	−0·1	0·6
11	2·8	1·9	2·8	1·7	1·9	2·4	2·9	2·7	+0·4	−0·6	0·9	0·3
12	2·5	+0·8	1·7	0·9	1·7	2·6	2·9	1·9	−0·9	1·7	1·6	0·2
13	2·1	−0·3	+0·4	0·4	1·4	2·9	3·1	1·2	1·9	2·6	1·9	0·2
14	1·6	1·1	−0·5	0·2	1·8	3·1	3·1	+0·1	2·7	3·2	1·9	0·6
15	0·9	1·5	1·1	0·4	2·2	3·5	2·8	−0·4	3·0	3·4	1·5	1·1
16	+0·2	1·6	1·2	0·6	2·3	3·6	2·3	1·2	2·9	2·8	0·9	1·3
17	−0·3	1·4	1·0	1·1	2·9	3·6	1·9	1·0	2·2	2·3	−0·2	1·9
18	0·5	0·8	−0·5	1·6	3·2	3·3	1·4	1·0	1·2	1·3	+0·8	2·2
19	0·5	−0·4	+0·2	2·1	3·4	3·0	1·2	−0·1	−0·4	−0·4	1·4	2·3
20	0·3	+0·2	0·6	2·3	3·3	2·6	1·4	+0·7	+0·8	+0·8	2·2	2·5
21	−0·1	0·7	1·1	2·6	3·1	2·4	1·5	1·3	1·6	1·9	2·7	2·8
22	+0·2	1·3	1·5	2·5	2·8	2·1	1·9	1·9	2·6	3·0	3·2	2·8
23	0·7	1·7	1·8	2·2	2·2	1·5	1·9	2·4	3·6	3·8	3·5	3·1
24	0·9	1·9	1·8	1·7	1·4	0·9	1·9	2·8	4·5	4·6	3·7	2·7
25	1·4	2·0	1·8	+0·9	+0·5	0·6	2·0	3·2	4·9	4·7	3·2	2·8
26	1·5	1·9	1·5	−0·1	−0·4	+0·1	1·8	3·9	4·8	4·6	3·0	2·2
27	1·7	1·7	0·9	0·7	0·9	−0·1	1·8	3·9	4·8	4·1	2·4	1·9
28	1·7	1·7	0·5	1·1	1·1	+0·2	2·2	3·8	4·1	3·6	2·1	1·5
29	1·8	+1·7	0·1	−0·9	0·7	0·6	1·8	3·5	3·6	3·1	1·8	1·1
30	1·8		0·2	0·0	−0·1	+0·3	1·7	3·1	+3·2	2·8	+1·9	0·6
31	+2·3		+0·7		+0·4		+1·5	+2·7		+2·9		+0·5

Day of the Month	1845 Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	"	"	"	"	"	"	"	"	"	"	"	"
1	+0·5	−0·5	+0·4	+0·7	+2·0	+2·8	+2·6	+0·8	−1·1	−2·0	−1·5	+1·0
2	0·4	0·8	0·0	−0·3	1·5	2·7	2·2	0·6	1·4	2·5	1·4	1·3
3	0·5	0·9	−0·6	0·6	1·4	2·4	1·9	+0·2	1·7	2·6	−0·9	1·7
4	0·7	0·8	1·2	−0·4	1·7	2·2	1·7	−0·2	1·7	2·3	+0·2	2·1
5	0·6	−0·2	1·4	+0·2	1·7	1·8	1·4	0·1	1·4	1·5	1·2	2·4
6	0·8	+0·4	1·0	0·9	1·7	1·8	1·1	−0·3	−0·5	−0·1	2·1	2·6
7	1·0	1·1	−0·1	1·5	1·8	1·4	0·7	0·0	+1·0	+1·5	3·0	2·7
8	1·3	2·0	+1·0	1·9	1·7	0·8	0·6	+0·6	2·4	2·8	3·5	2·9
9	1·4	2·8	1·8	2·3	1·7	0·8	0·7	1·7	3·8	4·2	4·1	2·7
10	1·7	3·6	2·6	2·4	1·1	+0·5	0·6	2·3	5·0	5·2	4·6	3·0
11	2·2	4·2	3·2	2·3	0·8	−0·1	1·1	3·1	5·9	5·9	5·0	2·8
12	2·7	4·1	3·8	1·8	+0·3	0·3	1·1	4·1	6·4	6·4	4·9	2·6
13	3·4	4·0	3·6	1·2	−0·2	0·9	1·3	4·7	6·7	6·6	4·7	2·1
14	3·3	3·5	3·3	+0·7	1·0	1·1	1·6	5·0	6·8	6·4	3·8	1·6
15	3·4	3·2	2·9	−0·2	1·5	1·1	1·9	5·5	6·2	5·8	2·9	0·8
16	3·3	2·6	2·3	0·8	2·1	0·9	2·2	5·4	5·3	4·9	1·8	+0·3
17	2·8	2·3	1·2	1·2	2·1	−0·1	3·0	4·8	4·3	3·6	0·8	−0·2
18	2·6	2·0	0·9	1·0	1·7	+1·0	3·4	3·8	3·0	2·1	0·2	0·3
19	2·5	2·2	0·8	−0·4	−0·7	2·1	3·3	3·0	1·8	1·2	0·0	−0·3
20	2·5	2·7	0·9	+0·7	+0·5	2·6	2·8	1·9	1·2	0·8	0·3	+0·2
21	2·7	3·4	1·5	1·8	1·8	2·9	2·3	1·3	0·6	0·5	0·5	0·7
22	3·0	3·5	2·3	3·0	2·9	2·8	1·9	1·0	0·4	0·5	1·1	1·0
23	3·1	3·4	3·2	3·8	3·8	2·8	1·7	0·9	0·6	0·7	1·3	1·1
24	2·9	3·2	3·5	4·3	4·1	3·0	1·8	1·0	0·6	0·8	1·3	1·6
25	2·5	2·5	3·8	4·7	4·4	3·3	2·0	1·1	0·5	1·1	1·3	1·8
26	1·9	2·1	3·8	4·5	4·5	3·6	2·3	1·0	0·4	0·9	1·2	2·1
27	1·3	1·5	3·9	4·6	4·6	3·6	2·5	0·8	+0·3	+0·5	1·0	2·4
28	0·5	+1·1	3·4	4·1	4·3	3·5	2·2	0·8	−0·2	0·0	0·8	3·1
29	+0·3		3·0	3·5	4·1	3·4	1·9	+0·3	0·8	−0·4	0·7	3·4
30	−0·1		2·3	+2·9	3·6	+2·8	1·5	−0·1	−1·4	1·0	+0·9	3·6
31	−0·5		+1·5		+3·2		+1·1	−0·6		−1·4		+3·8

Day of the Month	1846 Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	"	"	"	"	"	"	"	"	"	"	"	"
1	+3·9	+4·5	+4·1	+2·1	+0·4	−0·2	+1·4	+4·5	+6·7	+6·8	+4·7	+2·3
2	3·6	3·9	4·1	1·6	+0·1	0·0	1·8	4·6	6·6	6·5	3·9	2·2
3	3·7	3·3	3·8	1·0	−0·2	+0·1	2·2	4·7	6·3	5·8	2·8	1·6
4	3·2	2·4	3·1	+0·3	0·5	0·3	2·1	4·6	5·4	4·5	1·6	0·9
5	2·8	1·5	2·3	−0·1	0·6	0·6	2·4	4·8	4·2	2·8	+0·4	0·4
6	2·2	0·8	1·3	−0·4	−0·3	1·3	2·8	4·5	2·4	+1·0	−0·5	0·2
7	1·6	0·5	0·6	+0·1	+0·5	2·1	3·3	3·7	+0·5	−0·8	1·0	0·1
8	1·6	0·3	0·3	0·6	1·5	3·0	3·8	2·4	−1·0	1·7	1·2	0·5
9	1·2	0·6	0·2	1·7	2·7	3·9	3·7	+1·3	2·2	2·3	0·7	1·1
10	1·1	0·8	0·8	3·0	3·8	4·4	3·4	−0·2	2·4	2·4	−0·3	1·6
11	0·8	1·2	1·5	4·0	4·7	4·7	2·7	0·8	2·5	2·0	+0·5	2·3
12	0·6	1·3	2·4	4·7	5·2	4·8	2·1	1·2	2·1	1·5	1·1	2·8
13	0·5	1·4	3·1	5·0	5·5	4·6	1·6	0·8	1·6	0·9	1·5	3·0
14	+0·3	1·3	3·4	5·2	5·5	4·4	1·4	0·7	1·0	−0·3	1·8	3·2
15	0·0	1·4	3·6	4·8	5·5	4·0	1·5	−0·3	0·8	+0·2	2·0	3·0
16	−0·2	1·1	3·4	4·6	5·1	3·5	1·4	0·0	0·4	0·5	2·1	3·1
17	0·0	0·8	3·0	4·0	4·5	2·7	1·3	0·0	−0·1	0·5	2·0	2·8
18	+0·1	0·7	2·6	3·5	3·6	1·9	1·3	0·0	+0·1	0·9	1·7	2·7
19	0·2	0·5	2·0	2·2	2·4	1·2	1·0	+0·3	0·2	0·9	1·5	2·4
20	0·6	0·3	1·3	1·3	1·1	0·8	0·7	0·4	0·4	0·9	1·3	2·5
21	0·5	0·5	0·9	+0·1	+0·3	0·4	0·6	0·5	0·6	0·9	1·1	2·1
22	1·0	0·5	+0·3	−0·8	−0·4	0·3	0·7	0·7	0·9	1·0	1·1	2·2
23	1·2	0·8	−0·3	1·1	0·5	0·4	0·6	1·0	1·4	1·2	1·6	2·1
24	1·7	1·5	0·5	1·1	0·6	0·2	0·7	1·2	2·0	1·7	2·2	1·8
25	2·3	2·3	0·5	0·8	0·5	0·2	0·9	1·9	3·0	2·5	2·5	1·7
26	3·2	2·9	−0·2	−0·4	0·3	0·2	0·9	2·6	4·0	3·4	2·7	1·0
27	4·0	3·4	+0·6	+0·2	0·2	0·3	1·6	3·6	4·8	4·2	2·6	0·5
28	4·4	+3·7	1·1	0·5	0·2	0·5	2·0	4·5	5·8	4·8	2·6	0·4
29	4·7		1·6	0·7	0·1	0·8	2·9	5·4	6·6	5·4	2·4	0·4
30	4·6		2·1	+0·6	0·0	+1·2	3·2	6·1	+6·9	5·1	+2·4	0·6
31	+4·5		+2·4		−0·1		+4·1	+6·6		+5·0		+0·7

Day of the Month	1847 Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	"	"	"	"	"	"	"	"	"	"	"	"
1	+0·9	+1·1	+1·8	+4·4	+5·6	+3·9	+1·1	−1·4	−2·2	−1·5	+1·4	+3·0
2	0·9	1·4	2·3	4·5	5·3	3·1	0·9	1·0	−1·0	−0·2	2·1	3·3
3	0·7	1·5	2·7	4·5	4·6	2·4	1·0	−0·5	+0·2	+0·9	2·5	2·9
4	0·6	1·6	2·9	3·8	3·6	2·1	0·9	+0·3	1·4	1·9	2·7	2·6
5	0·8	1·7	2·8	3·1	2·8	1·5	0·9	0·9	2·3	2·4	2·6	2·4
6	1·1	1·8	2·7	2·1	1·8	0·9	0·9	1·6	3·0	2·9	2·5	2·1
7	1·5	1·9	2·3	+1·1	+0·7	+0·3	0·8	2·1	3·3	3·0	2·3	1·7
8	2·1	1·8	1·9	−0·1	−0·3	−0·5	0·6	2·5	3·6	2·9	1·8	1·2
9	2·5	1·6	1·1	1·1	1·2	1·1	0·7	3·0	3·6	2·7	1·4	0·8
10	2·6	1·3	+0·3	2·0	2·2	1·4	0·8	3·4	3·7	2·4	1·1	0·7
11	2·9	1·2	−0·3	2·5	2·8	1·4	1·3	3·5	3·4	2·5	1·1	0·5
12	2·9	1·2	0·8	2·5	2·9	1·0	1·9	3·5	3·4	2·3	1·1	0·6
13	2·9	1·6	1·2	2·2	2·7	−0·1	2·2	3·6	3·4	2·6	1·4	0·8
14	2·9	2·2	0·8	1·7	2·0	+0·5	2·4	3·5	3·7	2·8	1·8	1·0
15	3·0	3·1	−0·2	0·8	1·1	0·9	2·6	3·7	4·1	3·3	2·3	1·2
16	3·3	3·5	+0·8	−0·2	−0·3	1·6	2·8	4·0	4·8	3·7	2·6	1·3
17	3·6	4·0	1·6	+0·7	+0·6	2·2	3·0	4·2	4·6	4·0	2·9	1·4
18	3·7	4·0	2·3	1·7	1·5	2·5	3·4	4·3	4·7	4·1	3·0	1·8
19	3·8	3·7	2·6	2·3	2·2	2·8	3·5	4·5	4·6	4·2	3·3	1·9
20	3·6	3·3	3·0	2·8	2·4	2·9	3·6	4·3	4·0	4·1	3·4	2·1
21	3·2	2·7	3·3	2·7	2·4	2·8	3·6	3·7	3·6	3·9	3·2	2·4
22	2·7	2·1	3·2	2·6	2·4	2·9	3·3	3·0	3·1	3·4	2·5	2·5
23	1·8	1·3	2·6	2·2	2·3	2·9	3·0	2·4	2·3	2·8	1·7	2·3
24	0·9	0·6	2·2	1·9	2·4	2·9	2·6	1·8	+1·0	1·5	1·0	2·1
25	+0·2	0·3	1·6	2·1	2·8	3·0	2·2	+0·8	−0·2	+0·1	0·4	2·2
26	−0·1	0·3	1·2	2·6	3·3	3·1	1·8	−0·1	1·8	−1·2	0·6	2·4
27	0·5	0·7	1·3	3·2	3·9	3·0	1·2	1·1	3·1	2·0	0·9	2·6
28	−0·3	+1·0	1·6	4·0	4·4	2·8	+0·3	2·3	3·7	1·9	1·5	3·0
29	+0·3		2·2	4·9	4·8	2·3	−0·3	3·2	3·4	1·3	2·0	3·1
30	0·5		3·0	+5·3	5·0	+1·6	1·3	3·4	−2·4	−0·4	+2·7	3·2
31	+1·0		+3·8		+4·6		−1·6	−3·0		+0·7		+3·1

1848

Day of the Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	"	"	"	"	"	"	"	"	"	"	"	"
1	+2·8	+1·0	−0·5	−2·1	−1·1	+1·1	+2·5	+3·2	+2·4	+1·2	+1·4	+1·7
2	2·4	0·9	0·8	−0·5	+0·1	1·8	2·5	2·9	1·8	1·4	2·0	2·3
3	2·1	1·1	−0·3	+0·9	1·5	2·6	2·7	2·7	1·9	1·7	2·6	2·8
4	1·8	1·2	+0·5	2·6	2·7	3·3	3·1	2·3	1·6	1·9	3·0	3·2
5	1·7	1·9	1·8	3·8	3·7	4·0	3·3	1·9	1·6	2·3	3·5	3·5
6	1·2	2·2	3·1	5·0	4·7	4·1	2·9	1·5	1·5	2·4	3·8	3·7
7	1·1	2·8	4·1	5·7	5·2	4·2	2·6	1·4	1·4	2·6	3·8	4·3
8	0·9	3·2	5·0	5·9	5·3	3·8	2·1	1·0	1·2	2·5	4·0	4·7
9	1·2	3·4	5·3	5·7	5·3	3·2	1·7	0·7	0·9	2·3	4·1	4·8
10	1·2	3·0	5·2	5·3	4·7	2·4	1·1	0·2	0·5	2·2	4·1	4·6
11	1·3	2·6	4·5	4·4	4·0	2·0	0·8	+0·1	+0·3	1·7	3·3	3·7
12	1·5	1·6	3·6	3·7	3·3	1·8	0·8	0·0	−0·4	1·4	2·4	2·9
13	1·2	0·8	2·8	2·8	2·8	1·8	0·6	−0·5	0·9	+0·7	1·2	1·9
14	0·8	0·5	2·1	2·6	2·5	2·0	0·8	0·9	1·4	−0·2	0·3	1·5
15	0·2	0·4	1·6	2·4	2·6	2·2	0·7	1·4	1·9	1·1	0·1	1·3
16	0·4	0·9	1·7	2·7	2·8	2·1	+0·5	1·8	2·3	1·2	0·3	1·5
17	0·6	1·3	2·1	2·8	2·8	2·0	−0·1	2·2	1·8	1·0	1·0	1·6
18	1·1	2·1	2·4	2·7	2·6	1·5	0·4	2·0	−0·9	−0·1	1·4	1·6
19	1·8	2·4	2·4	2·4	2·5	0·8	0·8	−1·1	+0·5	+1·0	1·9	1·4
20	2·4	2·4	2·4	2·0	2·0	0·4	−0·5	+0·1	1·8	2·1	1·7	1·1
21	2·7	2·3	2·1	1·5	1·1	0·3	+0·1	1·8	3·0	2·6	1·8	0·8
22	2·9	1·9	1·7	+0·8	0·6	0·3	1·2	3·0	3·9	3·3	1·7	0·5
23	2·8	1·7	1·1	−0·2	+0·1	0·6	2·3	3·9	4·8	3·6	1·3	0·4
24	2·8	1·4	+0·6	0·8	−0·5	1·2	3·3	4·7	4·9	3·3	0·8	0·3
25	2·5	0·9	−0·3	1·5	0·8	1·7	3·7	5·2	4·8	2·9	0·3	0·4
26	2·5	0·5	0·8	2·2	1·1	1·8	4·0	5·4	4·5	2·3	0·3	0·5
27	2·5	+0·1	1·4	2·7	1·2	1·8	4·1	5·2	3·8	1·7	0·2	0·9
28	2·3	−0·3	2·1	2·9	1·0	1·8	4·2	5·0	3·0	1·3	0·2	1·1
29	2·0	−0·8	2·6	2·8	0·6	1·9	4·0	4·4	2·1	0·8	0·6	1·5
30	1·6		2·8	−2·2	−0·1	+2·2	4·0	3·7	+1·7	0·7	+1·2	1·8
31	+1·2		−2·6		+0·3		+3·8	+2·9		+0·8		+2·1

1849

Day of the Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	"	"	"	"	"	"	"	"	"	"	"	"
1	+2·3	+2·2	+4·2	+2·2	+1·8	+0·3	−0·1	+0·4	+1·4	+1·6	+1·1	+1·7
2	2·5	1·5	2·8	+0·9	0·8	0·1	+0·1	1·1	1·1	+0·8	+0·1	+0·7
3	2·6	1·0	1·5	−0·2	+0·1	0·3	0·6	1·4	0·9	0·0	−0·7	−0·2
4	2·6	0·8	+0·6	0·6	−0·2	0·5	1·2	1·4	0·3	−0·7	1·0	0·8
5	2·7	0·8	−0·3	0·7	−0·2	0·8	1·5	1·4	0·1	1·2	1·1	0·8
6	3·2	1·2	0·5	0·5	0·0	1·1	1·7	1·3	0·0	0·8	−0·6	−0·5
7	3·3	1·3	0·4	0·4	+0·1	1·0	1·8	1·3	0·6	−0·3	0·0	0·0
8	3·4	1·6	−0·1	0·4	+0·1	0·9	1·7	1·6	1·7	+0·7	+0·6	+0·6
9	3·3	1·4	+0·5	0·5	0·0	1·1	1·8	2·2	3·1	1·9	1·2	0·7
10	2·9	1·1	0·4	0·7	−0·2	1·0	2·0	3·2	4·2	2·6	1·8	0·9
11	2·1	0·9	+0·2	0·8	0·4	1·3	2·5	4·4	4·9	3·2	1·9	1·2
12	1·9	0·9	−0·1	1·0	0·4	1·4	3·1	5·3	5·1	3·6	2·2	1·2
13	1·6	0·8	0·2	1·1	0·3	1·8	3·9	5·5	5·1	3·6	2·2	1·3
14	1·6	0·7	0·2	1·0	−0·1	1·9	4·4	5·5	4·6	3·5	2·0	1·4
15	1·7	0·4	0·5	1·0	+0·1	2·1	4·4	5·0	3·7	3·1	1·7	1·6
16	1·6	0·4	0·5	0·7	0·2	2·2	4·3	4·2	3·0	2·6	1·7	2·0
17	1·2	0·3	0·5	0·7	0·2	2·4	3·9	3·2	2·1	1·9	1·7	2·2
18	0·5	0·4	0·4	−0·3	0·6	2·5	3·4	2·1	1·2	1·5	1·9	2·7
19	0·8	0·5	−0·3	+0·4	1·2	2·9	2·7	1·3	0·6	1·3	2·3	3·3
20	1·0	1·1	+0·1	1·2	2·0	3·0	2·0	+0·4	0·1	1·2	3·0	3·6
21	0·5	2·0	0·9	2·5	3·0	2·8	0·9	−0·3	0·2	1·6	3·5	3·9
22	0·7	2·9	1·7	3·8	4·0	2·6	0·5	0·6	0·4	2·2	3·9	4·0
23	1·3	3·8	3·0	5·2	4·7	2·2	+0·1	0·9	0·6	2·6	4·5	3·7
24	1·8	4·6	4·4	6·5	5·1	2·1	−0·1	0·8	1·0	3·2	4·6	3·1
25	1·9	5·1	5·5	7·1	5·2	1·9	0·4	0·6	1·4	3·7	4·2	3·0
26	2·8	5·4	6·5	7·5	4·8	1·4	0·6	0·6	1·6	3·9	4·0	2·8
27	2·9	5·5	7·2	6·9	4·4	1·0	0·7	−0·2	1·9	3·8	3·8	3·0
28	3·2	+5·2	7·4	6·2	3·7	+0·5	0·8	+0·1	2·0	3·5	3·6	3·1
29	3·3		6·8	4·8	2·8	0·0	0·8	0·5	2·2	3·3	3·3	2·9
30	3·2		5·7	+3·4	1·7	−0·3	0·5	0·8	+2·0	2·8	+2·6	2·4
31	+2·7		+4·2		+0·8		−0·1	+1·3		+2·1		+1·6

Day of the Month	1850 Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	″	″	″	″	″	″	″	″	″	″	″	″
1	+0·5	+0·9	+0·6	+1·4	+1·7	+2·3	+2·6	+3·2	+4·1	+4·5	+4·2	+3·7
2	0·2	1·2	1·1	1·8	2·2	2·4	2·7	3·2	4·3	4·6	4·4	4·0
3	0·1	1·5	1·7	2·2	2·3	2·4	2·5	3·2	4·0	4·4	4·3	4·1
4	0·3	1·6	1·9	2·5	2·4	2·1	2·3	2·8	3·4	3·7	4·1	3·9
5	0·6	1·6	2·3	2·5	2·6	2·0	1·9	2·1	2·3	3·0	3·6	3·8
6	0·9	1·9	2·4	2·9	2·4	1·7	1·4	1·5	1·3	2·4	3·5	3·6
7	0·9	1·8	2·7	3·0	2·6	1·5	1·0	+0·9	+0·5	1·8	3·1	3·5
8	1·0	2·1	3·0	3·3	2·7	1·1	0·7	−0·1	−0·4	1·4	3·1	3·7
9	1·1	2·4	3·1	4·0	2·7	1·3	0·5	0·9	0·6	1·4	3·4	3·6
10	1·2	3·0	3·5	4·4	3·1	1·7	+0·1	1·5	0·7	1·5	3·5	3·5
11	1·4	3·6	3·8	4·9	3·5	1·8	−0·3	1·6	−0·3	1·9	3·5	3·2
12	1·9	4·3	4·6	5·3	3·8	2·0	0·5	1·5	+0·3	2·2	3·4	2·7
13	2·2	4·7	5·3	5·7	4·1	1·9	0·6	0·9	0·8	2·4	3·1	2·1
14	2·7	4·8	5·7	5·7	4·2	2·0	0·4	−0·4	1·1	2·4	2·9	1·5
15	3·2	4·6	5·8	5·2	4·3	1·5	−0·2	+0·1	1·6	2·6	2·2	0·9
16	3·5	4·3	5·6	4·9	3·7	1·4	+0·1	0·7	2·0	2·6	2·0	0·7
17	3·5	3·5	5·2	4·1	3·1	0·9	0·4	1·1	2·2	2·3	1·2	0·4
18	3·5	2·5	4·3	3·0	1·9	0·3	0·8	1·7	2·3	2·1	+0·6	0·3
19	3·3	+1·3	3·2	+1·2	+0·7	0·4	1·2	2·2	2·4	1·7	−0·2	+0·2
20	2·9	0·0	1·5	−0·5	−0·6	0·7	1·7	2·7	2·0	1·0	0·9	−0·2
21	2·1	−1·2	+0·1	2·1	1·1	1·1	2·3	3·1	1·8	+0·1	1·8	0·2
22	1·3	1·9	−1·7	2·9	1·1	1·7	3·0	3·0	1·1	−0·8	2·3	0·3
23	0·6	2·2	3·0	2·9	0·8	2·3	3·4	2·8	+0·2	1·8	2·2	−0·2
24	0·3	2·0	3·6	2·5	−0·1	3·0	3·7	2·4	−0·6	2·3	1·8	+0·2
25	0·6	1·6	4·0	1·6	+0·9	3·3	3·4	1·9	0·7	2·1	−1·0	0·8
26	1·0	0·8	3·3	−0·6	1·4	3·1	3·2	1·4	−0·4	1·4	+0·3	1·4
27	1·2	−0·4	2·3	+0·1	1·8	2·9	2·8	1·3	+0·2	−0·1	1·2	2·1
28	1·3	+0·2	1·3	0·8	1·9	2·8	2·6	1·6	1·5	+1·4	2·0	2·4
29	1·0		−0·5	1·1	2·1	2·7	2·4	2·1	2·9	2·7	2·7	2·9
30	0·8		+0·1	+1·7	2·1	+2·6	2·7	3·2	+3·8	3·5	+3·2	3·2
31	+0·6		+0·8		+2·2		+2·7	+3·6		+4·0		+3·3

Day of the Month	1851 Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	″	″	″	″	″	″	″	″	″	″	″	″
1	+3·4	+2·9	+3·0	+3·0	+2·7	+1·5	+0·6	+0·5	+0·4	+0·5	+0·5	+0·4
2	3·4	2·7	3·1	3·1	2·8	1·9	1·3	0·9	1·0	0·8	0·2	−0·1
3	3·2	2·5	2·8	2·9	2·7	2·2	1·9	1·5	1·3	0·7	+0·1	0·5
4	3·1	2·1	2·6	2·7	2·6	2·5	2·2	1·9	1·8	0·6	−0·1	0·4
5	2·7	1·9	2·4	2·1	2·2	2·6	2·6	2·4	2·0	0·9	0·2	0·6
6	2·8	1·3	2·2	1·3	1·7	2·4	2·6	2·7	2·3	0·6	0·3	0·2
7	2·5	+0·6	1·5	+0·3	1·1	1·8	2·6	3·2	2·5	0·7	0·3	−0·1
8	1·5	−0·3	+0·5	−0·8	+0·5	1·3	2·7	3·4	2·4	0·4	0·5	+0·1
9	1·0	1·2	−0·6	1·8	−0·4	1·0	3·0	3·9	2·1	+0·4	0·6	0·2
10	0·6	2·0	1·7	2·8	1·3	1·2	3·3	3·8	1·8	0·0	0·9	0·1
11	+0·5	2·8	2·9	3·4	1·4	1·7	3·8	3·6	1·3	−0·5	1·3	0·0
12	−0·4	2·6	3·7	3·0	1·2	2·4	4·1	3·0	0·5	0·8	1·0	0·5
13	0·9	1·8	3·7	2·3	−0·3	3·0	4·0	2·3	0·0	1·0	−0·4	1·1
14	1·0	−0·4	3·1	−1·1	+0·7	3·2	3·3	1·5	0·0	−0·6	+0·6	2·1
15	−0·3	+0·8	1·9	+0·2	1·6	3·3	2·8	0·7	0·1	+0·1	1·9	3·2
16	+0·3	2·1	−0·2	1·5	2·3	2·8	1·8	0·2	0·8	1·3	3·4	3·9
17	1·1	3·0	+1·1	2·6	2·9	2·2	1·2	0·3	1·8	2·9	5·0	4·3
18	1·7	3·5	2·3	3·6	3·3	1·9	0·8	0·8	3·0	4·3	5·7	4·4
19	1·9	3·7	3·2	4·2	3·2	1·5	0·6	1·4	3·8	5·6	6·3	4·2
20	2·1	4·3	4·1	4·6	2·9	1·3	0·5	2·1	4·8	6·5	6·2	4·0
21	2·1	4·8	4·8	4·8	2·9	1·0	0·6	2·5	5·4	6·9	5·8	3·7
22	2·4	5·0	5·3	4·8	2·7	0·8	0·9	2·7	5·3	6·7	5·0	3·2
23	2·8	4·9	5·7	4·4	2·2	0·6	0·7	2·6	4·8	6·0	4·3	2·7
24	3·1	4·9	5·3	3·9	1·8	0·4	0·7	2·5	4·4	5·2	3·5	2·3
25	3·3	4·5	5·1	3·3	1·4	+0·1	+0·4	2·1	3·4	3·9	2·7	1·8
26	3·4	3·9	4·7	2·8	1·1	−0·3	0·0	1·6	2·2	2·9	2·0	1·3
27	3·4	3·4	4·0	2·5	1·0	0·3	−0·1	1·1	1·2	2·0	1·5	1·1
28	3·2	+3·2	3·5	2·4	1·0	0·4	0·3	0·5	0·4	1·4	1·3	0·7
29	3·1		3·2	2·5	1·1	−0·1	0·4	+0·1	0·3	1·0	0·9	0·5
30	3·1		3·4	+2·6	1·3	0·0	−0·1	−0·1	+0·2	1·0	+0·8	+0·2
31	+3·0		+3·1		+1·4		0·0	−0·2		+0·8		−0·1

Day of the Month	1852											
	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	″	″	″	″	″	″	″	″	″	″	″	″
1	−0·4	−0·7	−1·3	−0·5	+0·5	+1·3	+1·6	+1·6	+1·0	+1·3	+2·3	+2·6
2	0·5	−0·2	−1·0	+1·0	1·5	1·4	1·4	0·9	0·9	1·4	2·8	3·0
3	0·4	+0·7	+0·4	2·8	2·5	1·4	0·8	+0·2	0·8	2·1	4·0	3·5
4	−0·1	2·1	1·9	4·2	3·0	1·3	+0·3	−0·1	1·2	3·0	4·9	4·0
5	+0·6	3·4	4·0	5·3	3·4	1·1	0·0	−0·1	2·1	4·1	5·7	4·5
6	1·4	4·6	5·4	5·7	3·5	1·1	−0·4	+0·2	3·3	5·1	6·3	4·6
7	1·9	5·3	6·6	5·9	3·4	0·8	0·3	0·7	4·0	5·8	6·4	4·5
8	2·1	5·6	6·9	6·0	3·3	0·8	−0·1	1·7	4·6	6·3	6·0	3·9
9	2·4	5·8	7·0	5·6	3·2	0·6	+0·1	2·1	4·9	6·4	5·5	3·4
10	2·8	5·8	6·9	5·2	2·8	0·3	0·5	2·6	4·7	5·8	4·7	2·4
11	3·1	5·4	6·5	4·5	2·2	0·4	0·8	2·6	4·0	4·9	3·4	1·5
12	3·5	4·5	5·7	3·6	1·6	0·5	1·2	2·5	3·1	3·9	1·9	+0·5
13	4·0	4·0	4·8	2·6	1·2	0·6	1·2	2·2	2·3	2·6	+0·4	−0·1
14	3·8	3·4	3·8	1·7	0·7	0·8	1·4	2·0	1·2	+1·1	−0·7	0·4
15	3·5	2·4	2·7	0·9	0·5	1·3	1·5	1·2	+0·2	−0·4	1·4	0·6
16	3·2	1·4	1·6	0·5	0·9	1·8	1·6	0·5	−0·6	1·4	1·8	0·5
17	2·4	0·8	0·8	0·5	1·3	2·3	1·5	0·3	1·1	2·1	1·7	0·3
18	2·1	0·6	0·3	0·7	1·7	2·6	1·5	0·3	1·4	2·1	1·4	−0·1
19	1·8	0·3	0·1	1·1	2·3	2·7	1 6	0·3	1·1	2·1	1·3	+0·1
20	1·6	+0·1	0·0	1·3	2·7	3·0	1·9	0·6	0·8	1·6	1·0	0·2
21	1·2	−0·1	0·0	1·7	3·0	3·3	2·4	1·0	−0·3	1·1	0·7	0·4
22	1·0	0·2	0·0	2·1	3·1	3·5	2·9	1·3	+0·1	0·8	−0·3	0·9
23	0·8	0·3	0·4	2·0	3·2	3·6	3·1	1·4	0·6	−0·2	+0·3	1·2
24	0·5	0·3	0·4	1·7	2·9	3·6	3·2	1·9	1·4	+0·4	0·8	2·0
25	0·3	0·4	+0·4	1·5	2·7	3·3	3·0	2·2	1·7	0·7	1·2	2·6
26	+0·3	0·9	0·0	0·8	2·2	2·9	2·7	2·5	2·0	0·9	1·5	3·1
27	−0·1	1·1	−0·2	+0·2	1·6	2·2	2·6	2·8	2·1	1·4	1·9	3·3
28	0·4	1·2	0·8	−0·5	1·1	2·0	2·6	2·9	1·9	1·5	1·9	3·3
29	0·5	−1·6	1·3	0·6	0·8	1·6	2·7	2·5	1·5	1·4	2·1	2·9
30	1·0		1·6	−0·4	0·7	+1·7	2·5	2·2	+1·3	1·6	+2·3	2·8
31	−0·9		−1·4		+1·0		+2·0	+1·6		+1·8		+2·6

Day of the Month	1853											
	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	″	″	″	″	″	″	″	″	″	″	″	″
1	−0·4	−1·1	+0·5	−0·7	−0·4	−0·8	−0·2	+0·2	−1·0	−2·5	−3·8	−2·8
2	0·3	2·1	−0·6	1·7	1·1	0·6	0·0	0·0	2·0	3·6	4·9	3·2
3	0·4	2·9	1·7	2·5	1·5	0·4	+0·1	−0·4	3·2	5·0	5·5	3·4
4	0·8	3·8	3·0	3·1	1·8	−0·2	0·3	0·9	4·4	5·9	5·6	3·5
5	1·3	4·4	4·0	3·4	1·7	+0·3	0·5	1·5	5·0	6·4	5·5	3·2
6	1·8	4·5	4·7	3·3	1·3	0·5	0·5	2·3	5·4	6·3	4·7	2·8
7	2·3	4·4	4·6	2·9	0·8	0·8	+0·1	2·8	5·1	5·8	4·3	2·3
8	2·8	3·9	4·3	2·4	−0·4	0·9	−0·2	2·9	4·4	5·1	3·5	1·7
9	3·0	3·3	3·7	1·9	0·0	0·7	0·5	2·7	3·3	4·0	2·3	1·0
10	3·1	2·7	3·0	1·2	+0·2	0·6	0·6	2·3	2·5	3·1	1·2	0·3
11	3·0	2·1	2·4	0·9	0·4	+0·4	0·9	1·6	1·6	2·0	−0·5	−0·2
12	2·8	1·8	1·9	0·7	+0·2	−0·2	0·9	1·0	−0·8	−0·7	+0·2	+0·2
13	2·5	1·3	1·4	0·5	−0·1	0·8	1·0	0·8	+0·1	+0·2	0·9	0·5
14	2·2	1·0	1·0	0·7	0·6	1·6	1·0	0·4	0·7	1·3	1·3	0·6
15	1·8	0·9	0·8	1·0	1·5	2·4	1·2	0·3	1·4	2·0	1·3	0·5
16	1·5	1·0	0·7	1·7	2·5	3·0	1·5	−0·2	1·8	2·4	1·1	+0·1
17	1·5	1·0	1·0	2·6	3·4	3·2	1·6	+0·1	1·9	2·3	+0·5	−0·2
18	1·4	1·1	1·4	3 3	3·9	3·2	1·7	+0·1	1·7	1·7	0·0	0·6
19	1·3	0·8	1·6	3·4	4·2	3·0	1·6	0·0	1·2	1·2	−0·3	1·2
20	1·0	−0·4	1·8	3·3	3·7	2·8	1·8	−0·5	0·4	0·6	0·4	1·6
21	−0·6	+0·6	1·7	2·5	2·9	2·7	2·1	0·8	0·1	0·3	0·3	1·9
22	+0·1	1·5	1·3	1·2	2·1	2·7	2·3	1·0	0·0	0·5	0·3	1·9
23	0·8	2·1	−0·1	−0·1	1·4	2·8	2·6	1·2	0·2	0·6	0·2	1·9
24	1·4	2·6	+1·0	+0·7	1·2	2·8	2·4	1·0	0·7	1·0	0·1	1·9
25	1·9	2·7	1·9	1·2	1·4	2·5	2·2	0·7	1·1	1·2	0·3	2·1
26	1·9	2·5	2·5	1·2	1·1	2·0	1·7	−0·1	1·2	1·2	0·4	2·1
27	1·6	2·0	2·7	1·0	0·8	1·6	1·1	+0·4	1·1	0·9	1·0	2·1
28	1·2	+1·5	2·5	0·8	0·4	1·1	0·6	0·6	+0·9	+0·3	1·4	1·9
29	0·6		2·1	0·6	0·4	0·8	−0·2	0·7	0·0	−0·8	1·7	1·9
30	+0·1		1·4	+0·1	0·5	−0·6	+0·3	+0·5	−1·2	1·7	−2·2	1·8
31	−0·5		+0·4		−0·7		+0·2	−0·2		−2·8		−1·8

Day of the Month	1854											
	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	"	"	"	"	"	"	"	"	"	"	"	"
1	−1·8	+0·6	−0·1	+0·2	−0·8	−2·3	−2·3	−0·8	+2·0	+2·6	+2·6	+0·8
2	1·5	1·2	+0·7	+0·3	1·3	2·7	2·4	−0·2	2·6	3·3	2·9	1·1
3	0·9	1·5	1·3	−0·1	1·7	3·3	2·5	+0·2	3·2	3·7	2·9	0·9
4	0·4	1·5	1·6	0·7	2·4	3·9	2·5	0·5	3·4	4·0	2·6	+0·3
5	0·1	1·1	1·3	1·3	3·3	4·5	2·5	1·0	3·4	3·6	1·7	−0·6
6	0·1	0·6	0·7	2·1	4·0	4·7	2·4	1·3	2·8	3·1	+0·5	1·3
7	0·0	+0·2	+0·2	2·8	4·5	4·4	2·0	1·5	2·1	1·8	−0·7	2·2
8	0·1	−0·1	−0·5	3·1	4·5	3·6	1·2	1·1	+0·9	+0·7	1·7	2·6
9	0·1	0·4	1·1	2·9	4·3	2·6	0·7	+0·3	−0·2	−0·7	2·4	2·8
10	−0·1	−0·1	1·4	2·4	3·3	1·6	0·5	−0·4	1·2	1·6	2·5	2·7
11	0·0	+0·1	1·0	1·6	2·2	0·7	0·5	1·3	1·9	2·2	2·4	2·6
12	0·0	0·4	−0·5	−0·6	−1·2	−0·2	0·7	1·8	2·1	2·4	2·2	2·2
13	+0·2	0·5	+0·1	+0·2	0·0	0·0	1·1	1·7	2·2	2·3	1·9	1·8
14	+0·1	+0·4	0·6	0·8	+0·8	+0·3	1·0	1·6	2·0	2·1	1·7	1·5
15	−0·1	−0·1	1·0	1·1	1·2	0·6	0·6	1·3	2·1	2·1	1·7	1·4
16	0·5	0·9	0·8	1·4	1·4	0·9	−0·3	1·2	2·2	2·1	1·8	1·3
17	1·1	1·6	0·5	1·1	1·7	1·2	0·0	1·1	2·3	2·3	2·3	1·5
18	1·9	2·3	+0·1	1·0	1·8	1·2	+0·1	1·4	2·6	2·6	2·6	1·2
19	2·3	2·8	−0·3	+0·4	1·3	0·9	+0·1	1·7	2·8	3·2	2·9	0·9
20	2·8	3·2	0·8	−0·3	0·8	0·8	−0·1	2·1	3·4	3·6	3·3	0·5
21	3·1	3·7	1·4	1·1	+0·3	0·5	0·4	2·3	4·1	4·2	3·4	−0·3
22	3·4	4·2	2·3	1·8	−0·2	+0·1	0·8	2·7	4·5	4·7	3·3	0·0
23	3·4	4·4	3·4	2·2	0·3	−0·1	1·1	3·1	5·0	5·1	2·9	+0·1
24	3·6	4·6	3·9	2·2	0·4	0·4	1·4	3·5	5·2	5·0	2·2	0·3
25	3·4	4·2	4·3	1·9	0·6	0·7	1·9	3·7	4·9	4·4	1·5	0·2
26	3·1	3·3	3·9	1·4	0·7	1·1	2·1	3·8	4·3	3·3	0·8	+0·1
27	2·7	2·2	3·1	1·2	0·7	1·3	2·4	3·4	3·0	1·9	−0·4	−0·1
28	2·3	−1·2	2·3	0·8	0·8	1·7	2·4	2·8	−1·3	−0·7	+0·1	0·4
29	1·7		1·4	0·7	1·1	2·0	2·4	1·7	+0·3	+0·3	0·4	0·7
30	−0·9		0·7	−0·7	1·7	−2·1	1·9	−0·6	+1·6	1·4	+0·7	0·8
31	0·0		−0·1		−2·0		−1·4	+0·8		+1·9		−1·0

Day of the Month	1855											
	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
	"	"	"	"	"	"	"	"	"	"	"	"
1	−1·1	−2·0	−2·2	−0·1	+1·0	+1·5	+0·5	−2·3	−5·3	−5·6	−3·7	−1·7
2	1·3	2·2	1·7	+0·8	1·8	1·8	+0·4	2·9	5·0	5·3	3·1	1·1
3	1·8	2·0	1·0	1·2	2·2	1·9	0·0	3·2	4·5	4·6	2·5	0·5
4	2·3	2·0	0·7	1·4	2·4	1·9	−0·2	3·1	4·0	4·0	2·1	−0·2
5	2·7	2·2	0·4	1·4	2·3	1·9	0·2	2·7	3·4	3·5	1·7	+0·1
6	3·0	2·4	0·4	1·1	2·2	1·5	0·6	2·3	3·2	3·1	1·5	+0·1
7	3·2	2·4	0·6	+0·7	1·8	+0·9	0·7	2·1	2·7	2·7	1·2	0·0
8	3·2	2·5	0·8	0·0	1·3	0·0	0·9	2·0	2·7	2·4	1·1	−0·2
9	2·8	2·7	1·3	−0·5	+0·3	−0·9	1·2	2·2	2·3	2·3	1·4	0·6
10	2·6	2·8	1·7	1·5	−0·9	1·6	1·6	2·0	2·2	2·1	1·5	0·8
11	2·4	2·7	2·0	2·3	2·1	2·4	1·9	1·9	2·3	1·9	1·7	1·0
12	2·1	2·4	2·4	3·2	3·1	2·7	2·0	1·9	2·0	1·8	1·8	1·2
13	2·0	2·2	2·8	3·9	3·6	2·7	2·0	1·9	1·7	1·7	1·6	0·8
14	1·6	1·7	3·1	4·2	3·9	2·7	2·1	1·8	1·0	1·2	1·1	0·8
15	1·2	−0·7	3·1	3·9	3·8	2·7	2·2	1·5	−0·4	−0·6	−0·5	0·7
16	−0·6	+0·1	2·7	3·3	3·5	2·7	2·2	1·1	+0·6	+0·3	+0·1	1·0
17	+0·2	0·9	2·1	2·8	3·1	2·8	1·9	−0·5	1·6	1·3	0·4	1·4
18	1·1	1·2	1·3	2·4	2·9	2·7	1·7	+0·5	2·6	2·2	0·7	1·7
19	1·7	1·5	0·7	1·8	2·8	2·5	1·3	1·3	3·5	2·9	0·7	1·9
20	1·8	1·9	−0·1	1·4	2·6	2·4	0·8	2·1	3·9	3·2	0·6	1·9
21	1·9	1·7	+0·4	1·5	2·8	2·1	−0·2	2·8	4·1	3·3	0·4	1·8
22	1·9	1·3	0·7	1·8	2·8	2·0	+0·4	3·0	4·1	2·9	+0·1	1·8
23	1·6	+0·6	+0·4	2·2	3·0	1·9	0·8	3·1	3·9	2·4	−0·5	1·9
24	1·1	−0·3	−0·1	2·7	3·1	1·6	0·8	3·0	2·9	1·6	1·3	2·0
25	+0·5	1·0	0·9	3·0	3·1	1·4	0·7	2·8	+1·5	+0·2	2·0	2·3
26	−0·2	1·6	1·7	3·1	2·9	1·1	0·9	2·0	−0·1	−1·3	2·9	2·4
27	0·9	2·3	2·3	2·9	2·4	−0·7	1·0	+0·8	2·1	2·9	3·4	2·3
28	1·5	−2·5	2·7	2·3	1·7	0·0	0·8	−0·6	4·0	4·1	3·4	2·1
29	2·0		2·6	1·2	−0·7	+0·5	+0·5	2·4	5·1	4·4	3·0	1·4
30	2·2		2·1	−0·2	+0·2	+0·6	−0·3	4·1	−5·7	4·6	−2·4	0·9
31	−2·2		−1·2		+1·0		−1·4	−5·0		−4·3		−0·2

18.

ON THE CORRECTIONS TO BE APPLIED TO BURCKHARDT'S AND PLANA'S PARALLAX OF THE MOON, EXPRESSED IN TERMS OF THE MEAN ARGUMENTS.

[From the *Monthly Notices of the Royal Astronomical Society,* Vol. XIII. (1853).]

IN the Supplement to the *Nautical Almanac for* 1856, I have given new tables of the Moon's parallax, adapted to Burckhardt's form of the arguments. When the arguments have been already computed, these tables supply the most convenient means of finding the parallax, and they have, accordingly, been used in calculating the corrections to the *Nautical Almanac* Parallaxes since 1840, given in the paper above referred to.

When, however, Burckhardt's arguments are not previously known, it will be more simple to employ arguments increasing proportionably to the time, in order to calculate either the parallax itself immediately, or the correction to be applied to that found from Burckhardt's tables.

The following formulæ may be used for this purpose, the arguments being expressed in Damoiseau's notation.

The Moon's equatorial horizontal parallax, or, more strictly, the *sine* of that quantity converted into seconds of arc is equal to

$$\begin{aligned}
&3422''{\cdot}32 + 186''{\cdot}51 \cos x + 10''{\cdot}17 \cos 2x + 0''{\cdot}63 \cos 3x + 0''{\cdot}04 \cos 4x \\
&- 0''{\cdot}95 \cos t + 28''{\cdot}23 \cos 2t + 0''{\cdot}26 \cos 4t \\
&\qquad + 34''{\cdot}30 \cos (2t - x) + 0''{\cdot}37 \cos (4t - 2x) \\
&- 0''{\cdot}40 \cos z + 1''{\cdot}92 \cos (2t - z) + 1''{\cdot}45 \cos (2t - x - z) \\
&+ 1''{\cdot}16 \cos (x - z) - 0''{\cdot}71 \cos (2y - x) - 0''{\cdot}95 \cos (x + z) \\
&+ 0''{\cdot}01 \cos (x - t) - 0''{\cdot}31 \cos (2x - 2t) \\
&- 0''{\cdot}31 \cos (2t + z) - 0''{\cdot}23 \cos (2t - x + z) \\
&- 0''{\cdot}11 \cos (2y - 2t) + 0''{\cdot}22 \cos (2t + x - z) - 0''{\cdot}12 \cos (3x - 2t) \\
&+ 0''{\cdot}14 \cos (t + z) + 3''{\cdot}09 \cos (2t + x) + 0''{\cdot}60 \cos (4t - x) \\
&- 0''{\cdot}11 \cos (t + x) + 0''{\cdot}28 \cos (2t + 2x) \\
&+ 0''{\cdot}12 \cos (2x - z) - 0''{\cdot}10 \cos (2x + z) + 0''{\cdot}09 \cos (2t - 2z) \\
&- 0''{\cdot}09 \cos (2y + x - 2t) + 0''{\cdot}05 \cos (2t - x - 2z) \\
&+ 0''{\cdot}06 \cos (4t - x - z).
\end{aligned}$$

Also, the correction to be applied to the equatorial horizontal parallax found from Burckhardt's tables is

$$\begin{aligned}
&1''{\cdot}79 + 0''{\cdot}13 \cos x + 0''{\cdot}06 \cos 2x + 0''{\cdot}14 \cos 3x + 0''{\cdot}04 \cos 4x \\
&+ 0''{\cdot}06 \cos t + 0''{\cdot}05 \cos 2t - 0''{\cdot}29 \cos 3t + 0''{\cdot}17 \cos 4t \\
&\qquad - 0''{\cdot}18 \cos (2t - x) + 0''{\cdot}01 \cos (4t - 2x) \\
&+ 0''{\cdot}05 \cos z + 0''{\cdot}93 \cos (2t - z) + 1''{\cdot}15 \cos (2t - x - z) \\
&+ 0''{\cdot}07 \cos (x - z) - 1''{\cdot}50 \cos (2y - x) \\
&- 0''{\cdot}05 \cos (x - t) + 0''{\cdot}02 \cos (2x - 2t) \\
&- 0''{\cdot}90 \cos (2t + z) - 1''{\cdot}17 \cos (2t - x + z) \\
&- 0''{\cdot}12 \cos (2y - 2t) + 0''{\cdot}12 \cos (2t + x - z) + 0''{\cdot}10 \cos (3x - 2t) \\
&+ 0''{\cdot}14 \cos (t + z) + 0''{\cdot}09 \cos (2t - 2z) - 0''{\cdot}06 \cos (2y + x - 2t) \\
&+ 0''{\cdot}05 \cos (2t - x - 2z) + 0''{\cdot}07 \cos (2x + z - 2t) \\
&- 0''{\cdot}09 \cos (2t + x - 2y).
\end{aligned}$$

In both the above formulæ, quantities less than $0''{\cdot}05$ have been neglected, except where they can be included in the same table with larger terms.

When Burckhardt's parallax is known, it will be sufficient for ordinary purposes to calculate the correction to be applied to it, taking into account only the constant term, and the periodic terms depending on the arguments,

$$x,\ t,\ 2t - x,\ 2t - z,\ 2t + z,\ 2t - x - z,\ 2t - x + z,\ 2y - x,\ t + z.$$

If extreme accuracy be required, the parallax should be calculated afresh by means of the first of the above formulæ.

These formulæ, as well as my tables in the Supplement to the *Nautical Almanac for* 1856, give the value of the *sine* of the parallax, converted into seconds of arc, which is frequently more convenient for use than the parallax itself.

To find this latter quantity, we must add

$$0''{\cdot}16 + 0''{\cdot}03 \cos x.$$

Plana's formula for the parallax, as given in the Introduction to the Greenwich Lunar Reductions, also requires several corrections, partly in consequence of the developements not having been carried far enough, and partly from errors in the numerical conversion of the analytical expression.

The constant of parallax employed in the Lunar Reductions appears to be Henderson's, or $3421''{\cdot}8$; and the computed quantity is taken to be the parallax itself.

The correction to be applied to the parallax thus found, in order to make it agree with my determination, is given by the following formula:—

$$\begin{aligned}
&0''{\cdot}68 - 0''{\cdot}16 \cos x - 0''{\cdot}13 \cos 2x + 0''{\cdot}03 \cos 3x + 0''{\cdot}04 \cos 4x \\
&- 0''{\cdot}05 \cos t + 0''{\cdot}63 \cos 2t + 0''{\cdot}16 \cos 4t \\
&+ 0''{\cdot}40 \cos (2t - x) + 0''{\cdot}07 \cos (4t - 2x) \\
&- 0''{\cdot}28 \cos (2t - z) - 1''{\cdot}91 \cos (2y - x) + 0''{\cdot}29 \cos (2t + z) \\
&+ 0''{\cdot}01 \cos (x - t) - 0''{\cdot}51 \cos (2x - 2t) \\
&+ 0''{\cdot}09 \cos (2y - 2t) + 0''{\cdot}14 \cos (t + z) + 0''{\cdot}10 \cos (4t - x) \\
&- 0''{\cdot}11 \cos (t + x) - 0''{\cdot}02 \cos (2t + 2x) \\
&+ 0''{\cdot}12 \cos (2x - z) - 0''{\cdot}10 \cos (2x + z) + 0''{\cdot}09 \cos (2t - 2z) \\
&- 0''{\cdot}09 \cos (2y + x - 2t) + 0''{\cdot}05 \cos (2t - x - 2z) \\
&+ 0''{\cdot}06 \cos (4t - x - z).
\end{aligned}$$

As before, quantities less than $0''{\cdot}05$ have been neglected, except when they unite with larger terms.

In the American *Nautical Almanac for* 1855, recently published, Plana's formula for the parallax appears to have been employed; the constant, however, being slightly altered.

The following table, which Mr Farley has obligingly calculated at my request, shows the corrections to be applied to the parallaxes given in that work, in order to make them agree with those found from my tables.

Differences of Moon's Horizontal Parallax, as given in the American Nautical Almanac, from that obtained from my Tables.

1855. Greenwich Mean Noon of each Day.

Day of Mth.	Jan.	Feb.	Mar.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
	"	"	"	"	"	"	"	"	"	"	"	"
1	−1·5	−3·1	−2·2	−0·6	+0·8	+1·6	+1·4	−0·2	−2·3	−2·6	−3·3	−1·3
2	1·7	3·4	2·2	0·6	0·8	1·2	0·9	1·1	2·9	2·9	3·3	0·9
3	2·4	3·4	2·2	0·8	0·4	0·8	+0·2	1·9	3·6	3·2	3·1	−0·5
4	2·9	3·9	2·6	1·0	+0·2	0·3	−0·5	2·6	3·9	3·7	2·4	+0·1
5	3·3	4·0	2·8	1·2	0·0	+0·1	1·0	3·0	4·1	3·5	1·7	0·5
6	3·5	3·8	2·7	1·1	−0·2	−0·1	1·6	3·6	4·0	3·3	1·1	0·7
7	3·6	3·3	2·5	0·8	+0·2	0·2	1·8	3·8	3·8	2·8	−0·4	1·4
8	3·3	2·6	2·2	−0·2	0·5	0·4	2·1	3·8	3·3	2·0	+0·5	2·1
9	3·3	2·0	1·8	+0·5	0·8	0·6	2·4	3·8	2·8	1·2	1·0	2·2
10	3·0	1·5	1·1	1·1	0·8	0·7	2·5	3·4	2·3	−0·4	1·5	2·2
11	2·5	1·2	−0·4	1·3	0·7	0·9	2·5	3·1	1·8	+0·4	1·6	1·5
12	2·2	0·8	0·0	1·3	0·6	1·2	2·5	3·0	1·2	0·7	1·4	+0·6
13	1·8	−0·2	+0·4	1·4	0·5	1·5	2·6	2·7	0·6	0·8	0·8	−0·5
14	1·2	+0·4	0·8	1·2	+0·3	2·0	2·9	2·4	−0·2	0·9	+0·3	0·8
15	−0·4	1·1	1·2	1·2	0·0	2·4	2·9	2·0	+0·1	0·9	−0·2	1·1
16	+0·5	1·7	1·3	0·8	−0·6	2·7	3·0	1·7	0·3	0·8	0·6	1·7
17	1·1	1·8	1·5	+0·4	1·2	3·0	2·9	1·3	0·5	0·7	0·9	2·1
18	1·3	1·2	1·6	−0·3	1·9	3·0	2·6	0·8	0·5	0·4	1·4	2·3
19	1·2	+0·5	0·9	1·0	2·5	3·0	2·0	0·4	0·4	+0·1	1·7	2·6
20	0·7	−0·4	+0·5	1·9	2·8	2·7	1·8	0·4	0·1	−0·5	2·2	3·0
21	+0·1	1·2	−0·4	2·4	2·8	2·4	1·5	0·0	+0·1	1·0	2·4	3·1
22	−0·4	1·9	1·2	2·7	2·7	2·1	0·9	−0·1	−0·2	1·4	2·3	2·5
23	1·0	2·4	1·8	2·7	2·3	1·5	0·4	+0·3	0·1	1·3	2·2	2·0
24	1·3	2·8	1·9	2·5	2·0	1·0	−0·1	0·6	0·0	1·3	1·7	1·5
25	1·5	2·8	2·1	2·2	1·6	−0·2	+0·4	0·9	0·0	1·0	1·3	1·4
26	1·7	2·7	2·5	1·8	1·1	+0·6	1·1	1·3	0·1	1·1	0·8	1·2
27	1·8	2·5	2·5	1·2	−0·3	1·0	1·8	1·3	0·5	1·3	0·9	0·6
28	2·2	−2·5	2·1	0·7	+0·5	1·9	1·9	+0·7	0·9	1·5	1·3	0·9
29	2·3	...	1·7	−0·1	1·5	2·4	2·1	0·0	1·4	1·7	1·2	−0·5
30	2·6	...	1·4	+0·4	1·9	+2·1	1·6	−0·7	−1·9	2·4	−1·4	0·0
31	−2·8	...	−1·1	...	+1·9	...	+0·7	−1·6	...	−2·7	...	+0·8

The constant employed in the computations of the American *Nautical Almanac* does not appear to be mentioned in the Preface. It may, however, be determined in the following manner:—

The sum of the daily corrections given in the above table is $-370''\cdot4$. Now, I find that $-14''\cdot1$ of this is due to the corrections applied to the periodic terms, leaving $-356''\cdot3$ as the effect caused by the difference of the constants. This, divided by 365, gives $-0''\cdot98$ as the correction to be applied to the constant of the American *Nautical Almanac*, in order to make it agree with my own. Hence, this latter value being $3422''\cdot32$, it follows that the constant employed in the above work is $3423''\cdot30$.

19.

CONTINUATION OF TABLES I. AND III. OF DAMOISEAU'S TABLES OF JUPITER'S SATELLITES.

[From the *Nautical Almanac* (1881).]

DAMOISEAU'S Tables I. and III., the first containing the epochs of the Mean Conjunctions of Jupiter's Satellites and of the Arguments of the Inequalities, and the second containing the Inequalities due to the Perturbations of Jupiter, do not extend beyond the year 1880.

Hence it has now become necessary, in order to meet the requirements of the *Nautical Almanac*, that these Tables should be prolonged.

The perturbations of Jupiter employed by Damoiseau are those found from Bouvard's Tables of the planet, but since Le Verrier's new Tables are now used for computing the place of Jupiter given in the *Nautical Almanac*, it has been thought desirable to use the same Tables in order to form Table III. of Jupiter's Satellites.

The epochs of Mean Conjunction in Table I. are determined by the condition that when corrected for Le Verrier's value of the great inequality of Jupiter, they shall agree in the years 1750 and 1850 with the epochs given by Damoiseau when similarly corrected for Bouvard's value of the same inequality.

A further small correction has been applied to Damoiseau's epochs of Mean Conjunction of the first three Satellites, so as to make them exactly satisfy the theoretical relation known to exist between the mean longitudes of these Satellites, viz.:—

$$u_1 - 3u_2 + 2u_3 = 180°.$$

The long inequalities of the Satellites depending on the quantities $\Pi - \Lambda$ which enter into Table III. have been re-computed, the values given by Damoiseau being incorrect in consequence of his having omitted to take into account the modification of these inequalities caused by the mutual action of the first three Satellites.

Damoiseau's formulæ for the values of the mean arguments are not quite correctly derived from the fundamental data in p. iii of the Introduction. Small corrections have been accordingly applied to the arguments in order to make them consistent with the data and with each other.

These Tables have not been carried beyond the year 1890 as it is probable that new Tables of Jupiter's Satellites, founded on more accurate elements than those employed by Damoiseau, will appear before it becomes necessary to make the computations for the *Nautical Almanacs* of subsequent years.

FORMATION AND USE OF THE TABLES.

TABLE I.

Epochs of Mean Conjunction.

Le Verrier's value of the great inequality of Jupiter on January 1, 1750, exceeds Bouvard's value by 0°·00400. Hence, in order that the times of mean conjunction as affected by the great inequality may remain unaltered, we must increase Damoiseau's value of the excess of the mean longitude of each Satellite over the mean longitude of Jupiter by the above quantity.

If u_1, u_2, u_3 represent these excesses for the first three Satellites at any time, we know by the theory that

$$u_1 - 3u_2 + 2u_3 = 180° \text{ exactly.}$$

But if u_1, u_2, u_3 be derived for January 1, 1750, from the times given by Damoiseau for the first mean conjunctions in 1750, we find that

$$u_1 - 3\,u_2 + 2\,u_3 = 179^\circ{\cdot}98903.$$

Hence the theoretical condition will be satisfied if we increase u_1 and u_3 and diminish u_2 by one-sixth of the quantity 0·01097 or by 0·00183.

Therefore on the whole Damoiseau's values of u_1, u_2, u_3, u_4 for January 1, 1750, are increased respectively by

$$0^\circ{\cdot}00583,\quad 0^\circ{\cdot}00217,\quad 0^\circ{\cdot}00583, \text{ and } 0^\circ{\cdot}00400.$$

Hence the times of mean conjunction in January 1750 for the several Satellites will be diminished by

$$2^s{\cdot}48,\quad 1^s{\cdot}85,\quad 10^s{\cdot}03, \text{ and } 16^s{\cdot}09 \text{ respectively.}$$

Similarly on January 1, 1850, Le Verrier's value of the great inequality of Jupiter exceeds Bouvard's value by $0^\circ{\cdot}00435$.

At the same time the value of $u_1 - 3u_2 + 2u_3$ derived from Damoiseau's times for the first mean conjunctions in 1850 falls short of 180° by the quantity $0^\circ{\cdot}00834$, so that the theoretical condition will be satisfied by increasing u_1 and u_3 and diminishing u_2 by 0·00139.

Therefore, on the whole, Damoiseau's values of u_1, u_2, u_3, and u_4 for January 1, 1850, are increased by

$$0^\circ{\cdot}00574,\ 0^\circ{\cdot}00296,\ 0^\circ{\cdot}00574, \text{ and } 0^\circ{\cdot}00435 \text{ respectively.}$$

Hence the times of mean conjunction in January 1850 for the several Satellites will be diminished by

$$2^s{\cdot}44,\quad 2^s{\cdot}52,\quad 9^s{\cdot}87, \text{ and } 17^s{\cdot}48 \text{ respectively.}$$

The corresponding corrections to Damoiseau's times of mean conjunction in 1880 and 1890 will be as follows:

	Sat. I.	Sat. II.	Sat. III.	Sat. IV.
	s	s	s	s
1880	−2·42	−2·72	−9·82	−17·89
1890	−2·42	−2·79	−9·80	−18·03

The mean anomaly of Jupiter, which forms Arg^t 1 for each Satellite, has been found from Le Verrier's Tables of the planet. Corrections have been applied to Damoiseau's values of the other arguments so as to make them consistent with the data in p. iii of the Introduction.

These corrections for 1880 and 1890, expressed in decimals of a degree, are given in the following Table:

SAT. I.

	Argt 1	4	5	6	7	8	9	III.
1880	−·0011	−·007	−·002	−·224	·005	−·027	−·027	·003
1890	−·0034	−·008	−·001	−·241	·005	−·029	−·029	·003

SAT. II.

	Argt 1	2	3	4	5	6	7	8
1880	−·0011	·001	·001	·002	·003	·005	−·027	−·025
1890	−·0033	·001	·001	·002	·003	·006	−·029	−·027

	Argt I.	II.	III.	IV.
1880	·005	·023	·002	·003
1890	·005	·023	·002	·004

SAT. III.

	Argt 1	4	5	8	9	I.	IV.
1880	−·0010	·007	−·002	−·031	−·026	·112	·132
1890	−·0032	·009	−·002	−·033	−·028	·120	·142

SAT. IV.

	Argt 1	2	3	4	5	6	7
1880	−·0011	−·003	−·002	−·002	·007	·003	−·059
1890	−·0034	−·003	−·003	−·002	·007	·003	−·064

	Argt I.	II.	III.	IV.
1880	−·058	−·066	−·059	·003
1890	−·063	−·071	−·064	·002

Corrections of the Mean Arguments on account of the Perturbations of Jupiter.

J, which is the correction to be applied to Arg^t 1, is the great inequality of Jupiter, and is given in Table IX. of Le Verrier's Tables, where it is called δL.

The perturbations of longitude and of radius vector, which Damoiseau calls ϕ and ϕ_1, are to be found in the following manner:

Let v_0 denote the longitude and r_0 the radius vector, calculated from the mean longitude of Jupiter corrected by the secular term in Le Verrier's Table V., and the term δL in Table IX., and the longitude of the Perihelion corrected only by the secular term in Table V., employing the constant eccentricity

$$e = 0{\cdot}0480767, \quad \log e = 8{\cdot}6819346,$$

and the constant value of the mean distance

$$a = 5{\cdot}2025605, \quad \log a = 0{\cdot}7162171.$$

Also
$$E = 9916''{\cdot}53, \quad \log E = 3{\cdot}9963597,$$

$$\log \sqrt{\frac{1+e}{1-e}} = 0{\cdot}0208955.$$

These constant logarithms may be used when v_0 is found by passing through the eccentric anomaly. If we employ series and call A the mean anomaly we shall have

$$v_0 = L + \delta L + 19827''{\cdot}3 \sin A + 595''{\cdot}4 \sin 2A + 24''{\cdot}8 \sin 3A + 1''{\cdot}2 \sin 4A,$$

and then
$$r_0 = \frac{a(1-e^2)}{1+e\cos(v_0 - \varpi)},$$

where
$$\log a(1-e^2) = 0{\cdot}7152121.$$

Next, let v denote the longitude in the orbit and r the radius vector, as calculated from Le Verrier's Tables, and we shall have—

$$\phi = v - v_0,$$

$$\phi_1 = r - r_0.$$

The value thus found for ϕ is to be used instead of $\phi + \delta E$, and the value found for ϕ_1 is to be used instead of $\phi_1 + \delta r$, in Damoiseau's formula for Table III. of each Satellite. For J in the same formula, Le Verrier's value of δL in his Table IX. is to be used.

It should be remarked that in forming the complete arguments given in Table I. of each Satellite, wherever ϕ, or ϕ multiplied by a constant, occurs in Damoiseau's formula, $J + \phi$ must be substituted instead of ϕ.

The following corrections are special to each Satellite:

Satellite I.

Add to the formula for Table III.—

$$-4^s{\cdot}2 \sin(\Pi - \Lambda_{II}) + 0^s{\cdot}5 \sin(\Pi - \Lambda_{III}).$$

Satellite II.

Instead of the term $-9^s{\cdot}731 \sin(\Pi - \Lambda_{II})$ in Table III.,

Substitute the terms—

$$-2^s{\cdot}5 \sin(\Pi - \Lambda_{II}) - 1^s{\cdot}5 \sin(\Pi - \Lambda_{III}).$$

Satellite III.

Instead of the term $-5^s{\cdot}775 \sin(\Pi - \Lambda_{III})$ in Table III.,

Substitute the terms—

$$-0^s{\cdot}4 \sin(\Pi - \Lambda_{II}) - 5^s{\cdot}7 \sin(\Pi - \Lambda_{III}) + 0^s{\cdot}5 \sin(\Pi - \Lambda_{IV}).$$

Satellite IV.

In Table III. instead of the term $16^s{\cdot}694 \sin(\Pi - \Lambda_{IV})$,

Substitute the terms—

$$2^s{\cdot}0 \sin(\Pi - \Lambda_{III}) + 16^s{\cdot}9 \sin(\Pi - \Lambda_{IV}).$$

The terms which involve $\sin(5\bar{u} - 2u_0 - 34^\circ{\cdot}542)$ in Damoiseau's formulæ for Table III. of each Satellite are sufficiently accurate as they stand.

Damoiseau states that the values of J, ϕ, ϕ_1, δE and δr which he employs in the formation of the several Tables III., are taken from Bouvard's Tables of Jupiter. Mr Godward, however, has found that the numbers in these Tables do not accurately represent the results given by Damoiseau's formulæ. It may be remarked also that the value of Argument 1, or the mean anomaly of Jupiter, employed by Damoiseau slightly differs from Bouvard's value, except at the Epoch 1750, when the two coincide.

In order to be strictly accurate in forming the complete Arguments, the values of J and of $J+\phi$ corresponding to the actual time should be employed; whereas Table I. only includes the values of those quantities corresponding to the beginning of the year.

The following Table contains the yearly differences of the corrections thus applied to the several mean Arguments, and the correction of any Argument formed from Tables I. and II. will be found with sufficient accuracy by multiplying the corresponding value of Δ taken from this Table by the Fraction of the year.

	All the Satellites.		Sat^s I., II.	Sat. I.
	Arg^t 1.	Arg^t 3.	Arg^t 4.	Arg^t 5.
	Δ	Δ	Δ	Δ
1880	°	°	°	°
1881	−·0013	−·049	·024	·036
1882	−·0014	−·073	·037	·055
1883	−·0013	−·060	·030	·045
1884	−·0014	−·026	·013	·020
1885	−·0014	·007	−·003	−·005
1886	−·0014	·030	−·015	−·023
1887	−·0014	·042	−·021	−·031
1888	−·0014	·042	−·021	−·032
1889	−·0014	·032	−·016	−·024
1890	−·0015	·010	−·005	−·008

	Satellite III.		Satellite IV.		Sat^s. I., III., IV. Arg^{ts} 6, 7, Sat. II, Arg^{ts} 5, 6 and all the Satellites
	Arg^t 4.	Arg^t 5.	Arg^t 4.	Arg^t 5.	Arg^{ts} I., II., III., IV.
	Δ	Δ	Δ	Δ	Δ
1880	°	°	°	°	°
1881	·049	·028	·065	·180	·049
1882	·074	·042	·098	·272	·073
1883	·061	·034	·080	·222	·060
1884	·027	·015	·035	·098	·026
1885	−·007	−·004	−·009	−·026	−·007
1886	−·031	−·017	−·041	−·113	−·030
1887	−·042	−·024	−·056	−·155	−·042
1888	−·043	−·024	−·056	−·156	−·042
1889	−·032	−·018	−·042	−·117	−·032
1890	−·011	−·006	−·014	−·039	−·010

Suite de la TABLE I.

PREMIER SATELLITE DE JUPITER,

Époques des Conjonctions Moyennes et des Argumens des inégalités

Années	Conjonctions moyennes: Jours et parties du jour	Conjonctions moyennes: Fraction de l'année	1	2	3	4	5	6	7	8	9	I	II	III
	Janv. h ′ ″		s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °
1880 B	1 18 8 12·5	0·002	11 8·762	11 29·91	9 12·32	4 24·04	4 6·1	1 20·1	2 12·6	7 28·0	7 5·6	1 0·34	0 11·9	4 22·2
1881	2 2 48 13·2	0·003	0 9·201	0 0·99	8 12·93	3 24·75	2 22·2	2 18·0	3 12·3	5 1·5	4 7·2	2 0·83	1 24·5	5 25·3
1882	1 16 59 37·9	0·002	1 9·491	0 0·34	7 11·90	8 24·74	4 7·2	3 15·8	4 12·0	2 3·8	1 7·5	3 1·20	3 6·9	6 28·2
1883	1 7 11 2·7	0·001	2 9·781	11 29·68	6 10·89	1 24·73	5 22·2	4 13·5	5 11·7	11 6·0	10 7·8	4 1·55	4 19·3	8 1·1
1884 B	2 15 51 3·4	0·004	3 10·219	0 0·76	5 11·51	0 25·43	4 8·2	5 11·4	6 11·4	8 9·5	7 9·4	5 2·02	6 1·9	9 4·1
1885	1 6 2 28·1	0·001	4 10·511	0 0·10	4 10·57	5 25·38	5 23·1	6 9·1	7 11·0	5 11·7	4 9·7	6 2·31	7 14·3	10 7·0
1886	2 14 42 28·8	0·004	5 10·949	0 1·19	3 11·25	4 26·05	4 9·1	7 6·9	8 10·7	2 15·3	1 11·4	7 2·72	8 26·8	11 9·9
1887	2 4 53 53·6	0·003	6 11·239	0 0·53	2 10·34	9 25·98	5 24·0	8 4·5	9 10·3	11 17·5	10 11·7	8 2·97	10 9·1	0 12·7
1888 B	1 19 5 18·3	0·002	7 11·531	11 29·87	1 9·43	2 25·92	7 8·9	9 2·2	10 9·9	8 19·7	7 12·0	9 3·22	11 21·4	1 15·5
1889	2 3 45 19·0	0·003	8 11·968	0 0·96	0 10·11	1 26·59	5 24·9	10 0·0	11 9·6	5 23·3	4 13·6	10 3·63	1 3·9	2 18·5
1890	1 17 56 43·8	0·002	9 12·259	0 0·30	11 9·17	6 26·54	7 9·9	10 27·6	0 9·2	2 25·5	1 13·9	11 3·92	2 16·3	3 21·3

Suite de la TABLE III.

Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.
	′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″
1880·0	0 53·5	+0·9	1881·5	1 26·3	+3·1	1883·0	2 7·0	+1·9	1884·5	2 19·0	−0·3	1886·0	2 3·8	−1·6	1887·5	1 36·5	−1·8	1889·0	1 13·3	−1·0
·1	0 54·4	1·2	·6	1 29·4	3·1	·1	2 8·9	1·7	·6	2 18·7	0·4	·1	2 2·2	1·7	·6	1 34·7	1·9	·1	1 12·3	0·9
·2	0 55·6	1·3	·7	1 32·5	3·1	·2	2 10·6	1·6	·7	2 18·3	0·5	·2	2 0·5	1·7	·7	1 32·8	1·8	·2	1 11·4	0·9
·3	0 56·9	1·6	·8	1 35·6	3·0	·3	2 12·2	1·4	·8	2 17·8	0·7	·3	1 58·8	1·8	·8	1 31·0	1·7	·3	1 10·5	0·7
·4	0 58·5	1·8	·9	1 38·6	3·1	·4	2 13·6	1·2	·9	2 17·1	0·8	·4	1 57·0	1·8	·9	1 29·3	1·7	·4	1 9·8	0·6
·5	1 0·3	2·0	1882·0	1 41·7	2·9	·5	2 14·8	1·1	1885·0	2 16·3	0·8	·5	1 55·2	1·8	1888·0	1 27·6	1·8	·5	1 9·2	0·5
·6	1 2·3	2·1	·1	1 44·6	2·9	·6	2 15·9	0·9	·1	2 15·5	1·0	·6	1 53·4	1·9	·1	1 25·8	1·6	·6	1 8·7	0·4
·7	1 4·4	2·3	·2	1 47·5	2·8	·7	2 16·8	0·8	·2	2 14·5	1·0	·7	1 51·5	1·8	·2	1 24·2	1·6	·7	1 8·3	0·3
·8	1 6·7	2·4	·3	1 50·3	2·8	·8	2 17·6	0·6	·3	2 13·5	1·2	·8	1 49·7	1·9	·3	1 22·6	1·5	·8	1 8·0	0·1
·9	1 9·1	2·6	·4	1 53·1	2·6	·9	2 18·2	0·5	·4	2 12·3	1·2	·9	1 47·8	1·9	·4	1 21·1	1·5	·9	1 7·9	−0·1
1881·0	1 11·7	2·8	·5	1 55·7	2·5	1884·0	2 18·7	0·4	·5	2 11·1	1·4	1887·0	1 45·9	1·9	·5	1 19·6	1·4	1890·0	1 7·8	+0·1
·1	1 14·5	2·8	·6	1 58·2	2·4	·1	2 19·1	+0·2	·6	2 9·7	1·4	·1	1 44·0	1·8	·6	1 18·2	1·4	·1	1 7·9	
·2	1 17·3	3·0	·7	2 0·6	2·3	·2	2 19·3	0·0	·7	2 8·3	1·4	·2	1 42·2	1·9	·7	1 16·8	1·2			
·3	1 20·3	3·0	·8	2 2·9	2·1	·3	2 19·3	−0·1	·8	2 6·9	1·5	·3	1 40·3	1·9	·8	1 15·6	1·2			
·4	1 23·3	+3·0	·9	2 5·0	+2·0	·4	2 19·2	−0·2	·9	2 5·4	−1·6	·4	1 38·4	−1·9	·9	1 14·4	−1·1			

Suite de la Table I.

DEUXIÈME SATELLITE DE JUPITER,

Époques des Conjonctions Moyennes et des Argumens des inégalités

Années	Conjonctions Moyennes: Jours et parties du jour	Conjonctions Moyennes: Fraction de l'année	I	2	3	4	5	6	7	8	I	II	III	IV
	Janv. h ′ ″		s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °
1880 B	3 4 0 58·3	0·006	11 8·883	0 1·34	9 13·64	1 24·63	1 20·2	2 12·7	7 29·1	7 6·6	1 0·46	0 12·07	4 22·4	5 5·4
1881	3 5 44 13·0	0·006	0 9·296	0 2·14	8 13·97	0 25·22	2 18·1	3 12·4	5 2·4	·4 8·0	2 0·93	1 24·64	5 25·4	6 6·6
1882	4 7 27 27·8	0·009	1 9·710	0 2·94	7 14·29	11 25·82	3 16·0	4 12·2	2 5·7	1 9·4	3 1·42	3 7·24	6 28·4	7 7·8
1883	1 19 52 48·7	0·002	2 9·829	0 0·23	6 11·41	4 24·96	4 13·5	5 11·7	11 6·4	10 8·2	4 1·60	4 19·40	8 1·1	8 8·6
1884 B	2 21 36 3·5	0·005	3 10·242	0 1·04	5 11·77	3 25·54	5 11·4	6 11·5	8 9·7	7 9·6	5 2·05	6 1·95	9 4·1	9 9·8
1885	2 23 19 18·2	0·005	4 10·657	0 1·84	4 12·16	2 26·10	6 9·2	7 11·2	5 13·1	4 11·0	6 2·46	7 14·47	10 7·1	10 10·9
1886	4 1 2 32·9	0·008	5 11·070	0 2·64	3 12·58	1 26·65	7 7·0	8 10·9	2 16·4	1 12·5	7 2·84	8 26·96	11 10·1	11 12·0
1887	1 13 27 53·9	0·001	6 11·189	11 29·93	2 9·80	6 25·73	8 4·5	9 10·3	11 17·0	10 11·2	8 2·92	10 9·03	0 12·7	0 12·7
1888 B	2 15 11 8·6	0·004	7 11·603	0 0·73	1 10·22	5 26·27	9 2·2	10 9·9	8 20·4	7 12·7	9 3·30	11 21·51	1 15·6	1 13·8
1889	2 16 54 23·3	0·005	8 12·016	0 1·53	0 10·64	4 26·82	10 0·0	11 9·6	5 23·7	4 14·1	10 3·68	1 4·00	2 18·6	2 14·9
1890	3 18 37 38·1	0·008	9 12·430	0 2·33	11 11·04	3 27·38	10 27·8	0 9·3	2 27·0	1 15·5	11 4·09	2 16·52	3 21·5	3 16·0

Suite de la Table III.

Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.
	′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	′
1880·0	1 45·0	+2·2	1881·5	2 54·6	+6·3	1883·0	4 17·4	+3·7	1884·5	4 41·3	−0·6	1886·0	4 10·6	−3·2	1887·5	3 16·5	−3·6	1889·0	2 31·2	−1·9
·1	1 47·2	2·6	·6	3 0·9	6·4	·1	4 21·1	3·5	·6	4 40·7	0·9	·1	4 7·4	3·4	·6	3 12·9	3·6	·1	2 29·3	1·7
·2	1 49·8	3·0	·7	3 7·3	6·3	·2	4 24·6	3·1	·7	4 39·8	1·1	·2	4 4·0	3·4	·7	3 9·3	3·5	·2	2 27·6	1·5
·3	1 52·8	3·5	·8	3 13·6	6·2	·3	4 27·7	2·8	·8	4 38·7	1·3	·3	4 0·6	3·5	·8	3 5·8	3·5	·3	2 26·1	1·4
·4	1 56·3	3·8	·9	3 19·8	6·2	·4	4 30·5	2·5	·9	4 37·4	1·6	·4	3 57·1	3·6	·9	3 2·3	3·4	·4	2 24·7	1·2
·5	2 0·1	4·2	1882·0	3 26·0	6·1	·5	4 33·0	2·2	1885·0	4 35·8	1·7	·5	3 53·5	3·7	1888·0	2 58·9	3·3	·5	2 23·5	0·9
·6	2 4·3	4·5	·1	3 32·1	5·9	·6	4 35·2	1·9	·1	4 34·1	2·0	·6	3 49·8	3·6	·1	2 55·6	3·2	·6	2 22·6	0·7
·7	2 8·8	4·9	·2	3 38·0	5·7	·7	4 37·1	1·5	·2	4 32·1	2·1	·7	3 46·2	3·7	·2	2 52·4	3·1	·7	2 21·9	0·4
·8	2 13·7	5·2	·3	3 43·7	5·6	·8	4 38·6	1·2	·3	4 30·0	2·3	·8	3 42·5	3·7	·3	2 49·3	3·0	·8	2 21·5	−0·2
·9	2 18·9	5·4	·4	3 49·3	5·3	·9	4 39·8	1·0	·4	4 27·7	2·5	·9	3 38·8	3·7	·4	2 46·3	2·9	·9	2 21·3	+0·1
1881·0	2 24·3	5·7	·5	3 54·6	5·1	1884·0	4 40·8	0·6	·5	4 25·2	2·7	1887·0	3 35·1	3·7	·5	2 43·4	2·7	1890·0	2 21·4	+0·3
·1	2 30·0	6·0	·6	3 59·7	4·8	·1	4 41·4	0·4	·6	4 22·5	2·9	·1	3 31·4	3·7	·6	2 40·7	2·6	·1	2 21·7	
·2	2 36·0	6·1	·7	4 4·5	4·6	·2	4 41·8	+0·1	·7	4 19·6	2·9	·2	3 27·7	3·8	·7	2 38·1	2·5			
·3	2 42·1	6·2	·8	4 9·1	4·3	·3	4 41·9	−0·2	·8	4 16·7	3·0	·3	3 23·9	3·7	·8	2 35·6	2·3			
·4	2 48·3	+6·3	·9	4 13·4	+4·0	·4	4 41·7	−0·4	·9	4 13·7	−3·1	·4	3 20·2	−3·7	·9	2 33·3	−2·1			

TROISIÈME SATELLITE DE JUPITER,

Époques des Conjonctions Moyennes et des Argumens des inégalités

Suite de la Table I.

Années	Conjonctions moyennes: Jours et parties du jour	Conjonctions moyennes: Fraction de l'année	1	2	3	4	5	6	7	8	9	I	II	III	IV
	Janv. h ′ ″		s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °
1880 B	4 6 13 1·2	0·009	11 8·978	0 2·46	9 14·67	3 20·2	9 17·6	1 20·3	2 12·8	7 29·9	7 7·5	1 0·56	4 22·46	5 5·5	0 12·2
1881	3 17 52 29·8	0·008	0 9·343	0 2·69	8 14·47	1 20·9	11 24·1	2 18·1	3 12·5	5 2·8	4 8·5	2 0·97	5 25·44	6 6·6	1 24·7
1882	4 5 31 58·3	0·009	1 9·707	0 2·91	7 14·26	11 21·6	2 0·6	3 16·0	4 12·2	2 5·7	1 9·4	3 1·41	6 28·43	7 7·8	3 7·2
1883	4 17 11 26·9	0·010	2 10·073	0 3·13	6 14·06	9 22·3	4 7·1	4 13·8	5 12·0	11 8·6	10 10·4	4 1·84	8 1·42	8 8·9	4 19·7
1884 B	5 4 50 55·5	0·012	3 10·438	0 3·35	5 13·89	7 23·0	6 13·5	5 11·6	6 11·7	8 11·5	7 11·4	5 2·24	9 4·37	9 10·0	6 2·2
1885	4 16 30 24·1	0·010	4 10·803	0 3·58	4 13·75	5 23·6	8 20·0	6 9·3	7 11·3	5 14·4	4 12·4	6 2·60	10 7·29	10 11·0	7 14·7
1886	5 4 9 52·6	0·011	5 11·168	0 3·80	3 13·64	3 24·2	10 26·4	7 7·0	8 11·0	2 17·3	1 13·3	7 2·94	11 10·18	11 12·1	8 27·1
1887	5 15 49 21·2	0·013	6 11·534	0 4·02	2 13·54	1 24·8	1 2·8	8 4·8	9 10·6	11 20·1	10 14·3	8 3·27	0 13·06	0 13·1	10 9·5
1888 B	6 3 28 49·8	0·014	7 11·899	0 4·24	1 13·44	11 25·4	3 9·3	9 2·5	10 10·2	8 23·0	7 15·3	9 3·59	1 15·94	1 14·1	11 21·9
1889	5 15 8 18·3	0·013	8 12·263	0 4·47	0 13·32	9 26·0	5 15·7	10 0·2	11 9·9	5 25·9	4 16·3	10 3·93	2 18·84	2 15·1	1 4·4
1890	6 2 47 46·9	0·014	9 12·629	0 4·69	11 13·19	7 26·7	7 22·1	10 28·0	0 9·5	2 28·8	1 17·3	11 4·29	3 21·75	3 16·2	2 16·8

Suite de la Table III.

Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.
	′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″
1880·0	3 29·8	+4·7	1881·5	5 53·5	+12·9	1883·0	8 41·1	+7·5	1884·5	9 28·6	−1·3	1886·0	8 26·5	−6·5	1887·5	6 37·8	−7·2	1889·0	5 7·9	−3·7
·1	3 34·5	5·6	·6	6 6·4	12·9	·1	8 48·6	6·9	·6	9 27·3	1·8	·1	8 20·0	6·8	·6	6 30·6	7·2	·1	5 4·2	3·4
·2	3 40·1	6·3	·7	6 19·3	12·9	·2	8 55·5	6·3	·7	9 25·5	2·2	·2	8 13·2	6·9	·7	6 23·4	7·1	·2	5 0·8	3·0
·3	3 46·4	7·2	·8	6 32·2	12·7	·3	9 1·8	5·7	·8	9 23·3	2·7	·3	8 6·3	7·1	·8	6 16·3	6·8	·3	4 57·8	2·7
·4	3 53·6	8·0	·9	6 44·9	12·5	·4	9 7·5	5·0	·9	9 20·6	3·2	·4	7 59·2	7·3	·9	6 9·5	6·7	·4	4 55·1	2·2
·5	4 1·6	8·7	1882·0	6 57·4	12·3	·5	9 12·5	4·4	1885·0	9 17·4	3·6	·5	7 51·9	7·3	1888·0	6 2·8	6·6	·5	4 52·9	1·7
·6	4 10·3	9·4	·1	7 9·7	11·9	·6	9 16·9	3·7	·1	9 13·8	3·9	·6	7 44·6	7·3	·1	5 56·2	6·5	·6	4 51·2	1·2
·7	4 19·7	10·1	·2	7 21·6	11·6	·7	9 20·6	3·0	·2	9 9·9	4·3	·7	7 37·3	7·4	·2	5 49·7	6·2	·7	4 50·0	0·8
·8	4 29·8	10·7	·3	7 33·2	11·3	·8	9 23·6	2·4	·3	9 5·6	4·7	·8	7 29·9	7·4	·3	5 43·5	5·9	·8	4 49·2	−0·3
·9	4 40·5	11·2	·4	7 44·5	10·8	·9	9 26·0	1·9	·4	9 0·9	5·1	·9	7 22·5	7·5	·4	5 37·6	5·6	·9	4 48·9	+0·2
1881·0	4 51·7	11·7	·5	7 55·3	10·2	1884·0	9 27·9	1·2	·5	8 55·8	5·5	1887·0	7 15·0	7·4	·5	5 32·0	5·5	1890·0	4 49·1	+0·7
·1	5 3·4	12·1	·6	8 5·5	9·8	·1	9 29·1	0·7	·6	8 50·3	5·7	·1	7 7·6	7·4	·6	5 26·5	5·2	·1	4 49·8	
·2	5 15·5	12·5	·7	8 15·3	9·2	·2	9 29·8	+0·1	·7	8 44·6	5·8	·2	7 0·2	7·5	·7	5 21·3	4·8			
·3	5 28·0	12·7	·8	8 24·5	8·6	·3	9 29·9	−0·4	·8	8 38·8	6·1	·3	6 52·7	7·5	·8	5 16·5	4·5			
·4	5 40·7	+12·8	·9	8 33·1	+8·0	·4	9 29·5	−0·9	·9	8 32·7	−6·2	·4	6 45·2	−7·4	·9	5 12·0	−4·1			

Suite de la Table I.

QUATRIÈME SATELLITE DE JUPITER,

Époques des Conjonctions Moyennes et des Argumens des inégalités

Années	Conjonctions moyennes: Jours et parties du jour	Conjonctions moyennes: Fraction de l'année	1	2	3	4	5	6	7	I	II	III	IV
	Janv. h ′ ″		s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °
1880 B	0 20 28 51·0	0·000	11 8·710	11 29·29	9 11·76	6 10·9	4 9·2	2 12·49	1 20·0	0 29·44	5 4·41	4 21·3	0 11·0
1881	3 10 21 23·4	0·007	0 9·332	0 2·56	8 14·36	11 16·3	0 23·2	3 12·46	2 18·1	2 0·11	6 5·78	5 24·6	1 23·8
1882	7 0 13 55·8	0·016	1 9·953	0 5·83	7 16·93	4 21·7	9 7·4	4 12·45	3 16·2	3 0·81	7 7·18	6 27·9	3 6·7
1883	10 14 6 28·2	0·026	2 10·576	0 9·10	6 19·52	9 27·2	5 21·5	5 12·44	4 14·2	4 1·50	8 8·56	8 1·1	4 19·6
1884 B	14 3 59 0·6	0·036	3 11·198	0 12·37	5 22·14	3 2·5	2 5·5	6 12·38	5 12·2	5 2·15	9 9·91	9 4·3	6 2·4
1885	16 17 51 33·1	0·043	4 11·821	0 15·64	4 24·80	8 7·9	10 19·3	7 12·30	6 10·2	6 2·77	10 11·23	10 7·5	7 15·2
1886	3 13 38 58·5	0·007	5 11·050	0 2·40	3 12·36	9 11·6	10 16·1	8 10·83	7 6·9	7 1·97	11 11·10	11 9·2	8 26·1
1887	7 3 31 30·9	0·017	6 11·673	0 5·67	2 15·05	2 16·9	6 29·8	9 10·71	8 4·9	8 2·55	0 12·38	0 12·4	10 8·9
1888 B	10 17 24 3·4	0·027	7 12·295	0 8·94	1 17·74	7 22·2	3 13·5	10 10·59	9 2·8	9 3·14	1 13·66	1 15·5	11 21·6
1889	13 7 16 35·8	0·034	8 12·916	0 12·21	0 20·42	0 27·5	11 27·3	11 10·48	10 0·8	10 3·73	2 14·95	2 18·7	1 4·4
1890	0 3 4 1·2	−0·002	9 12·147	11 28·97	11 7·95	2 1·2	11 24·2	0 9·03	10 27·5	11 2·96	3 14·84	3 20·4	2 15·3

Suite de la Table III.

Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.
	′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″
1880·0	8 47·4	+11·2	1881·5	14 25·4	+30·3	1883·0	20 56·0	+17·3	1884·5	22 43·9	−3·4	1886·0	20 15·8	−15·4	1887·5	16 0·9	−17·0	1889·0	12 31·0	−8·7
·1	8 58·6	13·2	·6	14 55·7	30·1	·1	21 13·3	15·9	·6	22 40·5	4·4	·1	20 0·4	15·9	·6	15 43·9	16·9	·1	12 22·3	7·9
·2	9 11·8	15·2	·7	15 25·8	30·0	·2	21 29·2	14·5	·7	22 36·1	5·5	·2	19 44·5	16·3	·7	15 27·0	16·5	·2	12 14·4	6·9
·3	9 27·0	17·0	·8	15 55·8	29·6	·3	21 43·7	13·1	·8	22 30·6	6·5	·3	19 28·2	16·7	·8	15 10·5	16·1	·3	12 7·5	6·0
·4	9 44·0	18·8	·9	16 25·4	29·2	·4	21 56·8	11·5	·9	22 24·1	7·6	·4	19 11·5	17·0	·9	14 54·4	15·6	·4	12 1·5	5·1
·5	10 2·8	20·6	1882·0	16 54·6	28·6	·5	22 8·3	10·0	1885·0	22 16·5	8·5	·5	18 54·5	17·1	1888·0	14 38·8	15·3	·5	11 56·4	4·0
·6	10 23·4	22·1	·1	17 23·2	28·0	·6	22 18·3	8·3	·1	22 8·0	9·4	·6	18 37·4	17·3	·1	14 23·5	14·9	·6	11 52·4	2·8
·7	10 45·5	23·6	·2	17 51·2	27·1	·7	22 26·6	6·8	·2	21 58·6	10·3	·7	18 20·1	17·3	·2	14 8·6	14·5	·7	11 49·6	1·7
·8	11 9·1	25·0	·3	18 18·3	26·0	·8	22 33·4	5·4	·3	21 48·3	11·1	·8	18 2·8	17·4	·3	13 54·1	13·9	·8	11 47·9	−0·5
·9	11 34·1	26·3	·4	18 44·3	24·9	·9	22 38·8	4·1	·4	21 37·2	11·9	·9	17 45·4	17·5	·4	13 40·2	13·3	·9	11 47·4	+0·5
1881·0	12 0·4	27·5	·5	19 9·2	23·9	1884·0	22 42·9	2·7	·5	21 25·3	12·7	1887·0	17 27·9	17·5	·5	13 26·9	12·7	1890·0	11 47·9	+1·7
·1	12 27·9	28·5	·6	19 33·1	22·8	·1	22 45·6	1·4	·6	21 12·6	13·5	·1	17 10·4	17·5	·6	13 14·2	12·1	·1	11 49·6	
·2	12 56·4	29·2	·7	19 55·9	21·4	·2	22 47·0	+0·1	·7	20 59·1	14·0	·2	16 52·9	17·5	·7	13 2·1	11·2			
·3	13 25·6	29·7	·8	20 17·3	20·0	·3	22 47·1	−1·0	·8	20 45·1	14·4	·3	16 35·4	17·3	·8	12 50·9	10·4			
·4	13 55·3	+30·1	·9	20 37·3	+18·7	·4	22 46·1	−2·2	·9	20 30·7	−14·9	·4	16 18·1	−17·2	·9	12 40·5	−9·5			

CONTINUATION OF TABLES I. AND III. OF DAMOISEAU'S TABLES OF JUPITER'S SATELLITES FOR THE PERIOD 1890—1900.

[Appendix to the previous Paper *.]

On revising the above tables for 1880—1890, and continuing them for the period 1890—1900, it was found that some additional corrections should be applied to the terms which involve $\sin(5\bar{u} - 2u_0 - 34^\circ{\cdot}542)$ in Damoiseau's formulæ for Table III., and hence that the statement in the Introduction to the Tables 1880—1890 (see p. 118) as to the sufficient accuracy of these terms as they stand should be somewhat modified.

It appears that Damoiseau's values of these terms are sensibly erroneous both in the Argument and in the Coefficients, and in these tables for 1890—1900, revised expressions have been used for the inequalities in Table III. for Satellites II., III. and IV. depending on the terms referred to. In the case of Satellite I., this inequality is insensible. The approximate values of the adopted expressions appear to be

For Satellite II.	....	$+\ \overset{\text{s.}}{0{\cdot}84} \sin(5\bar{u} - 2u_0 - 16^\circ{\cdot}6)$,
Satellite III.	....	$+\ 2{\cdot}3 \sin(5\bar{u} - 2u_0 - 16^\circ{\cdot}6)$,
Satellite IV.	....	$+12{\cdot}6 \sin(5\bar{u} - 2u_0 - 16^\circ{\cdot}6)$,

where u_0 is the mean longitude of Jupiter and $\bar{u}$ that of Saturn.

The above expressions give corrections to times of Conjunctions in seconds of time. The corresponding corrections to the longitudes of the Satellites in seconds of arc would have for their coefficients for

Satellite II.	....	$-\ 3''{\cdot}5$
Satellite III.	....	$-\ 4{\cdot}8$
Satellite IV.	....	$-11{\cdot}3$

These agree closely with the expressions given by Souillart in his "Théorie des Satellites de Jupiter."

* [For this Appendix and the Tables, which were communicated to the *Nautical Almanac* Office in Jan. 1890, I am indebted to the kindness of Dr Downing, Superintendent of the *Nautical Almanac*. Ed.]

Suite de la Table I.

PREMIER SATELLITE DE JUPITER,

Époques des Conjonctions Moyennes et des Argumens des inégalités

Années	Conjonctions moyennes: Jours et parties du jour	Conjonctions moyennes: Fraction de l'année	1	2	3	4	5	6	7	8	9	I	II	III
	Janv. h ′ ″		s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °
1890	1 17 56 43·8	0·002	9 12·259	0 0·30	11 9·18	6 26·54	7 9·9	10 27·6	0 9·2	2 25·5	1 13·9	11 3·92	2 16·3	3 21·4
1891	1 8 8 8·6	0·001	10 12·549	11 29·64	10 8·21	11 26·50	8 24·8	11 25·4	1 8·8	11 27·7	10 14·3	0 4·23	3 28·7	4 24·2
1892 B	2 16 48 9·3	0·005	11 12·986	0 0·72	9 8·81	10 27·21	7 10·9	0 23·2	2 8·6	9 1·2	7 15·9	1 4·72	5 11·3	5 27·3
1893	1 6 59 34·0	0·001	0 13·276	0 0·06	8 7·81	3 27·19	8 25·9	1 21·0	3 8·2	6 3·5	4 16·2	2 5·07	6 23·7	7 0·2
1894	2 15 39 34·7	0·004	1 13·714	0 1·15	7 8·41	2 27·90	7 11·9	2 18·8	4 8·0	3 7·0	1 17·8	3 5·55	8 6·3	8 3·2
1895	2 5 50 59·4	0·003	2 14·005	0 0·49	6 7·44	7 27·87	8 26·9	3 16·6	5 7·6	0 9·2	10 18·1	4 5·87	9 18·6	9 6·1
1896 B	1 20 2 24·2	0·002	3 14·297	11 29·83	5 6·50	0 27·82	10 11·8	4 14·2	6 7·2	9 11·4	7 18·5	5 6·15	11 1·0	10 8·9
1897	2 4 42 24·9	0·003	4 14·734	0 0·92	4 7·18	11 28·49	8 27·8	5 12·0	7 6·9	6 15·0	4 20·1	6 6·57	0 13·5	11 11·9
1898	1 18 53 49·6	0·002	5 15·026	0 0·26	3 6·26	4 28·43	10 12·7	6 9·7	8 6·5	3 17·2	1 20·4	7 6·82	1 25·8	0 14·7
1899	1 9 5 14·3	0·001	6 15·316	11 29·60	2 5·35	9 28·36	11 27·6	7 7·4	9 6·0	0 19·4	10 20·7	8 7·08	3 8·1	1 17·5
1900	2 17 45 15·0	0·005	7 15·754	0 0·68	1 6·02	8 29·04	10 13·6	8 5·2	10 5·8	9 23·0	7 22·3	9 7·50	4 20·7	2 20·5

Suite de la Table III.

Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.
	′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″
1890·0	1 7·8	+0·1	1891·5	1 22·8	+2·0	1893·0	1 57·6	+2·3	1894·5	2 24·3	+0·9	1896·0	2 25·6	−0·7	1897·5	2 7·6	−1·5	1899·0	1 45·3	−1·3
·1	1 7·9	0·3	·6	1 24·8	2·0	·1	1 59·9	2·3	·6	2 25·2	0·8	·1	2 24·9	0·9	·6	2 6·1	1·6	·1	1 44·0	1·2
·2	1 8·2	0·3	·7	1 26·8	2·1	·2	2 2·2	2·3	·7	2 26·0	0·7	·2	2 24·0	0·9	·7	2 4·5	1·5	·2	1 42·8	1·2
·3	1 8·5	0·5	·8	1 28·9	2·2	·3	2 4·5	2·2	·8	2 26·7	0·5	·3	2 23·1	1·0	·8	2 3·0	1·5	·3	1 41·6	1·1
·4	1 9·0	0·6	·9	1 31·1	2·3	·4	2 6·7	2·1	·9	2 27·2	0·5	·4	2 22·1	1·1	·9	2 1·5	1·6	·4	1 40·5	1·0
·5	1 9·6	0·8	1892·0	1 33·4	2·3	·5	2 8·8	2·0	1895·0	2 27·7	0·3	·5	2 21·0	1·1	1898·0	1 59·9	1·6	·5	1 39·5	1·0
·6	1 10·4	0·9	·1	1 35·7	2·3	·6	2 10·8	1·9	·1	2 28·0	+0·2	·6	2 19·9	1·2	·1	1 58·3	1·5	·6	1 38·5	0·9
·7	1 11·3	1·0	·2	1 38·0	2·4	·7	2 12·7	1·9	·2	2 28·2	0·0	·7	2 18·7	1·2	·2	1 56·8	1·5	·7	1 37·6	0·9
·8	1 12·3	1·2	·3	1 40·4	2·4	·8	2 14·6	1·7	·3	2 28·2	−0·1	·8	2 17·5	1·3	·3	1 55·3	1·5	·8	1 36·7	0·7
·9	1 13·5	1·2	·4	1 42·8	2·5	·9	2 16·3	1·6	·4	2 28·1	0·1	·9	2 16·2	1·4	·4	1 53·8	1·5	·9	1 36·0	−0·8
1891·0	1 14·7	1·4	·5	1 45·3	2·4	1894·0	2 17·9	1·6	·5	2 28·0	0·3	1897·0	2 14·8	1·3	·5	1 52·3	1·5	1900·0	1 35·2	
·1	1 16·1	1·5	·6	1 47·7	2·5	·1	2 19·5	1·4	·6	2 27·7	0·4	·1	2 13·5	1·4	·6	1 50·8	1·4			
·2	1 17·6	1·6	·7	1 50·2	2·5	·2	2 20·9	1·2	·7	2 27·3	0·5	·2	2 12·1	1·5	·7	1 49·4	1·4			
·3	1 19·2	1·8	·8	1 52·7	2·4	·3	2 22·1	1·1	·8	2 26·8	0·5	·3	2 10·6	1·5	·8	1 48·0	1·4			
·4	1 21·0	+1·8	·9	1 55·1	+2·5	·4	2 23·2	+1·1	·9	2 26·3	−0·7	·4	2 9·1	−1·5	·9	1 46·6	−1·3			

Suite de la Table I.

DEUXIÈME SATELLITE DE JUPITER,

Époques des Conjonctions Moyennes et des Argumens des inégalités

Années	Conjonctions Moyennes: Jours et parties du jour	Conjonctions Moyennes: Fraction de l'année	1	2	3	4	5	6	7	8	I	II	III	IV
	Janv. h ′ ″		s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °
1890	3 18 37 38·1	0·008	9 12·430	0 2·33	11 11·04	3 27·38	10 27·8	0 9·3	2 27·0	1 15·5	11 4·09	2 16·52	3 21·5	3 16·0
1891	1 7 2 59·0	0·001	10 12·548	11 29·63	10 8·20	8 26·49	11 25·3	1 8·8	11 27·7	10 14·3	0 4·23	3 28·65	4 24·2	4 16·8
1892 B	2 8 46 13·8	0·004	11 12·961	0 0·43	9 8·54	7 27·08	0 23·2	2 8·5	9 1·0	7 15·7	1 4·69	5 11·21	5 27·2	5 18·0
1893	2 10 29 28·5	0·004	0 13·375	0 1·23	8 8·87	6 27·67	1 21·1	3 8·3	6 4·3	4 17·1	2 5·17	6 23·79	7 0·3	6 19·1
1894	3 12 12 43·2	0·007	1 13·788	0 2·03	7 9·21	5 28·26	2 18·9	4 8·1	3 7·7	1 18·5	3 5·63	8 6·36	8 3·3	7 20·3
1895	1 0 38 4·2	0·000	2 13·907	11 29·33	6 6·37	10 27·37	3 16·5	5 7·5	0 8·3	10 17·3	4 5·77	9 18·49	9 6·0	8 21·1
1896 B	2 2 21 18·9	0·003	3 14·321	0 0·13	5 6·77	9 27·93	4 14·3	6 7·2	9 11·7	7 18·7	5 6·18	11 1·00	10 8·9	9 22·2
1897	2 4 4 33·7	0·003	4 14·735	0 0·93	4 7·18	8 28·48	5 12·0	7 6·9	6 15·0	4 20·1	6 6·57	0 13·50	11 11·9	10 23·3
1898	3 5 47 48·4	0·006	5 15·149	0 1·73	3 7·60	7 29·03	6 9·8	8 6·6	3 18·3	1 21·5	7 6·95	1 25·99	0 14·8	11 24·4
1899	4 7 31 3·1	0·009	6 15·563	0 2·53	2 8·03	6 29·58	7 7·6	9 6·3	0 21·6	10 22·9	8 7·33	3 8·47	1 17·8	0 25·4
1900	1 19 56 24·1	0·002	7 15·681	11 29·83	1 5·23	11 28·67	8 5·1	10 5·7	9 22·3	7 21·7	9 7·42	4 20·56	2 20·4	1 26·2

Suite de la Table III.

Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.
	′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″
1890·0	2 21·2	+0·3	1891·5	2 52·3	+3·9	1893·0	4 1·3	+4·6	1894·5	4 52·3	+1·6	1896·0	4 51·7	−1·7	1897·5	4 12·9	−3·2	1899·0	3 26·7	−2·6
·1	2 21·5	0·5	·6	2 56·2	4·1	·1	4 5·9	4·5	·6	4 53·9	1·4	·1	4 50·0	1·9	·6	4 9·7	3·2	·1	3 24·1	2·4
·2	2 22·0	0·8	·7	3 0·3	4·3	·2	4 10·4	4·4	·7	4 55·3	1·1	·2	4 48·1	2·0	·7	4 6·5	3·2	·2	3 21·7	2·4
·3	2 22·8	1·1	·8	3 4·6	4·4	·3	4 14·8	4·2	·8	4 56·4	0·9	·3	4 46·1	2·2	·8	4 3·3	3·3	·3	3 19·3	2·3
·4	2 23·9	1·4	·9	3 9·0	4·5	·4	4 19·0	4·1	·9	4 57·3	0·7	·4	4 43·9	2·4	·9	4 0·0	3·2	·4	3 17·0	2·1
·5	2 25·3	1·6	1892·0	3 13·5	4·6	·5	4 23·1	3·9	1895·0	4 58·0	0·4	·5	4 41·5	2·4	1898·0	3 56·8	3·2	·5	3 14·9	2·0
·6	2 26·9	1·9	·1	3 18·1	4·7	·6	4 27·0	3·7	·1	4 58·4	+0·1	·6	4 39·1	2·6	·1	3 53·6	3·2	·6	3 12·9	1·8
·7	2 28·8	2·1	·2	3 22·8	4·7	·7	4 30·7	3·5	·2	4 58·5	−0·1	·7	4 36·5	2·7	·2	3 50·4	3·1	·7	3 11·1	1·7
·8	2 30·9	2·4	·3	3 27·5	4·8	·8	4 34·2	3·3	·3	4 58·4	0·3	·8	4 33·8	2·8	·3	3 47·3	3·1	·8	3 9·4	1·5
·9	2 33·3	2·6	·4	3 32·3	4·8	·9	4 37·5	3·1	·4	4 58·1	0·6	·9	4 31·0	2·8	·4	3 44·2	3·1	·9	3 7·9	−1·4
1891·0	2 35·9	2·8	·5	3 37·1	4·9	1894·0	4 40·6	2·9	·5	4 57·5	0·8	1897·0	4 28·2	2·9	·5	3 41·1	3·0	1900·0	3 6·5	
·1	2 38·7	3·0	·6	3 42·0	4·8	·1	4 43·5	2·6	·6	4 56·7	0·9	·1	4 25·3	3·0	·6	3 38·1	3·0			
·2	2 41·7	3·3	·7	3 46·8	4·9	·2	4 46·1	2·3	·7	4 55·8	1·2	·2	4 22·3	3·1	·7	3 35·1	2·9			
·3	2 45·0	3·6	·8	3 51·7	4·9	·3	4 48·4	2·0	·8	4 54·6	1·4	·3	4 19·2	3·1	·8	3 32·2	2·8			
·4	2 48·6	+3·7	·9	3 56·6	+4·7	·4	4 50·4	+1·9	·9	4 53·2	−1·5	·4	4 16·1	−3·2	·9	3 29·4	−2·7			

TROISIÈME SATELLITE DE JUPITER,

Époques des Conjonctions Moyennes et des Argumens des inégalités

Suite de la Table I.

Années	Conjonctions moyennes: Jours et parties du jour	Conjonctions moyennes: Fraction de l'année	1	2	3	4	5	6	7	8	9	I	II	III	IV
	Janv. h ′ ″		s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °
1890	6 2 47 46·9	0·014	9 12·629	0 4·69	11 13·20	7 26·7	7 22·1	10 28·0	0 9·5	2 28·8	1 17·2	11 4·29	3 21·75	3 16·2	2 16·9
1891	6 14 27 15·5	0·015	10 12·993	0 4·91	10 13·04	5 27·3	9 28·6	11 25·8	1 9·2	0 1·7	10 18·2	0 4·68	4 24·70	4 17·3	3 29·4
1892 B	7 2 6 44·0	0·016	11 13·358	0 5·14	9 12·85	3 28·0	0 5·1	0 23·6	2 8·9	9 4·6	7 19·2	1 5·09	5 27·67	5 18·4	5 11·9
1893	6 13 46 12·6	0·015	0 13·722	0 5·36	8 12·65	1 28·7	2 11·6	1 21·4	3 8·7	6 7·5	4 20·2	2 5·52	7 0·65	6 19·5	6 24·4
1894	7 1 25 41·2	0·016	1 14·087	0 5·58	7 12·46	11 29·4	4 18·0	2 19·2	4 8·4	3 10·4	1 21·2	3 5·93	8 3·62	7 20·6	8 6·9
1895	7 13 5 9·7	0·018	2 14·453	0 5·80	6 12·30	10 0·1	6 24·5	3 17·0	5 8·1	0 13·3	10 22·1	4 6·32	9 6·56	8 21·7	9 19·4
1896 B	8 0 44 38·3	0·019	3 14·819	0 6·02	5 12·17	8 0·7	9 0·9	4 14·7	6 7·7	9 16·2	7 23·1	5 6·68	10 9·47	9 22·7	11 1·8
1897	7 12 24 6·8	0·018	4 15·183	0 6·25	4 12·06	6 1·4	11 7·4	5 12·4	7 7·4	6 19·0	4 24·1	6 7·01	11 12·37	10 23·8	0 14·3
1898	8 0 3 35·4	0·019	5 15·549	0 6·47	3 11·95	4 2·0	1 13·8	6 10·2	8 7·0	3 21·9	1 25·1	7 7·35	0 15·25	11 24·8	1 26·7
1899	1 7 43 28·1	0·001	6 15·318	11 29·63	2 5·37	1 26·7	8 24·2	7 7·4	9 6·0	0 19·5	10 20·7	8 7·08	1 17·50	0 25·2	3 8·3
1900	1 19 22 56·7	0·002	7 15·683	11 29·86	1 5·26	11 27·3	11 0·7	8 5·1	10 5·7	9 22·4	7 21·7	9 7·43	2 20·40	1 26·2	4 20·7

Suite de la Table III.

Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.
	′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″
1890·0	4 48·4	+0·7	1891·5	5 52·3	+7·9	1893·0	8 10·8	+9·2	1894·5	9 51·2	+3·1	1896·0	9 47·1	−3·6	1897·5	8 26·8	−6·5	1899·0	6 52·7	−5·2
·1	4 49·1	1·2	·6	6 0·2	8·2	·1	8 20·0	9·0	·6	9 54·3	2·6	·1	9 43·5	4·0	·6	8 20·3	6·6	·1	6 47·5	5·0
·2	4 50·3	1·7	·7	6 8·4	8·6	·2	8 29·0	8·6	·7	9 56·9	2·1	·2	9 39·5	4·3	·7	8 13·7	6·6	·2	6 42·5	4·8
·3	4 52·0	2·3	·8	6 17·0	8·8	·3	8 37·6	8·4	·8	9 59·0	1·6	·3	9 35·2	4·6	·8	8 7·1	6·6	·3	6 37·7	4·5
·4	4 54·3	2·8	·9	6 25·8	9·1	·4	8 46·0	8·1	·9	10 0·6	1·1	·4	9 30·6	4·8	·9	8 0·5	6·6	·4	6 33·2	4·3
·5	4 57·1	3·3	1892·0	6 34·9	9·3	·5	8 54·1	7·7	1895·0	10 1·7	0·6	·5	9 25·8	5·2	1898·0	7 53·9	6·6	·5	6 28·9	4·0
·6	5 0·4	4·0	·1	6 44·2	9·4	·6	9 1·8	7·3	·1	10 2·3	+0·1	·6	9 20·6	5·4	·1	7 47·3	6·5	·6	6 24·9	3·7
·7	5 4·4	4·4	·2	6 53·6	9·5	·7	9 9·1	6·9	·2	10 2·4	−0·4	·7	9 15·2	5·5	·2	7 40·8	6·4	·7	6 21·2	3·4
·8	5 8·8	4·8	·3	7 3·1	9·6	·8	9 16·0	6·5	·3	10 2·0	0·8	·8	9 9·7	5·7	·3	7 34·4	6·3	·8	6 17·8	3·0
·9	5 13·6	5·2	·4	7 12·7	9·7	·9	9 22·5	6·1	·4	10 1·2	1·4	·9	9 4·0	5·9	·4	7 28·1	6·2	·9	6 14·8	−2·9
1891·0	5 18·8	5·7	·5	7 22·4	9·7	1894·0	9 28·6	5·6	·5	9 59·8	1·8	1897·0	8 58·1	6·0	·5	7 21·9	6·1	1900·0	6 11·9	
·1	5 24·5	6·3	·6	7 32·1	9·8	·1	9 34·2	5·1	·6	9 58·0	2·1	·1	8 52·1	6·1	·6	7 15·8	6·0			
·2	5 30·8	6·8	·7	7 41·9	9·7	·2	9 39·3	4·4	·7	9 55·9	2·6	·2	8 46·0	6·3	·7	7 9·8	5·9			
·3	5 37·6	7·2	·8	7 51·6	9·7	·3	9 43·7	4·0	·8	9 53·3	2·9	·3	8 39·7	6·4	·8	7 3·9	5·7			
·4	5 44·8	+7·5	·9	8 1·3	+9·5	·4	9 47·7	+3·5	·9	9 50·4	−3·3	·4	8 33·3	−6·5	·9	6 58·2	−5·5			

Suite de la TABLE I.

QUATRIÈME SATELLITE DE JUPITER,

Époques des Conjonctions Moyennes et des Argumens des inégalités

ANNÉES	CONJONCTIONS MOYENNES: JOURS et parties du jour	CONJONCTIONS MOYENNES: FRACTION de l'année	I	2	3	4	5	6	7	I	II	III	IV
	Janv. h ′ ″		s °	s °	s °	s °	s °	s °	s °	s °	s °	s °	s °
1890	0 3 4 1·3	−0·002	9 12·417	11 28·97	11 7·95	2 1·2	11 24·2	0 9·03	10 27·5	11 2·96	3 14·84	3 20·4	2 15·3
1891	3 16 56 33·7	0·007	10 12·768	0 2·24	10 10·58	7 6·6	8 8·1	1 8·97	11 25·5	0 3·61	4 16·18	4 23·6	3 28·1
1892 B	7 6 49 6·1	0·017	11 13·389	0 5·51	9 13·19	0 12·0	4 22·2	2 8·94	0 23·6	1 4·28	5 17·55	5 26·9	5 11·0
1893	9 20 41 38·5	0·024	0 14·011	0 8·78	8 15·78	5 17·4	1 6·3	3 8·92	1 21·6	2 4·96	6 18·93	7 0·1	6 23·8
1894	13 10 34 11·0	0·034	1 14·633	0 12·05	7 18·38	10 22·8	9 20·3	4 8·88	2 19·7	3 5·63	7 20·30	8 3·4	8 6·7
1895	17 0 26 43·4	0·044	2 15·255	0 15·32	6 21·01	3 28·2	6 4·3	5 8·82	3 17·7	4 6·27	8 21·64	9 6·6	9 19·5
1896 B	3 20 14 8·9	0·008	3 14·486	0 2·08	5 8·55	5 1·9	6 1·1	6 7·37	4 14·4	5 5·50	9 21·53	10 8·3	11 0·4
1897	6 10 6 41·3	0·015	4 15·108	0 5·35	4 11·23	10 7·2	2 14·9	7 7·26	5 12·4	6 6·10	10 22·83	11 11·4	0 13·2
1898	9 23 59 13·7	0·025	5 15·730	0 8·62	3 13·91	3 12·5	10 28·6	8 7·15	6 10·3	7 6·68	11 24·11	0 14·6	1 25·9
1899	13 13 51 46·1	0·034	6 16·352	0 11·90	2 16·60	8 17·8	7 12·4	9 7·04	7 8·3	8 7·28	0 25·40	1 17·8	3 8·7
1900	17 3 44 18·5	0·044	7 16·974	0 15·17	1 19·27	1 23·1	3 26·2	10 6·93	8 6·3	9 7·88	1 26·70	2 21·0	4 21·5

Suite de la TABLE III.

Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.	Ann.	Perturb.	Diff.
	′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″		′ ″	″
1890·0	11 43·8	+1·8	1891·5	14 14·2	+18·4	1893·0	19 37·1	+21·4	1894·5	23 29·0	+6·9	1896·0	23 15·7	−8·6	1897·5	20 5·5	−15·5	1899·0	16 24·4	−12·3
·1	11 45·6	2·9	·6	14 32·6	19·2	·1	19 58·5	20·8	·6	23 35·9	5·9	·1	23 7·1	9·4	·6	19 50·0	15·5	·1	16 12·1	11·7
·2	11 48·5	4·2	·7	14 51·8	20·1	·2	20 19·3	20·1	·7	23 41·8	4·6	·2	22 57·7	10·2	·7	19 34·5	15·5	·2	16 0·4	11·1
·3	11 52·7	5·3	·8	15 11·9	20·7	·3	20 39·4	19·4	·8	23 46·4	3·5	·3	22 47·5	11·0	·8	19 19·0	15·5	·3	15 49·3	10·6
·4	11 58·0	6·6	·9	15 32·6	21·1	·4	20 58·8	18·7	·9	23 49·9	2·3	·4	22 36·5	11·6	·9	19 3·5	15·6	·4	15 38·7	10·1
·5	12 4·6	7·9	1892·0	15 53·7	21·7	·5	21 17·5	17·8	1895·0	23 52·2	1·2	·5	22 24·9	12·3	1898·0	18 47·9	15·4	·5	15 28·6	9·2
·6	12 12·5	9·2	·1	16 15·4	22·0	·6	21 35·3	16·9	·1	23 53·4	+0·1	·6	22 12·6	12·7	·1	18 32·5	15·2	·6	15 19·4	8·6
·7	12 21·7	10·4	·2	16 37·4	22·2	·7	21 52·2	16·1	·2	23 53·5	−1·2	·7	21 59·9	13·1	·2	18 17·3	15·1	·7	15 10·8	7·9
·8	12 32·1	11·2	·3	16 59·6	22·4	·8	22 8·3	15·0	·3	23 52·3	2·2	·8	21 46·8	13·5	·3	18 2·2	14·9	·8	15 2·9	7·2
·9	12 43·3	12·4	·4	17 22·0	22·5	·9	22 23·3	13·9	·4	23 50·1	3·4	·9	21 33·3	13·9	·4	17 47·3	14·5	·9	14 55·7	−6·6
1891·0	12 55·7	13·4	·5	17 44·5	22·7	1894·0	22 37·2	12·8	·5	23 46·7	4·4	1897·0	21 19·4	14·2	·5	17 32·8	14·4	1900·0	14 49·1	
·1	13 9·1	14·7	·6	18 7·2	22·7	·1	22 50·0	11·7	·6	23 42·3	5·3	·1	21 5·2	14·5	·6	17 18·4	14·1			
·2	13 23·8	15·8	·7	18 29·9	22·7	·2	23 1·7	10·3	·7	23 37·0	6·3	·2	20 50·7	14·9	·7	17 4·3	13·7			
·3	13 39·6	16·9	·8	18 52·6	22·5	·3	23 12·0	9·0	·8	23 30·7	7·1	·3	20 35·8	15·0	·8	16 50·6	13·4			
·4	13 56·5	+17·7	·9	19 15·1	+22·0	·4	23 21·0	+8·0	·9	23 23·6	−7·9	·4	20 20·8	−15·3	·9	16 37·2	−12·8			

20.

ON PROFESSOR CHALLIS'S NEW THEOREMS RELATING TO THE MOON'S ORBIT.

[From the *Philosophical Magazine*, Vol. VIII. (1854).]

In the June Number of your valuable Journal, Professor Challis calls attention to some circumstances connected with his withdrawal of a paper, relating to the Moon's motion, which he had communicated to the Cambridge Philosophical Society, and of the principal results of which he had given an account in your Number for April (p. 278).

Professor Challis mentions that one of the reporters, whose unfavourable judgement led to this withdrawal, had of his own accord communicated to him some of the reasons on which this judgement was based. Professor Challis, however, thinks these reasons to be very unsatisfactory, and consequently invites the reporter to discuss with him the questions on which they are at issue, in the pages of the *Philosophical Magazine*.

As I am the reporter thus referred to, I beg that you will allow me to state some reasons which appear to me sufficient to prove, beyond a doubt, that the principal conclusions of Professor Challis's paper are erroneous, in order that he may have the opportunity, which he desires, of replying publicly to my objections*. At the same time, I must decline to enter

* It may be proper to mention that the opinion of the other reporter on the paper perfectly agreed with my own.

into any prolonged controversy on the subject, submitting with confidence what I have now to say to those who are competent to form a judgement respecting it.

The principal results of Professor Challis's paper are embodied in two theorems, which, as already stated, form the subject of an article in the *Philosophical Magazine* for April last. As my main objections to the paper relate to these theorems, I shall confine my observations almost entirely to the article in question.

It will be convenient, however, to make a few preliminary remarks on the nature of the process usually followed in the lunar theory. Professor Challis objects to the *logic* of this process, on the ground that the introduction of the quantities usually denoted by c and g into the first approximation to the Moon's motion is only suggested by observation. He therefore considers the results of the ordinary process to be *hypothetical*, until they are confirmed by observation.

But surely the *sufficient* and the *only* test of the correctness of any solution is, that it should satisfy the differential equations of motion at the same time that it contains the proper number of arbitrary constants to fulfil any given initial conditions.

Any process which does this, no matter how it may be *suggested* to us, must be logical; and if the results obtained by it should not agree with observation, the conclusion would be that the law of gravitation, which was assumed in forming the original differential equations, is not really the law of nature.

If we begin with the supposition that the Moon's orbit is an *immoveable ellipse*, the differential equations cannot be satisfied, without adding, to the first approximate expressions for the Moon's coordinates, quantities which are capable of indefinite increase; and this proves, as is stated by Professor Challis, that an immoveable ellipse is not, or rather does not continue to be, an approximation to the real orbit.

But if we introduce the quantities usually denoted by c and g, having assigned values slightly differing from unity, which amounts to supposing the apse and node to have certain mean motions, we find that the differential equations are satisfied by adding to the first approximate expressions for the Moon's coordinates, terms, which always remain *small*; and we thus know that our first approximation was a good one, and that the *true* and the *only true* solution of the differential equations has been obtained.

On the other hand, no solution can be a true one, which does not contain the proper number of arbitrary constants; and any person who asserts that one of the constants usually considered *arbitrary* is not so, is bound to show by what other really arbitrary constant the former is replaced.

I will now proceed to consider Professor Challis's two theorems, which are thus enunciated by him.

Theorem I. All small quantities of the second order being taken into account, the relation between the radius-vector and the time in the Moon's orbit is the same as that in an orbit described by a body acted upon by a force tending to a fixed centre.

Theorem II. The eccentricity of the Moon's orbit is a function of the ratio of her periodic time to the Earth's periodic time, and the first approximation to its value is that ratio divided by the square root of 2.

I will endeavour, in the first place, to show that these theorems cannot possibly be true; and secondly, to point out the fallacies in the argument by which Professor Challis attempts to establish them.

The problem will be simplified by supposing the Moon to move in the plane of the ecliptic, and the Earth's orbit to be a circle. On these suppositions, Professor Challis's fundamental equations become

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{r^3} + \frac{m'x}{2a'^3} + \frac{3m'r}{2a'^3}\cos(\theta - 2\overline{n't+\epsilon'}),$$

$$\frac{d^2y}{dt^2} = -\frac{\mu y}{r^3} + \frac{m'y}{2a'^3} - \frac{3m'r}{2a'^3}\sin(\theta - 2\overline{n't+\epsilon'}).$$

Multiply these equations by y and x respectively, and subtract the results; and again multiply by x and y, and add the results together; thus we obtain, after expressing x and y by means of polar coordinates,

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -\frac{3m'r^2}{2a'^3}\sin(2\theta - 2\overline{n't+\epsilon'}) \quad \text{..........................(1)},$$

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2} + \frac{m'r}{2a'^3} + \frac{3m'r}{2a'^3}\cos(2\theta - 2\overline{n't+\epsilon'}) \text{...........(2)}.$$

Now these equations, which are equivalent to the former, are satisfied to terms of the second order inclusive by putting

$$r = a\left\{1 - \frac{m^2}{6} + \frac{1}{2}e^2 - e\cos(cnt + \epsilon - \varpi) - \frac{1}{2}e^2\cos 2(cnt + \epsilon - \varpi)\right.$$
$$- m^2\cos(2\overline{nt + \epsilon} - 2\overline{n't + \epsilon'})$$
$$\left. - \frac{15}{8}me\cos(2\overline{nt+\epsilon} - 2\overline{n't+\epsilon'} - \overline{cnt + \epsilon - \varpi})\right\}$$

$$\theta = nt + \epsilon + 2e\sin(cnt + \epsilon - \varpi) + \frac{5}{4}e^2\sin 2(cnt + \epsilon - \varpi)$$
$$+ \frac{11}{8}m^2\sin(2\overline{nt+\epsilon} - 2\overline{n't+\epsilon'})$$
$$+ \frac{15}{4}me\sin(2\overline{nt+\epsilon} - 2\overline{n't+\epsilon'} - \overline{cnt+\epsilon-\varpi}),$$

where $$n^2 = \frac{\mu}{a^3},\quad n'^2 = \frac{m'}{a'^2},\quad m = \frac{n'}{n},\quad c = 1 - \frac{3}{4}m^2,$$

and a, ϵ, e, and ϖ are the four arbitrary constants required by the complete solution.

The fact that the differential equations are satisfied by these expressions for r and θ, whatever be the value of e, is quite sufficient to shew that Professor Challis is mistaken in restricting e to one particular value.

The terms of the *second order* in the value of r, which depend on the arguments

$$2\overline{nt+\epsilon} - 2\overline{n't+\epsilon'} \text{ and } 2\overline{nt+\epsilon} - 2\overline{n't+\epsilon'} - \overline{cnt+\epsilon-\varpi},$$

and which constitute the well-known inequalities called the "variation" and "evection," prove the incorrectness of Professor Challis's Theorem I.; since in an orbit described by a body acted on by a force tending to a fixed centre, and varying, as Professor Challis supposes, as some function of the distance, the expression for the radius-vector in terms of the time cannot possibly contain any terms dependent on the *Sun's longitude.*

I now come to consider the reasoning by which Professor Challis arrives at his theorems. All this reasoning is based on his equation

$$\left(\frac{dr}{dt}\right)^2 + \frac{h^2}{r^2} - \frac{2\mu}{r} - \frac{m'r^2}{2a'^3} + C = 0 \text{ ....(C)},$$

the truth of which, he says, cannot be contested. In speaking of the *truth* of this equation, Professor Challis cannot mean that it is anything more than an *approximation* to the truth, since in forming it he avowedly neglects all quantities of orders superior to the second.

Now what I assert is, *first*, that the *degree of approximation* attained by the equation (C) is not sufficient to justify Professor Challis in inferring Theorem I. from it; and *secondly*, that Theorem II. does not follow from that equation at all.

To prove the first of these assertions, I remark that the equation (C) gives an approximate value of $\left(\frac{dr}{dt}\right)^2$ in terms of r, but that it does not profess to include terms of the third order. Now $\frac{dr}{dt}$ is itself a quantity of the first order, and consequently an error of the third order in $\left(\frac{dr}{dt}\right)^2$ leads to one of the second order in $\frac{dr}{dt}$, and therefore to one of the same order in the value of r expressed in terms of t. Hence Professor Challis is not entitled to infer that the relation between the radius-vector and the time in the Moon's orbit is the same, to quantities of the second order, as that which would be given by the equation (C).

We may test the degree of accuracy to be attained by the use of this equation in the following manner.

By differentiation, the constant C disappears, and the resulting equation becomes divisible by $\frac{dr}{dt}$; dividing out, we obtain

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} + \frac{\mu}{r^2} - \frac{m'r}{2a'^3} = 0.$$

This is a strict deduction from Professor Challis's equation; we will now obtain directly from the equations of motion given above, an expression to be compared with it.

Integrating equation (1), and putting, with Professor Challis, $nt + \epsilon$ for θ, and a for r in the term of the second order, we find

$$r^2 \frac{d\theta}{dt} = h + \frac{3}{4} \frac{m'}{a'^3} \frac{a^2}{n} \cos\left(2\overline{nt+\epsilon} - 2\overline{n't+\epsilon'}\right).$$

The value of the constant h, expressed in terms of the system of constants before used, is

$$h = na^2\left(1 - \frac{m^2}{3} - \frac{e^2}{2}\right).$$

Hence

$$r^4\left(\frac{d\theta}{dt}\right)^2 = h^2 + \frac{3}{2}\frac{m'}{a'^3}a^4\cos(2\overline{nt+\epsilon} - 2\overline{n't+\epsilon'}),$$

and

$$r\left(\frac{d\theta}{dt}\right)^2 = \frac{h^2}{r^3} + \frac{3}{2}\frac{m'}{a'^3}a\cos(2\overline{nt+\epsilon} - 2\overline{n't+\epsilon'}),$$

putting, as before, a for r in the small term. Substituting this value of $r\left(\frac{d\theta}{dt}\right)^2$ in equation (2), we find

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} + \frac{\mu}{r^2} - \frac{m'r}{2a'^3} - 3\frac{m'}{a'^3}a\cos(2\overline{nt+\epsilon} - 2\overline{n't+\epsilon'}) = 0.$$

The equation above deduced from Professor Challis's differs from this by the omission of the last term, which gives rise to the *variation* inequality. In order to find the *evection*, which is also an inequality of the second order, it would be necessary to carry the approximation one step still further than we have here done.

This shews how unfitted equation (C) is for giving any accurate information respecting the Moon's orbit.

As a matter of fact, it may be observed that this equation would make the Moon's apsidal distances to be *constant*. A simple inspection of the calculated values of the Moon's horizontal parallax, given in the *Nautical Almanac*, is sufficient to shew how far this is from the truth.

I now proceed to make good my second assertion, viz. that Professor Challis's Theorem II. cannot be inferred from his equation (C). The process by which he attempts so to infer it is of the following nature. He first finds that a method, apparently legitimate, of treating the equation (C) leads to a difficulty. To get rid of this difficulty, he makes the strange supposition that the equation (C) contains the disturbing force as a factor, and then tries to shew that, in order that this condition may be satisfied, the arbitrary constants h and C must have a certain relation to each other, from which it would immediately follow that the eccentricity must have the value assigned to it in Theorem II.

Now it is remarkable that every one of the steps of this process is unwarranted. The difficulty to which Professor Challis is led is purely imaginary; the supposition that the equation (C) contains the disturbing force as a factor is wholly unsupported by any proof; and even if that supposition were well founded, it would not follow that the constants h and C must have the relation assigned to them by Professor Challis.

The supposed difficulty is founded on the inference at the bottom of p. 280 of Professor Challis's paper, "Hence we must conclude that the mean distance and mean periodic time in this approximation to the Moon's orbit are the same as those in an elliptic orbit described by the action of the central force $\frac{\mu}{r^2}$." But this is not a correct conclusion: if h and C be supposed to have the same values in equation (C) and in that obtained from it by putting a for r in the small term, the values of the mean distances in the two cases would not be the same, but would differ by a quantity of the second order.

This may be readily shewn in the following manner.

At the apsides $\frac{dr}{dt}=0$, and therefore the equation (C) gives the following equation for finding the apsidal distances,

$$h^2-2\mu r+Cr^2-\frac{m'}{2a'^3}r^4=0.$$

Now if a be the mean distance, and e the eccentricity, the apsidal distances are $a(1+e)$ and $a(1-e)$.

Substituting these values for r in the above equation, and developing the small term to quantities of the fourth order, we obtain

$$h^2-2\mu a(1+e)+Ca^2(1+2e+e^2)-\frac{m'}{2a'^3}a^4(1+4e+6e^2)=0,$$

and

$$h^2-2\mu a(1-e)+Ca^2(1-2e+e^2)-\frac{m'}{2a'^3}a^4(1-4e+6e^2)=0;$$

whence it follows that

$$h^2-2\mu a+Ca^2(1+e^2)-\frac{m'}{2a'^3}a^4(1+6e^2)=0$$

and

$$\mu a-Ca^2+\frac{m'}{a'^3}a^4=0.$$

These equations give the relations between the arbitrary constants h and C, and the new constants a and e by which the former may be replaced.

From the second of them, we find

$$a = \frac{\mu}{C} + \frac{m'}{a'^3}\frac{a^3}{C};$$

or, putting for a in the small term its first approximate value $\frac{\mu}{C}$,

$$a = \frac{\mu}{C} + \frac{m'}{a'^3}\frac{\mu^3}{C^4},$$

which agrees with Professor Challis's expression in p. 281.

Now apply a similar process to the equation

$$\left(\frac{dr}{dt}\right)^2 + \frac{h^2}{r^2} - \frac{2\mu}{r} - \frac{m'a^2}{2a'^3} + C = 0,$$

which differs from the equation (C) in having a put for r in the small term. In this case, we find

$$h^2 - 2\mu a + Ca^2(1+e^2) - \frac{m'}{2a'^3}a^4(1+e^2) = 0,$$

and

$$\mu a - Ca^2 + \frac{m'}{2a'^3}a^4 = 0;$$

from the latter of which equations it follows that

$$a = \frac{\mu}{C} + \frac{m'}{2a'^3}\frac{a^3}{C},$$

or

$$a = \frac{\mu}{C} + \frac{m'}{2a'^3}\frac{\mu^3}{C^4},$$

to the same degree of approximation as before.

Hence we see that the values of a, in the two cases supposed, differ by a quantity of the second order. Consequently the difficulty into which Professor Challis is led by the conclusion that these values are the same, disappears, and the solution of the difficulty with it.

But even if we were to suppose, with Professor Challis, that the equation (C) contains the disturbing force as a factor (of which, as already remarked, no proof whatever is given), it would not follow, as is inferred by him, that h^2C must be equal to μ^2. On the contrary, it is evident that the required condition would be satisfied if h^2C differed from μ^2 by any quantity involving the disturbing force as a factor; whence it would follow that e must be *some* function, indeed, of the disturbing force, but it could not be decided *what* function.

Professor Challis attempts to find the relation between r and t by direct integration of the equation

$$dt = \frac{-dr}{\sqrt{-C - \frac{h^2}{r^2} + \frac{2\mu}{r} + \frac{m'r^2}{2a'^3}}}.$$

Now it may be remarked that $\left(\frac{dr}{dt}\right)^2$ is a small quantity of the second order which vanishes twice in each revolution, and that the difference between the complete value of $\left(\frac{dr}{dt}\right)^2$ and the approximate value

$$-C - \frac{h^2}{r^2} + \frac{2\mu}{r} + \frac{m'r^2}{2a'^3}$$

which is used instead of it in the above equation, is a periodic quantity of the third order.

Hence it follows that the quantity

$$-C - \frac{h^2}{r^2} + \frac{2\mu}{r} + \frac{m'r^2}{2a'^3}$$

may vanish for values of r different from those which make $\left(\frac{dr}{dt}\right)^2$ vanish, and that it may even become negative for actual values of r, which $\left(\frac{dr}{dt}\right)^2$ itself can never do.

Therefore the coefficient of dr in the above differential equation may become infinite, or even imaginary, within the limits of integration, so that it is not surprising that Professor Challis should have met with such difficulties in performing the integration.

The relations between r, θ, and t, given in page 281 (which profess to include all small quantities of the second order), are said to be derived from the equations (B) and (C). It is easy to see, however, that they do not satisfy the first of those equations, since the term of the second order

$$\frac{3m'\rho^2}{2a'^3}\cos 2\overline{\theta - \theta'}$$

in the right-hand member of that equation involves the *longitude of the Sun*, which does not occur at all in the relations in question.

The contradiction to Professor Challis's theory, which is presented by the eccentricity of the orbit of *Titan*, is supposed by him to be occasioned by the large inclination of that orbit to the plane of the orbit of *Saturn*. But in page 280 it is remarked that the inclination of the orbit is taken into account; and even if this were not the case, no proof is offered that the taking it into account would tend to reconcile the discrepancy.

At the bottom of page 282, Professor Challis attempts to shew, *à priori*, that the eccentricity of the Moon's orbit must be a function of the disturbing force in the following manner.

If there were no disturbing force, the value of the radius-vector drawn from the Earth's centre in a given direction, would be constantly the same in different revolutions. But if a disturbing force act in such a manner as to cause the apsidal line to make complete revolutions, the value of the above-mentioned radius-vector would fluctuate in different revolutions, between the two apsidal distances. Hence it is argued that, since if there were no disturbing force there would be no such fluctuation of distance, therefore the total amount of such fluctuation, and consequently the eccentricity, must be a function of the disturbing force.

But, on consideration, it will appear that this argument is fallacious. No doubt it may be inferred that *some of the circumstances* of this fluctuation of distance will depend on the disturbing force which causes it, but it cannot be asserted, without investigation, that the *total amount* of such fluctuation must necessarily depend on the disturbing force.

As a simple example, we will suppose the principal force to vary inversely as the square of the distance, and a central disturbing force to be introduced which varies inversely as the cube of that distance. In this case we know, by Newton's 9th section, that the motion would be accurately represented by supposing it to take place in a revolving ellipse, the angular velocity of the orbit being always proportional to that of the body at the same instant; and the eccentricity of the orbit might be any whatever, and would not at all depend on the disturbing force.

Now, since the orbit would be fixed, were it not for the disturbing force, it might be argued in exactly the same manner as is done by Professor Challis in the passage above referred to, that the eccentricity of the orbit must be a function of the force which causes the orbit to revolve, but this we know to be a false conclusion.

What would depend on the disturbing force in this case, would be, not the *total amount* of the fluctuation of distance in different revolutions, but the *number of revolutions of the body in which such fluctuation would take place*, or *the time of revolution of the apse.* If the disturbing force were increased, the total fluctuation in the value of the radius-vector in question would be the same as before, but the change from one of the extreme values to the other would occupy a shorter time.

The objection mentioned by Professor Challis at the top of page 283, is alone quite fatal to the supposition that the eccentricity of the Moon's orbit must have a particular value. Where is the proof that the eccentricity would *settle down* to such a value, as Professor Challis imagines, if it were initially different?

In fact, it is easy to shew, by the method of variation of elements, that there would be no such settlement, but that the non-periodic part of the eccentricity would remain constant.

21.

ON THE SECULAR VARIATION OF THE MOON'S MEAN MOTION.

[From the *Philosophical Transactions of the Royal Society,* Vol. CXLIII. (1853). Abstract of same, *Proceedings of the Royal Society,* June 16, 1853 and *Monthly Notices of the Royal Astronomical Society,* Vol. XIV. (1853).]

1. IN treating a great problem of approximation, such as that presented to us by the investigation of the Moon's motion, experience shows that nothing is more easy than to neglect, as insignificant, considerations which ultimately prove to be of the greatest importance. One instance of this occurs with reference to the secular acceleration of the Moon's mean motion. Although this acceleration, and the diminution of the eccentricity of the Earth's orbit, on which it depends, had been made known by observation as separate facts, yet many of the first geometers altogether failed to trace any connexion between them, and it was only after making repeated attempts to explain the phenomenon by other means, that Laplace himself succeeded in referring it to its true cause.

2. The accurate determination of the amount of the acceleration is a matter of very great importance. The effect of an error in any of the periodic inequalities upon the Moon's place, is always confined within certain limits, and takes place alternately in opposite directions within very moderate intervals of time, whereas the effect of an error in the acceleration goes on increasing for an almost indefinite period, so that the calculation of the Moon's place for a very distant epoch, such as that of the eclipse of Thales, may be seriously vitiated by it.

In the *Mécanique Céleste*, the approximation to the value of the acceleration is confined to the principal term, but in the theories of Damoiseau and Plana the developments are carried to an immense extent, particularly in the latter, where the multiplier of the change in the square of the eccentricity of the Earth's orbit, which occurs in the expression of the secular acceleration, is developed to terms of the seventh order.

As these theories agree in principle, and only differ slightly in the numerical value which they assign to the acceleration, and as they passed under the examination of Laplace, with especial reference to this subject, it might be supposed that at most only some small numerical corrections would be required in order to obtain a very exact determination of the amount of this acceleration.

It has therefore not been without some surprise, that I have lately found that Laplace's explanation of the phenomenon in question is essentially incomplete, and that the numerical results of Damoiseau's and Plana's theories, with reference to it, consequently require to be very sensibly altered.

3. Laplace's explanation may be briefly stated as follows. He shews that the mean central disturbing force of the Sun, by which the Moon's gravity towards the Earth is diminished, depends not only on the Sun's mean distance, but also on the eccentricity of the Earth's orbit. Now this eccentricity is at present, and for many ages has been, diminishing, while the mean distance remains unaltered. In consequence of this the mean disturbing force is also diminishing, and therefore the Moon's gravity towards the Earth at a given distance is, on the whole, increasing. Also, the area described in a given time by the Moon about the Earth is not affected by this alteration of the central force; whence it readily follows that the Moon's mean distance from the Earth will be diminished in the same ratio as the force at a given distance is increased, and that the mean angular motion will be increased in double the same ratio.

4. This is the main principle of Laplace's analytical method, in which he is followed by Damoiseau and Plana; but it will be observed, that this reasoning supposes that the area described by the Moon in a given time is not permanently altered, or in other words, that the tangential disturbing force produces no permanent effect. On examination, however, it will be found that this is not strictly true, and I will endeavour briefly to point out the manner in which the inequalities of the Moon's motion are modified by a gradual change of the central disturbing force, so as to give rise to such an alteration of the areal velocity.

As an example, I will take the *Variation*, the most direct effect of the disturbing force.

In the ordinary theory, the orbit of the Moon as affected by this inequality only, would be symmetrical with respect to the line of conjunction with the Sun, and the areal velocity generated while the Moon was moving from quadrature to syzygy, would be exactly destroyed while it was moving from syzygy to quadrature, so that no permanent alteration of areal velocity would be produced.

In reality, however, the magnitude of the disturbing force by which this inequality is caused, depends in some degree on the eccentricity of the Earth's orbit, and as this is continually diminishing, the central disturbing forces at equal angular distances on opposite sides of conjunction will not be exactly equal. Hence the orbit will no longer be symmetrically situated with respect to the line of conjunction. Now the change of areal velocity produced by the tangential force at any point, depends partly on the value of the radius vector at that point, and consequently the effects of the tangential force before and after conjunction will no longer exactly balance each other.

The other inequalities of the Moon's motion will be similarly modified, especially those which depend, more directly, on the eccentricity of the Earth's orbit, so that each of them gives rise to an uncompensated change of the areal velocity.

Since the distortion in the form of the orbit just pointed out is due to the alteration of the disturbing force consequent upon a change in the eccentricity of the Earth's orbit, and it is by virtue of this distortion that the tangential force produces a permanent change in the rate of description of areas, it follows that this alteration of the areal velocity will be of the order of the square of the disturbing force multiplied by the rate of change of the Earth's eccentricity.

It is evident that the amount of the acceleration of the Moon's mean motion will be directly affected by this alteration of areal velocity.

5. Having thus briefly indicated the way in which the effect now treated of originates, I will proceed with the analytical investigation of its amount.

In the present communication, however, I shall confine my attention to the principal term of the change thus produced in the acceleration of the Moon's motion, deferring to another, though I hope not a distant, opportunity, the fuller development of this subject, as well as the consideration of the secular variations of the other elements of the Moon's orbit arising from the same cause.

In what follows, the notation, except when otherwise explained, is the same as that of Damoiseau's *Théorie de la Lune.*

6. If we suppose the Moon to move in the plane of the ecliptic, and also neglect the terms depending on the Sun's parallax, the differential equations of the Moon's motion become

$$0=\frac{d^2u}{d\nu^2}+u-\frac{1}{h^2}+\frac{m'u'^3}{2h^2u^3}+\frac{3}{2}\frac{m'u'^3}{h^2u^3}\cos(2\nu-2\nu')$$

$$-\frac{3}{2}\frac{m'u'^3}{h^2u^4}\frac{du}{d\nu}\sin(2\nu-2\nu')-\frac{3m'}{h^2}\left(u+\frac{d^2u}{d\nu^2}\right)\int\frac{u'^3\,d\nu}{u^4}\sin(2\nu-2\nu')$$

$$\frac{dt}{d\nu}=\frac{1}{hu^2}+\frac{3}{2}\frac{m'}{h^3u^2}\int\frac{u'^3\,d\nu}{u^4}\sin(2\nu-2\nu')+\frac{27}{8}\frac{m'^2}{h^5u^2}\left[\int\frac{u'^3\,d\nu}{u^4}\sin(2\nu-2\nu')\right]^2.$$

In the solution usually given of these equations, u is expressed by means of a constant part and a series involving *cosines* of angles composed of multiples of $2\nu-2m\nu$, $c\nu-\varpi$, and $c'm\nu-\varpi'$; also t is expressed by means of a part proportional to ν and a series involving *sines* of the same angles; the coefficients of the periodic terms being functions of m, e and e'. Now if e' be a constant quantity, this is the true form of the solution, but if e' be variable, it is impossible to satisfy the differential equations without adding to the expression for u a series of small supplementary terms depending on the *sines* of the angles whose cosines are already involved in it, and to that for t, similar terms depending on the *cosines* of the same angles, the coefficients of these new terms involving $\frac{de'}{dt}$ as a factor.

The quantity $\int\frac{u'^3\,d\nu}{u^4}\sin(2\nu-2\nu')$, which occurs in the above equations, is proportional to the variable part of the square of the areal velocity, and consists, in the ordinary theory, of a series of periodic terms involving *cosines* of the angles above mentioned. In consequence, however, of the existence of the new terms just described, there will be added to it a series of small terms involving *sines* of the same angles, together with a non-periodic part of the form $\int He'de'$ or $\frac{1}{2}He'^2$. The introduction of this term will evidently change the relation between the non-periodic part of $\frac{dt}{dv}$ and e'^2, upon which the secular acceleration depends.

7. We must commence by finding the new terms to be added to the ordinary expression for u.

For the sake of simplification we will neglect the eccentricity of the Moon's orbit.

Let $\frac{1}{a}$ denote the non-periodic part of u, and $\frac{1}{a}+\delta u$ the complete value.

Then by substitution in the equation for u, making use of Damoiseau's developments of the undisturbed values of the several functions of u, u', and $v-v'$ which occur in it, putting $h^2=a_{,}$, and writing, for convenience, mv instead of $\int mdv+\lambda$, and $c'mv$ instead of $c'\int mdv+\lambda-\varpi'$ (as in Plana, vol. I. p. 322), we obtain

$$0=\frac{d^2\left(\frac{1}{a}\right)}{dv^2}+\frac{1}{a}-\frac{1}{a_{,}}+\frac{d^2\delta u}{dv^2}+\delta u$$

$$+\frac{1}{2}\frac{\overline{m}^2}{a_{,}}\left(1+\frac{3}{2}e'^2\right)+\frac{3}{2}\frac{\overline{m}^2}{a_{,}}a'\delta u'+\frac{3}{2}\frac{\overline{m}^2}{a_{,}}e'\cos c'mv-\frac{3}{2}\frac{\overline{m}^2}{a_{,}}\{1+3e'\cos c'mv\}a\delta u$$

$$-\frac{3}{2}\overline{m}^2\frac{a}{a_{,}}\frac{d\left(\frac{1}{a}\right)}{dv}\sin(2v-2mv)+\frac{3}{2}\frac{\overline{m}^2}{a_{,}}\left(1-\frac{5}{2}e'^2\right)\cos(2v-2mv)$$

$$+\frac{21}{4}\frac{\overline{m}^2}{a_{,}}e'\cos(2v-2mv-c'mv)-\frac{3}{4}\frac{\overline{m}^2}{a_{,}}e'\cos(2v-2mv+c'mv)$$

$$-\frac{3\overline{m}^2}{a_{,}}\int dv\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{7}{2}e'\sin(2v-2mv-c'mv)\right.$$

$$\left.-\frac{1}{2}e'\sin(2v-2mv+c'mv)\right\}$$

$$-\frac{9}{2}\frac{\overline{m}^2}{a_,}\left\{\left(1-\frac{5}{2}e'^2\right)\cos(2\nu-2m\nu)+\frac{7}{2}e'\cos(2\nu-2m\nu-c'm\nu)\right.$$
$$\left.-\frac{1}{2}e'\cos(2\nu-2m\nu+c'm\nu)\right\}a\,\delta u$$

$$-\frac{3}{2}\frac{\overline{m}^2}{a_,}\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2\nu-2m\nu)+\frac{7}{2}e'\sin(2\nu-2m\nu-c'm\nu)\right.$$
$$\left.-\frac{1}{2}e'\sin(2\nu-2m\nu+c'm\nu)\right\}\frac{d(a\,\delta u)}{d\nu}$$

$$+12\frac{\overline{m}^2}{a_,}\int d\nu\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2\nu-2m\nu)+\frac{7}{2}e'\sin(2\nu-2m\nu-c'm\nu)\right.$$
$$\left.-\frac{1}{2}e'\sin(2\nu-2m\nu+c'm\nu)\right\}a\,\delta u$$

$$-\frac{3\overline{m}^2}{a_,}\left\{\frac{d^2(a\,\delta u)}{d\nu^2}+a\,\delta u\right\}\int d\nu\left\{\left(1-\frac{5}{2}e'^2\right)\sin(2\nu-2m\nu)\right.$$
$$\left.+\frac{7}{2}e'\sin(2\nu-2m\nu-c'm\nu)-\frac{1}{2}e'\sin(2\nu-2m\nu+c'm\nu)\right\}.$$

8. Also, assume

$$a\,\delta u=m^2\left(1-\frac{5}{2}e'^2\right)\cos(2\nu-2m\nu)+\alpha_{30}\frac{e'de'}{ndt}\sin(2\nu-2m\nu)$$
$$-\frac{3}{2}m^2e'\cos c'm\nu+\alpha_{16}\frac{de'}{ndt}\sin c'm\nu$$
$$+\frac{7}{2}m^2e'\cos(2\nu-2m\nu-c'm\nu)+\alpha_{33}\frac{de'}{ndt}\sin(2\nu-2m\nu-c'm\nu)$$
$$-\frac{1}{2}m^2e'\cos(2\nu-2m\nu+c'm\nu)+\alpha_{34}\frac{de'}{ndt}\sin(2\nu-2m\nu+c'm\nu),$$

where the coefficients of the terms involving cosines are those given by the ordinary theory, and α_{30}, α_{16}, α_{33}, and α_{34} are numerical quantities to be determined.

9. In developing the terms of the above equation, by the substitution of this value of $a\,\delta u$, the quantity $\frac{de'}{dt}$ may be considered constant, and $\frac{de'}{d\nu}$ must be expressed in terms of it.

Thus $\dfrac{de'}{dv} = \dfrac{ndt}{dv}\dfrac{de'}{ndt}$

$$= \frac{de'}{ndt}\left\{1 - \frac{11}{4} m^2 \cos(2v - 2mv) - \frac{77}{8} m^2 e' \cos(2v - 2mv - c'mv) + \frac{11}{8} m^2 e' \cos(2v - 2mv + c'mv)\right\}.$$

Also, integrating by parts, and putting 2 instead of $2-2m$, $2-3m$, and $2-m$ in the divisors introduced by integration, since we only want to find the terms of the lowest order which are multiplied by $\dfrac{de'}{dt}$, we obtain

$$-\frac{3\overline{m}^2}{a_{\prime}}\int dv \left\{\left(1 - \frac{5}{2}e'^2\right)\sin(2v - 2mv) + \frac{7}{2}e' \sin(2v - 2mv - c'mv) - \frac{1}{2}e' \sin(2v - 2mv + c'mv)\right\}$$

$$= \frac{3}{2}\frac{\overline{m}^2}{a_{\prime}}\left(1 - \frac{5}{2}e'^2\right)\cos(2v - 2mv) + \frac{21}{4}\frac{\overline{m}^2}{a_{\prime}} e' \cos(2v - 2mv - c'mv) - \frac{3}{4}\frac{\overline{m}^2}{a_{\prime}} e' \cos(2v - 2mv + c'mv)$$

$$+ \frac{15}{2}\frac{\overline{m}^2}{a_{\prime}}\int dv \frac{e'de'}{ndt}\frac{ndt}{dv}\cos(2v - 2mv) - \frac{21}{4}\frac{\overline{m}^2}{a_{\prime}}\int dv \frac{de'}{ndt}\frac{ndt}{dv}\cos(2v - 2mv - c'mv)$$

$$+ \frac{3}{4}\frac{\overline{m}^2}{a_{\prime}}\int dv \frac{de'}{ndt}\frac{ndt}{dv}\cos(2v - 2mv + c'mv).$$

And
$$a'\delta u' = 3m^2 e' \sin c'mv\,[-e' \sin c'mv]$$
$$= -\frac{3}{2} m^2 e'^2,$$

retaining only the term which will be required.

10. When the proper substitutions are made, the terms involving cosines destroy each other, as in the usual theory, and by equating to zero the terms involving the sines, we obtain

$$20m^2 - 3a_{30} + \frac{15}{4} m^2 = 0,$$

or
$$3a_{30} = \frac{95}{4} m^2 \quad \therefore \; a_{30} = \frac{95}{12} m^2$$

$$3m^3 + a_{16} = 0 \quad \therefore \; a_{16} = -3m^3$$

$$-14m^2 - 3a_{33} - \frac{21}{8} m^2 = 0,$$

or
$$3a_{33} = -\frac{133}{8} m^2 \quad \therefore \; a_{33} = -\frac{133}{24} m^2$$

$$2m^2 - 3a_{34} + \frac{3}{8} m^2 = 0,$$

or
$$3a_{34} = \frac{19}{8} m^2 \quad \therefore \; a_{34} = \frac{19}{24} m^2.$$

11. In order to obtain the relation between a and $a_{,}$, we must substitute the value just found for $a\,\delta u$, in the same equation, and equate to zero the non-periodic part, observing that the terms

$$12 \frac{\overline{m}^2}{a_{,}} \int dv \left\{ \left(1 - \frac{5}{2} e'^2\right) \sin(2v - 2mv) + \frac{7}{2} e' \sin(2v - 2mv - c'mv) \right.$$
$$\left. - \frac{1}{2} e' \sin(2v - 2mv + c'mv) \right\} a\,\delta u$$

give

$$\frac{12\overline{m}^2}{a_{,}} \int dv \left\{ \frac{95}{24} m^2 \frac{e'de'}{ndt} - \frac{931}{96} m^2 \frac{e'de'}{ndt} - \frac{19}{96} m^2 \frac{e'de'}{ndt} \right\}$$
$$= -\frac{285}{4} \frac{m^4}{a_{,}} \int ndt \, \frac{e'de'}{ndt} \text{ nearly,}$$
$$= -\frac{285}{8} \frac{m^4}{a_{,}} e'^2 \text{ as their non-periodic part.}$$

Also the terms

$$\frac{15}{2} \frac{\overline{m}^2}{a_{,}} \int dv \frac{e'de'}{ndt} \frac{ndt}{dv} \cos(2v - 2mv) - \frac{21}{4} \frac{\overline{m}^2}{a_{,}} \int dv \frac{de'}{ndt} \frac{ndt}{dv} \cos(2v - 2mv - c'mv)$$
$$+ \frac{3}{4} \frac{\overline{m}^2}{a_{,}} \int dv \frac{de'}{ndt} \frac{ndt}{dv} \cos(2v - 2mv + c'mv)$$

of Art. 9, similarly give

$$\frac{15}{2}\frac{\overline{m}^2}{a_,}\int dv\left(-\frac{11}{8}m^2\frac{e'de'}{ndt}\right)-\frac{21}{4}\frac{\overline{m}^2}{a_,}\int dv\left(-\frac{77}{16}m^2\frac{e'de'}{ndt}\right)+\frac{3}{4}\frac{\overline{m}^2}{a_,}\int dv\left(\frac{11}{16}m^2\frac{e'de'}{ndt}\right)$$

$$=-\frac{165}{32}\frac{m^4}{a_,}e'^2+\frac{1617}{128}\frac{m^4}{a_,}e'^2+\frac{33}{128}\frac{m^4}{a_,}e'^2 \text{ nearly}$$

$$=\frac{495}{64}\frac{m^4}{a_,}e'^2 \text{ as their non-periodic part.}$$

12. Hence we obtain

$$0=\frac{1}{a}-\frac{1}{a_,}+\frac{1}{2}\frac{\overline{m}^2}{a_,}\left(1+\frac{3}{2}e'^2\right)-\frac{9}{4}\frac{m^4}{a_,}e'^2+\frac{495}{64}\frac{m^4}{a_,}e'^2+\frac{27}{8}\frac{m^4}{a_,}e'^2$$

$$-\frac{9}{4}\frac{m^4}{a_,}(1-5e'^2)-\frac{441}{16}\frac{m^4}{a_,}e'^2-\frac{9}{16}\frac{m^4}{a_,}e'^2$$

$$+\frac{3}{2}\frac{m^4}{a_,}(1-5e'^2)+\frac{147}{8}\frac{m^4}{a_,}e'^2+\frac{3}{8}\frac{m^4}{a_,}e'^2-\frac{285}{8}\frac{m^4}{a_,}e'^2$$

$$-\frac{9}{4}\frac{m^4}{a_,}(1-5e'^2)-\frac{441}{16}\frac{m^4}{a_,}e'^2-\frac{9}{16}\frac{m^4}{a_,}e'^2,$$

or

$$0=\frac{1}{a}-\frac{1}{a_,}\left\{1-\frac{1}{2}\overline{m}^2-\frac{3}{4}\overline{m}^2e'^2+3m^4+\frac{3153}{64}m^4e'^2\right\}.$$

Now $\overline{m}^2=\dfrac{m^2}{(1+p)^3}$ in Plana's notation, or (substituting the value of p given in Plana, Vol. ii. p. 855),

$$\overline{m}^2=m^2\left(1-\frac{1}{2}m^2-\frac{3}{4}m^2e'^2\right) \text{ nearly,}$$

$$\therefore\ \frac{1}{a}=\frac{1}{a_,}\left\{1-\frac{1}{2}m^2+\frac{13}{4}m^4-\frac{3}{4}m^2e'^2+\frac{3201}{64}m^4e'^2\right\}$$

and

$$a^2=a_,^2\left\{1+m^2-\frac{23}{4}m^4+\frac{3}{2}m^2e'^2-\frac{3129}{32}m^4e'^2\right\}.$$

13. Again, by substitution in the equation for $\dfrac{dt}{dv}$, we obtain

$$\frac{dt}{d\nu} = \frac{a^2}{\sqrt{a_{\prime}}} \left\{ 1 - 2a\,\delta u + \frac{3}{2} m^4 (1 - 5e'^2) + \frac{27}{8} m^4 e'^2 + \frac{147}{8} m^4 e'^2 + \frac{3}{8} m^4 c'^2 \right.$$

$$+ \frac{3}{2} \overline{m}^2 \frac{a}{a_{\prime}} \int d\nu \left[\left(1 - \frac{5}{2} e'^2\right) \sin (2\nu - 2m\nu) + \frac{7}{2} e' \sin (2\nu - 2m\nu - c'm\nu) \right.$$

$$\left. - \frac{1}{2} e' \sin (2\nu - 2m\nu + c'm\nu) \right]$$

$$- 3\overline{m}^2 \frac{a}{a_{\prime}} a\,\delta u \int d\nu \left[\left(1 - \frac{5}{2} e'^2\right) \sin (2\nu - 2m\nu) + \frac{7}{2} e' \sin (2\nu - 2m\nu - c'm\nu) \right.$$

$$\left. - \frac{1}{2} e' \sin (2\nu - 2m\nu + c'm\nu) \right]$$

$$- 6\overline{m}^2 \frac{a}{a_{\prime}} \int d\nu \left[\left(1 - \frac{5}{2} e'^2\right) \sin (2\nu - 2m\nu) + \frac{7}{2} e' \sin (2\nu - 2m\nu - c'm\nu) \right.$$

$$\left. - \frac{1}{2} e' \sin (2\nu - 2m\nu + c'm\nu) \right] a\,\delta u$$

$$+ \frac{27}{8} \overline{m}^4 \left(\frac{a}{a_{\prime}}\right)^2 \left\{ \int d\nu \left[\left(1 - \frac{5}{2} e'^2\right) \sin (2\nu - 2m\nu) + \frac{7}{2} e' \sin (2\nu - 2m\nu - c'm\nu) \right. \right.$$

$$\left. \left. \left. - \frac{1}{2} e' \sin (2\nu - 2m\nu + c'm\nu) \right] \right\}^2 \right\}.$$

14. Develope this equation as before, retaining m^4 only when it occurs in the non-periodic part, and we have

$$\frac{dt}{d\nu} = \frac{a^2}{\sqrt{a_{\prime}}} \left\{ 1 - 2a\,\delta u + \frac{3}{2} m^4 + \frac{3}{4} m^4 (1 - 5e'^2) + \frac{27}{64} m^4 (1 - 5e'^2) - \frac{495}{128} m^4 e'^2 \right.$$

$$+ \frac{117}{8} m^4 e'^2 + \frac{147}{16} m^4 e'^2 + \frac{3}{16} m^4 e'^2 + \frac{285}{16} m^4 e'^2 + \frac{1323}{256} m^4 e'^2 + \frac{27}{256} m^4 e'^2$$

$$- \frac{3}{4} m^2 \left(1 - \frac{5}{2} e'^2\right) \cos (2\nu - 2m\nu) - \frac{21}{8} m^2 e' \cos (2\nu - 2m\nu - c'm\nu)$$

$$+ \frac{3}{8} m^2 e' \cos (2\nu - 2m\nu + c'm\nu)$$

$$- \frac{15}{8} m^2 \frac{e'de'}{ndt} \sin (2\nu - 2m\nu) + \frac{21}{16} m^2 \frac{de'}{ndt} \sin (2\nu - 2m\nu - c'm\nu)$$

$$\left. - \frac{3}{16} m^2 \frac{de'}{ndt} \sin (2\nu - 2m\nu + c'm\nu) \right\},$$

$$\text{or}\quad \frac{dt}{dv}=\frac{a^2}{\sqrt{a_,}}\left\{1+\frac{171}{64}m^4+\frac{2391}{64}m^4e'^2\right.$$

$$-\frac{11}{4}m^2\left(1-\frac{5}{2}e'^2\right)\cos(2v-2mv)-\frac{425}{24}m^2\frac{e'de'}{ndt}\sin(2v-2mv)$$

$$+3m^2e'\cos c'mv+6m^3\frac{de'}{ndt}\sin c'mv$$

$$-\frac{77}{8}m^2e'\cos(2v-2mv-c'mv)+\frac{595}{48}m^2\frac{de'}{ndt}\sin(2v-2mv-c'mv)$$

$$\left.+\frac{11}{8}m^2e'\cos(2v-2mv+c'mv)-\frac{85}{48}m^2\frac{de'}{ndt}\sin(2v-2mv+c'mv)\right\}.$$

15. Substitute the value before found for a^2 in terms of $a_,^2$;

$$\therefore\quad \frac{dt}{dv}=a_,^{\frac{3}{2}}\left\{1+m^2-\frac{197}{64}m^4+\frac{3}{2}m^2e'^2-\frac{3867}{64}m^4e'^2\right.$$

$$-\frac{11}{4}m^2\left(1-\frac{5}{2}e'^2\right)\cos(2v-2mv)-\frac{425}{24}m^2\frac{e'de'}{ndt}\sin(2v-2mv)$$

$$+3m^2e'\cos c'mv+6m^3\frac{de'}{ndt}\sin c'mv$$

$$-\frac{77}{8}m^2e'\cos(2v-2mv-c'mv)+\frac{595}{48}m^2\frac{de'}{ndt}\sin(2v-2mv-c'mv)$$

$$\left.+\frac{11}{8}m^2e'\cos(2v-2mv+c'mv)-\frac{85}{48}m^2\frac{de'}{ndt}\sin(2v-2mv+c'mv)\right\}.$$

16. Now, put $\dfrac{1}{n}=a_,^{\frac{3}{2}}\left\{1+m^2-\dfrac{197}{64}m^4+\dfrac{3}{2}m^2e'^2-\dfrac{3867}{64}m^4e'^2\right\}$,

multiply by n, and integrate;

$$\therefore\quad \int ndt=v-\frac{11}{8}m^2\left(1-\frac{5}{2}e'^2\right)\sin(2v-2mv)+\frac{295}{24}m^2\frac{e'de'}{ndt}\cos(2v-2mv)$$

$$+3me'\sin c'mv+3\frac{de'}{ndt}\cos c'mv$$

$$-\frac{77}{16}m^2e'\sin(2v-2mv-c'mv)-\frac{413}{48}m^2\frac{de'}{ndt}\cos(2v-2mv-c'mv)$$

$$+\frac{11}{16}m^2e'\sin(2v-2mv+c'mv)+\frac{59}{48}m^2\frac{de'}{ndt}\cos(2v-2mv+c'mv).$$

17. In the expression for $\frac{1}{n}$ just found, $a_{,}$ is absolutely constant, but e' is variable, consequently n will vary, and therefore m likewise, which is connected with it by the equation $m = \frac{n'}{n}$.

Taking the variation of the equation for n, and observing that

$$\frac{\delta m}{m} = -\frac{\delta n}{n},$$

we have
$$0 = \frac{\delta n}{n}(1 - m^2) + \left(\frac{3}{2}m^2 - \frac{3867}{64}m^4\right)\delta(e'^2),$$

$$\therefore \frac{\delta n}{n} = -\left(\frac{3}{2}m^2 - \frac{3771}{64}m^4\right)\delta(e'^2).$$

Therefore, if N be the initial value of n, and E' the corresponding value of e',

$$n = N - \left(\frac{3}{2}m^2 - \frac{3771}{64}m^4\right)n(e'^2 - E'^2),$$

and
$$\int n\,dt = Nt + \epsilon - \left(\frac{3}{2}m^2 - \frac{3771}{64}m^4\right)\int (e'^2 - E'^2)\,n\,dt.$$

Hence the expression for the true longitude in terms of the mean, contains the secular equation

$$-\left(\frac{3}{2}m^2 - \frac{3771}{64}m^4\right)\int (e'^2 - E'^2)\,n\,dt.$$

18. According to Plana, the corresponding terms in the expression for the secular equation are

$$-\left(\frac{3}{2}m^2 - \frac{2187}{128}m^4\right)\int (e'^2 - E'^2)\,n\,dt.$$

Hence we see that the terms now taken into consideration have the effect of making the second term of the secular equation more than three times as great as it would otherwise be. Of course, the succeeding terms will also be materially changed.

The principal term of the correction to be applied to Plana's value of the secular acceleration is therefore

$$\frac{5355}{128}m^4\int (e'^2 - E'^2)\,n\,dt.$$

Now $$\int (e'^2 - E'^2)\, n\, dt = -1270'' \left(\frac{t}{100}\right)^2 \text{ nearly,}$$
where t is expressed in years; therefore the numerical value of this term is
$$-1''\cdot 66 \left(\frac{t}{100}\right)^2.$$
This result will serve to give an idea of the numerical importance of the new terms to be added to the received value of the secular acceleration, and probably will not differ widely from the complete correction; though in order to obtain a value sufficiently accurate to be definitely used in the calculation of ancient eclipses, the approximation must be carried considerably further.

The new periodic terms added to the Moon's longitude are perfectly insignificant, the coefficient of that involving $\cos c'm\nu$, which is by far the largest of them, only amounting to $0''\cdot 003$.

19. Transforming the expressions found above, so as to obtain the Moon's longitude and radius vector in terms of the time, and writing for convenience nt instead of $\int n\,dt + \epsilon$, mnt instead of $mnt + \epsilon'$, and $c'mnt$ instead of $c'mnt + \epsilon' - \varpi'$, we have
$$\begin{aligned}
\nu = nt &+ \frac{11}{8} m^2 \left(1 - \frac{5}{2} e'^2\right) \sin (2 - 2m)\, nt - \frac{74}{3} m^2 \frac{e'\,de'}{n\,dt} \cos (2 - 2m)\, nt \\
&- 3me' \sin c'mnt - 3 \frac{de'}{ndt} \cos c'mnt \\
&+ \frac{77}{16} m^2 e' \sin (2 - 2m - c'm)\, nt + \frac{215}{48} m^2 \frac{de'}{n\,dt} \cos (2 - 2m - c'm)\, nt \\
&- \frac{11}{16} m^2 e' \sin (2 - 2m + c'm)\, nt - \frac{257}{48} m^2 \frac{de'}{n\,dt} \cos (2 - 2m + c'm)\, nt
\end{aligned}$$
$$\begin{aligned}
\frac{a}{r} = au = 1 &- \frac{11}{8} m^4 - \frac{201}{16} m^4 e'^2 \\
&+ m^2 \left(1 - \frac{5}{2} e'^2\right) \cos (2 - 2m)\, nt + \frac{203}{12} m^2 \frac{e'\,de'}{n\,dt} \sin (2 - 2m)\, nt \\
&- \frac{3}{2} m^2 e' \cos c'mnt - 3m^3 \frac{de'}{n\,dt} \sin c'mnt \\
&+ \frac{7}{2} m^2 e' \cos (2 - 2m - c'm)\, nt - \frac{61}{24} m^2 \frac{de'}{n\,dt} \sin (2 - 2m - c'm)\, nt \\
&- \frac{1}{2} m^2 e' \cos (2 - 2m + c'm)\, nt + \frac{91}{24} m^2 \frac{de'}{n\,dt} \sin (2 - 2m + c'm)\, nt.
\end{aligned}$$

20. The existence of the new terms in the expressions for the Moon's coordinates occurred to me some time since, when I was engaged in thinking over a new method of treating the lunar theory, though I did not then perceive their important bearing on the value of the secular equation.

My attention was first directed to this latter subject while endeavouring to supply an omission in the theory of the Moon given by Pontécoulant in his *Théorie Analytique.* In this valuable work, the author, following the example originally set by Sir J. Lubbock in his Tracts on the Lunar Theory, obtains directly the expressions for the Moon's coordinates in terms of the time, which are found in Plana's theory by means of the reversion of series. With respect to the secular acceleration of the mean motion, however, Pontécoulant unfortunately adopts Plana's result without examination. On performing the calculation requisite to complete this part of the theory, I was surprised to find that the second term of the expression for the secular acceleration thus obtained, not only differed totally in magnitude from the corresponding term given by Plana, but was even of a contrary sign. My previous researches, however, immediately led me to suspect what was the origin of this discordance, and when both processes were corrected by taking into account the new terms whose existence I had already recognized, I had the satisfaction of finding a perfect agreement between the results.

[*Abstract.*]

THE author remarks, that in treating a great problem of approximation, such as that presented to us by the investigation of the Moon's motion, experience shews that nothing is more easy than to neglect, on account of their apparent insignificance, considerations which ultimately prove to be of the greatest importance. One instance of this occurs with reference to the secular acceleration of the Moon's mean motion. Although this acceleration and the diminution of the eccentricity of the Earth's orbit, on which it depends, had been made known by observation as separate facts, yet many of the first geometers altogether failed to trace any connexion between them, and it was not until he had made repeated attempts to explain the phenomenon by other means, that Laplace himself succeeded in referring it to its true cause.

The accurate determination of the amount of the acceleration is a matter of very great importance. The effect on the Moon's place, of an error in any of the periodic inequalities, is always confined within certain limits, and takes place alternately in opposite directions within very moderate intervals of time, whereas the effect of an error in the acceleration goes on increasing for an almost indefinite period, so as to render it impossible to connect observations made at very distant times.

In the *Mécanique Céleste*, the approximation to the value of the acceleration is confined to the principal term, but in the theories of Damoiseau and Plana, the developments are carried to an immense extent, particularly in the latter, where the multiplier of the change in the square of the eccentricity of the Earth's orbit, which occurs in the expression of the secular acceleration, is given to terms of the seventh order.

As these theories agree in principle, and only differ slightly in the numerical value which they assign to the acceleration, and as they passed under the examination of Laplace, with especial reference to this subject, it might be supposed that only some small numerical rectifications would be required in order to obtain a very exact determination of this value.

It has not been, therefore, without surprise, which he has no doubt will be shared by the Society, that the author has lately found that Laplace's explanation of the phenomenon in question is essentially incomplete, and that the numerical results of Damoiseau's and Plana's theories, with reference to it, consequently require to be very sensibly altered.

Laplace's explanation may be briefly stated as follows. He shews that the mean central disturbing force of the Sun, by which the Moon's gravity towards the Earth is diminished, depends not only on the Sun's mean distance, but also on the eccentricity of the Earth's orbit. Now this eccentricity is at present (and for many ages has been) diminishing, while the mean distance remains unaltered. In consequence of this, the mean disturbing force is also diminishing, and therefore the Moon's gravity towards the Earth at a given distance, is, on the whole, increasing. Also the area described in a given time by the Moon about the Earth is not affected by this alteration of the central force; whence it readily follows that the Moon's mean distance from the Earth will be diminished in the same ratio as the force at a given distance is increased, and the mean angular motion will be increased in double the same ratio.

This, the author states, is the main principle of Laplace's analytical method, in which he is followed by Damoiseau and Plana; but it will be observed that this reasoning supposes that the area described by the Moon in a given time is not permanently altered, or, in other words, that the tangential disturbing force produces no permanent effect. On examination, however, he remarks it will be found that this is not strictly true, and he proceeds briefly to point out the manner in which the inequalities of the Moon's motion are modified by a gradual change of the disturbing force, so as to give rise to such an alteration of the areal velocity.

As an example, he takes the case of the *Variation*, the most direct effect of the disturbing force. In the ordinary theory, the orbit of the Moon, as affected by this inequality only, would be symmetrical with respect to the line of conjunction with the Sun, and the areal velocity generated while the Moon was moving from quadrature to syzygy, would be exactly destroyed while it was moving from syzygy to quadrature, so that no permanent alteration would be produced.

In reality, however, the magnitude of the disturbing force by which this inequality is caused, depends in some degree on the eccentricity of the

Earth's orbit; and as this is continually diminishing, the disturbing forces at equal intervals before and after conjunction will not be exactly equal. Hence the orbit will no longer be symmetrically situated with respect to the line of conjunction, and therefore the effects of the tangential force before and after conjunction no longer exactly balance each other.

The other inequalities of the Moon's motion will be similarly modified, especially those which depend, more directly, on the eccentricity of the Earth's orbit, so that each of them will give rise to an uncompensated change of the areal velocity, and all of these must be combined in order to ascertain the total effect.

Since the distortion of the orbit just pointed out is due to the change of the disturbing force consequent upon a change in the eccentricity of the Earth's orbit, and the action of the tangential force, permanently to change the rate of description of areas, is only brought into play by means of this distortion, it follows that the alteration of the areal velocity will be of the order of the square of the disturbing force multiplied by the rate of change of the square of the eccentricity. It is evident that this alteration of areal velocity will have a direct effect in changing the acceleration of the Moon's mean motion.

Having thus briefly indicated the way in which the effect now treated of originates, the author proceeds with the analytical investigation of its amount. In the present communication, however, he proposes to confine his attention to the principal term of the change thus produced in the acceleration of the Moon's mean motion, deferring to another, though he hopes not a distant opportunity, the fuller treatment of this subject, as well as the determination of the secular variations of the other elements of the Moon's motion, which, arising from the same cause, have also been hitherto overlooked.

In the usual theory, the reciprocal of the Moon's radius vector is expressed by means of a series of *cosines* of angles formed by combinations of multiples of the mean angular distance of the Moon from the Sun, of the mean anomalies of the Moon and Sun, and of the Moon's mean distance from the node; and the Moon's longitude is expressed by means of a series of *sines* of the same angles, the coefficients of the periodic terms being functions of the ratio of the Sun's mean motion to that of the Moon, of the eccentricities of the two orbits and of their mutual inclination.

Now, if the eccentricity of the Earth's orbit be supposed to remain constant, this is the true form of the expressions for the Moon's coordinates; but if that eccentricity be variable, the author shews that the differential equation cannot be satisfied without adding to the expression for the reciprocal of the radius vector, a series of small supplementary terms depending on the *sines* of the angles whose *cosines* are already involved in it, and to the expression for the longitude, a series of similar terms depending on the *cosines* of the same angles; all the coefficients of these new terms containing as a factor the differential coefficient of the eccentricity of the Earth's orbit taken with respect to the time.

The author first determines as many of these terms as are necessary in the order of approximation to which he restricts himself, and then takes them into account in the investigation of the secular acceleration. The expression which he thus obtains for the first two terms of this acceleration, is,

$$-\left(\frac{3}{2}m^2 - \frac{3771}{64}m^4\right)\int (e'^2 - E'^2)\, n\, dt.$$

According to Plana, the corresponding expression is

$$-\left(\frac{3}{2}m^2 - \frac{2187}{128}m^4\right)\int (e'^2 - E'^2)\, n\, dt.$$

It will be observed that the coefficient of the second term has been completely altered in consequence of the introduction of the new terms.

The numerical effect of this alteration is to diminish by $1''\cdot 66$ the coefficient of the square of the time in the expression for the secular acceleration; the time being, as usual, expressed in centuries.

It will, of course, be necessary to carry the approximation much further, in order to obtain such a value of this coefficient as may be employed with confidence in the calculation of ancient eclipses.

22.

ON THE SECULAR VARIATION OF THE ECCENTRICITY AND INCLINATION OF THE MOON'S ORBIT.

[From the *Monthly Notices of the Royal Astronomical Society* (1859). Vol. XIX.]

IN a memoir read before the Royal Society in June, 1853, I shewed that the secular variation of the Moon's mean motion is given by means of the equation

$$\frac{dn}{ndt} = \frac{e'de'}{dt}\left\{-3m^2 + \frac{3771}{32}m^4\right\},$$

in which the coefficient of m^4 is totally different from that in Plana's result.

I have since carried the approximation to the seventh order in m, and find that

$$\frac{dn}{ndt} = \frac{e'de'}{dt}\left\{-3m^2 + \frac{3771}{32}m^4 + \frac{34047}{32}m^5 + \frac{306865}{48}m^6 + \frac{17053741}{576}m^7\right\}.$$

This reduces the coefficient of $\left(\frac{t}{100}\right)^2$, in the expression for the acceleration to $5''{\cdot}7$, only about one-half of the value hitherto received*. M. Delaunay has recently verified my coefficient of m^4; and he informs me that he shall very soon have carried the approximation to the eighth order in m, and included the terms depending on e^2 and γ^2.

In my memoir above referred to I mentioned that other elements of the Moon's orbit suffer secular changes which had been overlooked.

* The first part of this Paper was communicated to the French Institute in January, 1859, and was published in the *Comptes Rendus*.

I find the following expressions for the secular variation of the eccentricity and inclination of the Moon's orbit, adopting Plana's definitions of e and γ:—

$$\frac{de}{dt} = ee'\frac{de'}{dt}\left\{\frac{235}{64}m^2\right\},$$

$$\frac{d\gamma}{dt} = \gamma e'\frac{de'}{dt}\left\{-\frac{221}{64}m^2 + \frac{779}{256}m^3 + \frac{199631}{4096}m^4\right\}.$$

I am engaged in carrying on the approximation to the value of $\frac{de}{dt}$ to the same extent as I have done in the case of $\frac{d\gamma}{dt}$, and in finding the part of the secular variation of the mean motion which depends on e^2 and γ^2. These terms, however, can only very slightly affect the numerical value of the secular acceleration.

Supplement to the foregoing.

Since I sent my result respecting the secular variations of the eccentricity and inclination of the Moon's orbit to the Society the other day, I have found the leading terms of the secular acceleration of the mean motion which depend on the eccentricity and inclination of the orbit. The result is one of remarkable simplicity, considering the nature of the calculations which have led to it; and I should be glad if you would let it appear in the *Monthly Notices* as soon as you conveniently can, as a supplement or a note to my former communication. The result is,

$$\frac{dn}{n\,dt} = \frac{e'de'}{dt}\left\{-3m^2 + \frac{3771}{32}m^4 + \text{\&c.} - \frac{27}{8}m^2e^2 + \frac{27}{8}m^2\gamma^2\right\}.$$

[I have not written down the coefficients of higher powers of m, as given in my former note.]

It is curious that the coefficients of e^2 and γ^2, in this expression, are equal and of contrary signs, although they are found by totally distinct processes. The effect of the terms in e^2 and γ^2 on the magnitude of the secular acceleration is, as I anticipated, very insignificant. The term in e^2 increases the coefficient of the square of the number of centuries by $0''\cdot036$, and that in γ^2 diminishes the same coefficient by $0''\cdot097$; so that, on the whole, the coefficient $5''\cdot70$, which I previously found, must be diminished by $0''\cdot06$, or reduced to $5''\cdot64$. This value I believe to be within one-tenth of a second of the true *theoretical* value of the coefficient of the secular acceleration. Whether ancient observations admit of such a small value of the acceleration is a different question.

23.

REPLY TO VARIOUS OBJECTIONS AGAINST THE THEORY OF THE SECULAR ACCELERATION OF THE MOON'S MEAN MOTION (WITH POSTSCRIPT.)

[From the *Monthly Notices of the Royal Astronomical Society* (1860). Vol. xx.]

IF I have hitherto published no reply to the "Observations" of M. de Pontécoulant, contained in the *Monthly Notices* of July last, it is not because the task presented any difficulty, for the fallacies which pervade M. de Pontécoulant's communication were perfectly evident to me from the very first. I thought that any competent person who chose to look into my Memoir "On the Secular Acceleration," and into these observations upon it, might be safely left to form his own judgment on the matter. Again, I had some hopes that M. de Pontécoulant might be led to see and acknowledge the errors into which he had fallen, and with that object in view I sent to him, on more than one occasion, through a friend, communications which appeared to me amply sufficient to expose the fallacies contained not only in his printed "Observations," but also in several private letters which he subsequently wrote upon the subject. I find, however, that M. de Pontécoulant, in a letter which he has lately caused to be circulated among the members of the French Institute, has ventured to ignore these communications of mine altogether, and to speak as if his observations had been admitted without dispute. Under these circumstances, as my further silence might be misconstrued, I beg leave to offer to the Society the following remarks.

In order to give a more complete view of the subject, however, and to obviate the necessity of my returning to it in a controversial manner, I shall not confine myself to the observations of M. de Pontécoulant, but shall likewise say a few words in reply to the objections of M. Plana and those of M. Hansen. I shall also take the opportunity of making some preliminary remarks which may tend to remove certain misapprehensions, which I have reason to believe exist in some minds with respect to the real nature of the matter in dispute.

First, then, I would call attention to the fact that the question is a purely mathematical one, with the decision of which observation has nothing whatever to do. It may be simply stated thus: if the eccentricity of the Earth's orbit be supposed to change at a given uniform rate and very slowly, what will be the corresponding rate of change, according to the theory of gravitation, in the mean motion of the Moon? Now the solution of this question is effected by means of a purely algebraical process, the validity of each step of which admits of being placed beyond all possible doubt.

What conclusion must be drawn, then, supposing that ancient observations should shew that the secular variation of the Moon's mean motion is different from that which, according to theory, is due to the known change of the eccentricity of the Earth's orbit?

Why, simply this; that the mean motion of the Moon is affected by some other cause or causes, besides the variation of eccentricity which has been taken into account. This fact, if established, would be a most interesting one, and might put us on the traces of an important physical discovery. It is not difficult to imagine the existence of causes which may affect the mean motion of the Moon, but whether it were so or not, any question respecting the validity of a mathematical process must be decided on mathematical grounds alone, quite independently of the agreement or disagreement of theory and observation.

In the case before us the mathematical question as stated above may be greatly simplified, without its ceasing to involve the point which is in dispute. The values of the secular acceleration given by M. Plana's theory and mine, differ in terms which are independent of the eccentricity and inclination of the Moon's orbit; consequently in deciding which of the theories is right, we may suppose the eccentricity and inclination to vanish.

In the next place I would remark that the error which I attribute to M. Plana's theory on this point is not one of calculation which might require long and complicated numerical processes to be gone through for its correction, but that it is an error of principle, about which a mathematician ought not to have much difficulty in making up his mind. I am therefore inclined entirely to agree with M. de Pontécoulant's opinion, that the prolonged discussion of this subject would not be creditable to science, and indeed, considering the importance of the question, and the length of time which has passed since the publication of my Memoir, I cannot but think it strange that any controversy respecting it should still exist at all.

Some persons appear to be under the impression that the contest lies between two values of the secular acceleration, that M. Delaunay and I agree in one value, and that MM. Plana, de Pontécoulant, and Hansen, agree in a larger value; but this is by no means the true state of the case. Between M. Delaunay's result and my own, indeed, there is a perfect agreement. He has carried the approximation much further than I have done, but all of the terms which I have calculated have been confirmed by him. Again, before publishing my Memoir in 1853, I had obtained my result by two different methods, and I have since confirmed and extended it by means of a third. M. Delaunay arrived at his result by an independent method of his own, and he has lately found exactly the same result by following the method given by Poisson.

On the other hand, among our opponents there is far from being the same satisfactory agreement.

In his theory of the Moon, M. Plana obtained one value of the secular acceleration. In 1856 he printed a paper in which he admitted that his theory was wrong on this point, and actually deduced my result from his own equations. Soon afterwards, however, M. Plana retracted his admission of the correctness of my result, and obtained a third result, differing both from his former one and from my own.

Again, M. de Pontécoulant, in the last communication which I received from him, gives two different values of the secular acceleration, one of which he has obtained by using the time, and the other by using the Moon's longitude as the independent variable. Strange to say, however, he does not appear at all startled at obtaining two contradictory values, but seems fully inclined to defend both. Indeed, judging from the last paragraph of his letter in the *Monthly Notices*, he appears to have expected that the results of the two methods would differ from each other. One of the values which M. de Pontécoulant thus obtains agrees with that given in M. Plana's theory, as of course it must do, being found by means of the same principles. But he seems to be quite unaware that this value has been abandoned by M. Plana himself in his last paper above referred to, which is contained in the eighteenth volume of the Turin *Memoirs*.

M. Hansen's value of the secular acceleration is not given in an analytical form, like those of MM. Plana and de Pontécoulant, and therefore we can only compare the final numerical results. This comparison, which I shall presently give, shews that M. Hansen's value of the acceleration considerably exceeds either of those found by M. Plana.

Here then we find nothing to inspire confidence; certainly nothing like the cumulative testimony which there is in support of M. Delaunay's result and mine.

I may now be permitted to make some remarks on another point. In the introduction to my Memoir of 1853, I gave some general reasoning to shew that a change in the eccentricity of the Earth's orbit had a tendency to produce a change in the mean areal velocity of the Moon, and that M. Plana was therefore wrong in assuming this velocity to be constant, as in his theory he does. Now this seems to have led some persons to imagine that my analysis in the following part of the memoir depended in some way or other on the validity of the general reasoning which had gone before, and therefore that my conclusions could not be regarded as established with mathematical strictness. But this is quite a mistaken view of the case. I make no assumption respecting the variability of the mean areal velocity. I prove mathematically that this velocity does vary by finding the amount of its variation, and the general reasoning given in the introduction is simply the translation, so to speak, of my analysis into ordinary language, in order to make the nature of my correction to M. Plana's theory more generally intelligible. It may be remarked too that even if I had started with the assumption that the mean areal velocity was variable, no error could have been caused thereby, for if this velocity had been really constant I should have found its variation equal to zero. In mathematics the terms "constant" and "variable" are not looked upon as opposed to each other, but a constant is regarded as a particular case of a variable quantity.

It may be as well to guard against the idea that the extreme minuteness of the quantities which we have to deal with in this investigation, gives rise to any uncertainty in the result. The present rate of approach of the Moon to the Earth which accompanies the acceleration of its motion, is less than one inch per annum, but the theory can determine this minute quantity to within, say, a thousandth part of its true amount, just as easily and certainly as if the quantity to be found had been any number of times greater.

I will now proceed briefly to explain the principles which I employ in determining the secular acceleration, and to point out the errors which vitiate the several results of MM. Plana and de Pontécoulant which have been already referred to.

The principle of my method is simply this, viz., that the differential equations must be satisfied, and that quantities which really vary must be treated as variable in all the differentiations and integrations which occur throughout the investigation.

Now if e', the eccentricity of the Earth's orbit, be variable, the differentiation or integration of any term which involves e' in its coefficient will produce, in addition to the term which would result if e' were constant, another term involving $\frac{de'}{dt}$ in its coefficient, supposing t to be the independent variable.

In consequence of the existence of these supplementary terms, the ordinary expressions for the Moon's coordinates when substituted in the differential equations will not satisfy them, but will leave terms multiplied by $\frac{de'}{dt}$ outstanding. In order to destroy these terms, it is necessary to add terms of the same form to the usual expressions for the Moon's coordinates. The values of these new terms may, if we please, be easily found by the method of indeterminate coefficients, each of the coefficients being obtained by means of a simple equation.

If n, the Moon's mean motion, be variable, the double differentiation of the Moon's coordinates will produce in the differential equations, terms involving $\frac{dn}{dt}$ of the same form as those already mentioned which involve $\frac{de'}{dt}$.

Thus the same system of simultaneous simple equations that gives the values of the indeterminate coefficients, determines likewise the value of $\frac{dn}{dt}$; which is what we want to find.

If the Moon's longitude ν be taken as the independent variable, we must proceed according to the same principles, but there is one additional circumstance to be attended to.

In the former case, since e' is supposed to vary uniformly with the time, $\frac{de'}{dt}$ is considered constant, or $\frac{d^2e'}{dt^2}=0$. In the latter case the terms which are introduced by the consideration of the variability of e' will involve $\frac{de'}{d\nu}$ instead of $\frac{de'}{dt}$ as before; and since the Moon's motion in longitude is not uniform, the value of $\frac{de'}{d\nu}$ cannot be considered constant, or $\frac{d^2e'}{d\nu^2}$ cannot be neglected. To take this into account we must substitute for $\frac{de'}{d\nu}$ its value $\frac{de'}{dt}\frac{dt}{d\nu}$, in which $\frac{dt}{d\nu}$ is a known function of ν, and then the remainder of the process will be exactly similar to that before described.

Let us now consider the method followed in M. Plana's theory, and also by M. de Pontécoulant.

In this method the terms above described involving $\frac{de'}{dt}$ are ignored, and consequently the differential equations as developed by these astronomers furnish no materials whatever for determining the value of $\frac{dn}{dt}$. Hence they are forced to supply the lack of data by means of an assumption, which is that one of the so-called constants introduced by integration is absolutely constant.

The value of any one of the constants so employed can be expressed in terms of n, e' and known quantities. If then this so-called constant were really so, we should be able by differentiating this relation to obtain $\frac{dn}{dt}$ in terms of $\frac{de'}{dt}$. But if on the other hand this supposed constant be really variable, we must take its variation into account, in order to obtain the true value of $\frac{dn}{dt}$ in terms of $\frac{de'}{dt}$.

In M. Plana's theory, in which ν is taken as the independent variable, the constant so employed is h^2, which is added to complete the integral $2\int r^2 \frac{dR}{d\nu} d\nu$, in the equation

$$r^4 \left(\frac{d\nu}{dt}\right)^2 = \mathrm{h}^2 + 2\int r^2 \frac{dR}{d\nu}\, d\nu,$$

in which $2\int r^2 \frac{dR}{d\nu} d\nu$ is supposed to consist of a series of cosines of multiples of ν.

The quantity $r^2 \frac{d\nu}{dt}$ is equal to twice the area described in a unit of time, or to twice the areal velocity, so that h^2 is the non-periodic part of the square of twice the areal velocity, the periodic part being supposed developed in cosines of multiples of ν.

In M. de Pontécoulant's theory, the constant h is introduced to complete the integral $\int \frac{dR}{d\nu} dt$ in the equation

$$r^2 \frac{d\nu}{dt} = h + \int \frac{dR}{d\nu}\, dt,$$

in which $\int \frac{dR}{d\nu} dt$ is supposed to consist of a series of cosines of multiples of t.

M. de Pontécoulant's h is not identical with M. Plana's h, but there is a simple relation between these quantities.

M. de Pontécoulant, however, does not employ the constant h in finding the value of the secular acceleration, but another constant $\frac{1}{a}$, which is introduced to complete the integral in the equation

$$\frac{1}{2}\frac{d^2(r^2)}{dt^2} - \frac{1}{r} + \frac{1}{a} = 2\int d'R + r\frac{dR}{dr},$$

all the periodic terms of which are supposed to consist of cosines of multiples of t.

If we neglect the eccentricity and inclination of the Moon's orbit, and also omit all powers of m above the fourth, the relations between these several constants and the mean motion n will be expressed as follows:

$$\mathrm{h} = n^{-\frac{1}{3}} \left\{ 1 - \frac{1}{3} m^2 + \frac{719}{576} m^4 + e'^2 \left[-\frac{1}{2} m^2 + \frac{2635}{384} m^4 \right] \right\},$$

$$h = n^{-\frac{1}{3}} \left\{ 1 - \frac{1}{3} m^2 + \frac{11}{144} m^4 + e'^2 \left[-\frac{1}{2} m^2 - \frac{185}{96} m^4 \right] \right\},$$

$$\frac{1}{a} = n^{\frac{2}{3}} \left\{ 1 + \frac{2}{3} m^2 - \frac{1253}{288} m^4 + e'^2 \left[m^2 - \frac{5593}{192} m^4 \right] \right\},$$

the sum of the masses of the Earth and Moon being supposed to be unity.

From these relations we find by differentiation

$$\frac{dn}{ndt} = -3 \frac{d\mathrm{h}}{\mathrm{h}dt} + \frac{d(e'^2)}{dt} \left\{ -\frac{3}{2} m^2 + \frac{2187}{128} m^4 \right\},$$

$$\frac{dn}{ndt} = -3 \frac{dh}{hdt} + \frac{d(e'^2)}{dt} \left\{ -\frac{3}{2} m^2 - \frac{297}{32} m^4 \right\},$$

$$\frac{dn}{ndt} = -\frac{3}{2} \frac{da}{adt} + \frac{d(e'^2)}{dt} \left\{ -\frac{3}{2} m^2 + \frac{5337}{128} m^4 \right\},$$

having taken care to observe that, since $m = \dfrac{n'}{n}$ and n' is constant, we have $\dfrac{dm}{mdt} = -\dfrac{dn}{ndt}$.

If $\dfrac{d\mathrm{h}}{\mathrm{h}dt}$ be neglected in the first of these expressions, we obtain the value of $\dfrac{dn}{ndt}$ found in M. Plana's theory, and one of those found by M. de Pontécoulant. If $\dfrac{dh}{hdt}$ be neglected in the second, the resulting value of $\dfrac{dn}{ndt}$ is what would have been found by M. de Pontécoulant, if he had taken his own h to be constant instead of M. Plana's h.

If in the third expression $\dfrac{da}{adt}$ be neglected, we obtain the value of $\dfrac{dn}{ndt}$ which M. de Pontécoulant communicated to me as the result which he had found by using t as the independent variable.

It is obvious that these several values of $\frac{dn}{ndt}$ contradict each other, and the reason is that the quantities h, h, and a are really variable, and that therefore $\frac{d\mathrm{h}}{\mathrm{h}dt}$, $\frac{dh}{hdt}$, and $\frac{da}{adt}$ have been wrongly neglected. In order to find the true value of $\frac{dn}{ndt}$ we must therefore determine the values of these last-mentioned differential coefficients, and substitute them in the several expressions for $\frac{dn}{ndt}$ given above.

Now the supplementary terms involving $\frac{de'}{dt}$ which I have shewn to exist in the expressions for the Moon's coordinates, will introduce into the integral

$$2\int r^2 \frac{dR}{dv}\,dv,$$

besides periodic terms, a non-periodic one of the form

$$\int H\frac{d(e'^2)}{dt}\,dt, \text{ or } He'^2,$$

consequently, since in the equation

$$r^4\left(\frac{dv}{dt}\right)^2 = \mathrm{h}^2 + 2\int r^2\frac{dR}{dv}\,dv,$$

M. Plana considers h^2 to denote the whole of the non-periodic part of $r^4\left(\frac{dv}{dt}\right)^2$, h^2 must consist of an absolutely constant part together with the variable quantity He'^2 just mentioned

$$\text{and } \therefore \frac{d(\mathrm{h}^2)}{dt} \text{ must be equal to } H\frac{d(e'^2)}{dt}.$$

Similarly $\frac{dh}{dt}$ may be found by determining the non-periodic term which is in the same way introduced into the integral

$$\int \frac{dR}{dv}\,dt$$

in the equation

$$r^2\frac{dv}{dt} = h + \int\frac{dR}{dv}\,dt;$$

and $\dfrac{d\left(\frac{1}{a}\right)}{dt}$ may be similarly found by means of the non-periodic terms introduced into the integral $\int d'R$, in the equation

$$\frac{1}{2}\frac{d^2(r^2)}{dt^2} - \frac{1}{r} + \frac{1}{a} = 2\int d'R + r\frac{dR}{dr}.$$

When all this has been done, and the proper substitutions made, the three expressions for $\dfrac{dn}{ndt}$ are found to agree in giving

$$\frac{dn}{ndt} = \frac{d(e'^2)}{dt}\left\{-\frac{3}{2}m^2 + \frac{3771}{64}m^4\right\},$$

which is the result obtained by M. Delaunay and myself.

The supplementary terms in the Moon's coordinates which involve $\dfrac{de'}{dt}$ are of the order of the disturbing force, and therefore the terms which they introduce into the integrals,

$$\int r^2\frac{dR}{dv}\,dv, \quad \int\frac{dR}{dv}\,dt, \text{ and } \int d'R,$$

will be the order of the square of the disturbing force.

This is the reason why $\dfrac{d\mathrm{h}}{\mathrm{h}dt}$, $\dfrac{dh}{hdt}$, and $\dfrac{da}{adt}$ are all of the order m^4.

It may be well to mention, in order to prevent any misapprehension, that in my Memoir of 1853, h has not the same signification as the h of M. Plana's theory.

It is proved in Art. 11 of the Memoir that

$$2\int r^2\frac{dR}{dv}\,dv$$

contains the non-periodic terms

$$\mathrm{h}^2\left\{-\frac{285}{8}m^4e'^2 + \frac{495}{64}m^4e'^2\right\},$$

$$= \mathrm{h}^2\left\{-\frac{1785}{64}m^4e'^2\right\}$$

and the h^2 employed in the Memoir is the absolutely constant quantity added to complete the integral, so that if for the sake of distinction h_0^2 be written for the h^2 of the Memoir, we shall have

$$\mathrm{h}^2 = h_0^2 + \mathrm{h}^2 \left\{ -\frac{1785}{64} m^4 e'^2 \right\}$$

$$\text{or } \mathrm{h}^2 = h_0^2 \left\{ 1 - \frac{1785}{64} m^4 e'^2 \right\}.$$

The following relation exists between the h of M. Plana and the h of M. de Pontécoulant:—

$$\frac{\mathrm{h}}{h} = 1 + \frac{75}{64} m^4 + e'^2 \left[\frac{1125}{128} m^4 \right].$$

Now this relation at once shews that if e' be variable, h and h cannot both be constant; and since no *à-priori* reason can be given why one of these quantities should be constant rather than the other, we are not justified in assuming that either of them is so.

This argument, however, does not appear convincing to M. de Pontécoulant.

In the two methods which, as I mentioned before, I employed previously to the publication of my Memoir of 1853, the value of $\frac{dn}{ndt}$ was deduced from those of $\frac{d\mathrm{h}}{\mathrm{h}dt}$ and $\frac{dh}{hdt}$ respectively. In the method which I now employ, $\frac{dn}{ndt}$ is determined by direct substitution in the differential equations, without introducing either the quantity h or h, that is, without taking into consideration the mean areal velocity at all.

In M. Plana's Memoir, contained in the eighteenth volume of the Turin Memoirs, he no longer maintains the constancy of his quantity h, but he determines its variation incorrectly, only taking into account part of the terms which produce this variation. M. Plana here recognises the reality of the supplementary terms involving $\frac{de'}{dt}$, which I have proved to exist in the expressions for the Moon's coordinates; and he finds values for δu and δnt in pp. 14 and 20 of the Memoir, which coincide with mine, except in the terms with the argument $c'mv$, in which a mistake occurs in his coefficients, which, however, does not affect the coefficient of m^4 in the expression for the secular acceleration. It is very remarkable, however, that although he finds these values of δu and δnt, he does not substitute them in his equations, but puts $\delta u = 0$ and $\delta nt = 0$ instead of them. It is only by this strange process of suppressing part of the results which he himself has found, that M. Plana arrives at a different value of the secular acceleration from mine. Indeed, in the first form of this Memoir, as I have already mentioned, M. Plana did actually obtain a value coincident with mine.

M. Plana is led to make this suppression of his own results by a supposed *à-priori* proof that a certain integral which is equivalent to

$$2\int r^2 \frac{dR}{dv}\,dv$$

can contain no such terms as those which would arise from the substitution in it of the true values of δu and δnt. Now, even if this proof had been ever so convincing, M. Plana was surely bound to shew in what manner the terms thus arising from δu and δnt were destroyed, as the different parts of his investigation would otherwise contradict each other.

In fact, however, this proof is entirely fallacious, for it rests on the assumption made at the top of p. 43 of the Memoir, that the terms multiplied by p, p^2, &c., in the equation given on the preceding page, may be neglected; and these are precisely the terms which are equivalent to those which M. Plana suppresses.

It may be as well to make another remark on this part of the investigation. In p. 42, M. Plana puts

$$e'^g \cos g\tau = \Sigma M \cos(pv + q),$$

$$e'^g \sin g\tau = \Sigma M \sin(pv + q),$$

and he assumes that all the coefficients p will be small quantities. But this will not be the case when $e'^g \cos g\tau$ and $e'^g \sin g\tau$ are thus expressed in terms of the Moon's longitude. If these functions were similarly expressed in terms of the time, viz., if we were to put

$$e'^g \cos g\tau = \Sigma M \cos(pt + q),$$

$$e'^g \sin g\tau = \Sigma M \sin(pt + q),$$

all the coefficients p would be small.

The result which M. Plana obtains in this Memoir is

$$\frac{dn}{n\,dt} = \frac{d\,(e'^2)}{dt}\left\{-\frac{3}{2}m^2 + \frac{351}{64}m^4\right\},$$

and the difference between this result and mine arises in the way I have explained, viz., from his having neglected to take into account the term

$$\mathrm{h}^2\left\{-\frac{285}{8}m^4e'^2\right\}$$

which is shewn in Art. 11 of my Memoir to constitute part of the non-periodic term of $2\int r^2 \frac{dR}{dv}\,dv$.

M. Hansen's value of the secular acceleration is not exhibited in an analytical form, like those of MM. Plana and de Pontécoulant, and we can therefore only compare his numerical result with theirs. These differ considerably, and, in fact, much more than appears at first sight, on account of a reason which I will explain.

If we put
$$\frac{dn}{n\,dt} = K\frac{d\,(e'^2)}{dt},$$

where K is the coefficient found from theory, the secular equation to be applied to the mean longitude will be

$$K\int(e'^2 - E'^2)\,n\,dt,$$

E' being the eccentricity of the Earth's orbit at the epoch from which t is reckoned.

Now I find that M. Hansen uses a smaller value of the integral

$$\int(e'^2 - E'^2)\,n\,dt$$

than M. Plana does; that is, he supposes a slower change in the eccentricity of the Earth's orbit: and yet his resulting value of the secular equation is larger than those of M. Plana.

It may be inferred, either from the data in the Introduction to M. Hansen's Solar Tables, or from other data in the Introduction to his Lunar Tables, that the value of the integral $\int(e'^2 - E'^2)\,n\,dt$ which he employs is $-1212''\cdot 5t^2$, t being expressed, as usual, in centuries.

Now M. Plana, in his *Theory of the Moon*, supposes the value of the above integral to be $-1264''\cdot 1t^2$, and in his Memoir in vol. xviii. of the Turin Memoirs he gives it the value $-1297''\cdot 7t^2$.

If, then, we reduce the coefficients of the secular equation given by these authors, so as to make them correspond with the value $-1270''t^2$ of the above integral, which is that employed in my Memoir of 1853, they will become

Coefficient	according to	M. Plana's theory	$10''\cdot 60$,
,,	,,	M. Plana's memoir (1856)	$11\cdot 24$,
,,	,,	M. Hansen's theory	$12\cdot 76$.

The difference between M. Hansen's coefficient and either of M. Plana's is much greater than could possibly have arisen if both values had been found on correct principles, and they had differed merely in consequence of the approximations not being carried far enough.

My value of the same coefficient, which was communicated to the French Institute in January, 1859, is $5''\cdot 70$. And M. Delaunay, while perfectly agreeing with me in the terms which I have calculated, has added a great number of others depending on the eccentricity and inclination of the Moon's orbit, and thus increases the coefficient to $6''\cdot 11$.

As M. Hansen's method of obtaining his coefficient has not yet appeared, it is, of course, impossible for me to point out the reason of the difference between it and my own, as I have done in reference to the results of MM. Plana and de Pontécoulant. I have very little doubt, however, that it arises from M. Hansen having tacitly assumed, like M. Plana, that one of his constants introduced by integration is an absolutely constant quantity.

M. Hansen has suggested that the difference between his result and that obtained by M. Delaunay and myself may arise from want of convergency in the series proceeding according to powers of m, by means of which we determine the coefficient denoted above by K.

If we confine our attention to the terms of K which are independent of the eccentricity and inclination of the Moon's orbit, and which are admitted by all to constitute by far the largest part of that quantity, we find that the terms involving the successive powers of m taken into account by me give rise to the following parts of the coefficient of the secular equation:—

m^2	$10''\cdot66$,
m^4	$-\ 2\cdot34$,
m^5	$-\ 1\cdot58$,
m^6	$-\ 0\cdot71$,
m^7	$-\ 0\cdot25$.

The sum of these is $5''\cdot78$. The convergence, although slow at starting, becomes more rapid in the later terms; and I inferred, in my communication to the French Institute above mentioned, that the remainder of the series would be very nearly equal to $-0''\cdot08$.

Now M. Delaunay has since calculated the next term of the series, and finds it $= -0''\cdot06$, which is in exact accordance with my anticipations.

Although I think that there can remain no doubt with respect to the convergency of the series, yet, in order to remove all possible objection, I have calculated the value of K by a method which does not require any expansion in powers of m, and the resulting coefficient of the secular equation is $5''\cdot70$, exactly agreeing with that found by means of the series of powers of m.

A very few words will now suffice in reply to the objections which M. de Pontécoulant brings forward in his observations in the *Monthly Notices*. In fact, almost all of them have been virtually answered in what I have said before.

At the outset of his paper, M. de Pontécoulant rightly describes the difference between my method of finding the secular acceleration and all preceding ones, as arising from the consideration of the variability of the eccentricity of the Earth's orbit in the differential equations of the Moon's motion, in which this element had hitherto been considered as constant. He then refers to the statement in my Memoir, that when this consideration was introduced into the formulæ, I found exactly the same result whether the time or the Moon's longitude was taken as the independent variable. But, adds M. de Pontécoulant, "il n'y a qu'une petite difficulté dans cette assertion, c'est qu'elle énonce un fait mathématiquement *inadmissible*."

Now I confess that I cannot see M. de Pontécoulant's "petite difficulté." I am far from looking upon the agreement between the results of different methods as a fact mathematically inadmissible. On the contrary, it appears to me a palpable absurdity to suppose that the result of a mathematical investigation can be different according as one independent variable or another is employed in obtaining it, or that two methods of solving the same problem may both be correct and yet lead to contradictory results.

In order, however, to shew this mathematical inadmissibility, M. de Pontécoulant goes on to say, "En effet, M. Adams convient quelque part, je crois, et d'ailleurs, je le démontrerais bientôt jusqu'à l'évidence, que la considération de la variabilité de l'orbe terrestre, n'exerce aucune influence sur la détermination de l'inégalité séculaire, lorsqu'on emploie pour l'obtenir les formules directes que j'ai adoptées dans ma théorie."

In thus stating that I admit that one of the methods of determining the secular acceleration is unaffected by the consideration of the variability of the eccentricity of the Earth's orbit, M. de Pontécoulant overlooks "une petite difficulté," viz., that instead of admitting this, I assert, in so many words, the exact contrary. In the concluding sentence of my Memoir I say, "when both processes were corrected by taking into account the new terms whose existence I had already recognized, I had the satisfaction of finding a perfect agreement between the results."

For M. de Pontécoulant's demonstration "jusqu'à l'évidence," I am not responsible, and indeed, I think his paper tends to shew that he has peculiar ideas as to what constitutes demonstration.

In the next place M. de Pontécoulant offers "une réflexion très simple," which he thinks ought to have struck me. "Qui est-ce après tout que le coefficient de l'équation séculaire?—une certaine fonction des éléments des orbites de l'astre troublé et de l'astre perturbateur, qui se déduit des formules différentielles du mouvement; cette fonction est la même, selon M. Adams, par quelque méthode qu'on l'obtienne, dans le cas où l'on considère comme variable l'excentricité de l'orbe terrestre; à plus forte raison elle doit l'être dans le cas où l'on regarde cette excentricité comme constante." I am at a loss to imagine what can be the meaning of this last clause, since the secular equation in question is entirely due to the variability of the eccentricity of the Earth's orbit, and would not exist at all if this eccentricity were constant.

It must be admitted that my new determination of the secular acceleration has, as M. de Pontécoulant says, "l'inconvénient d'altérer profondément l'expression analytique admise jusqu'à présent, du coefficient de cette équation," but truth must not be sacrificed to convenience.

In the algebraical portion of his paper, M. de Pontécoulant is not happier than in his introductory remarks. Indeed, throughout the paper he expressly leaves out of consideration all the terms which give rise to the difference between M. Plana's result and mine.

Thus, at the bottom of p. 311, having found from an assumed term in $\frac{dR}{d\nu}$, that

$$\int \frac{dR}{d\nu} dt = -\frac{A}{f} e' \cos(ft + l) + \frac{A}{f} \frac{de'}{dt} \sin(ft + l),$$

he incorporates the term involving $\frac{de'}{dt}$ with the preceding under the form

$$-\frac{A}{f} e' \cos\left(ft + l + \frac{de'}{e' dt}\right),$$

and then remarks :—

"On voit donc que la considération de la variation de l'excentricité de l'orbite terrestre ne fait qu'altérer d'une manière insensible la partie constante des angles des diverses inégalités lunaires multipliées par e', elle ne change en rien la forme des séries qui déterminent les coordonnées du mouvement troublé..."

Now these alterations of the constant part of the angles on which the several lunar inequalities depend, which are neglected as insensible by M. de Pontécoulant, actually give rise to the terms in the Moon's coordinates involving $\frac{de'}{dt}$, which I have been the first to take into account, and thus do change the form of the expressions for those coordinates.

The term $\frac{A}{f} \frac{de'}{dt} \sin(ft + l)$ is not destroyed by being incorporated with the preceding term $-\frac{A}{f} e' \cos(ft + l)$, as M. de Pontécoulant seems to suppose.

Again, in order to shew that the integral $\int \frac{dR}{d\nu} dt$ can contain no non-periodic term depending on e', M. de Pontécoulant assumes, at the foot of p. 310, that $\frac{dR}{d\nu}$ is made up of terms of the form

$$Ae' \sin(ft + l).$$

But $\frac{dR}{d\nu}$ is a function of r and ν; and since these quantities contain terms depending on the disturbing force and multiplied by $\frac{de'}{dt}$, $\frac{dR}{d\nu}$ will contain, in addition to the terms of the form considered by M. de Pontécoulant, other terms of the order of the square of the disturbing force, and of the form

$$B\frac{de'}{dt}\cos(ft+l);$$

among these there will be a term in which the angle $ft+l$ vanishes; viz., one of the form

$$Ce'\frac{de'}{dt},$$

and consequently $\int\frac{dR}{d\nu}dt$ will contain the non-periodic term $\frac{1}{2}Ce'^2$.

M. de Pontécoulant characterises the process which I have employed at the bottom of p. 402 in my Memoir (*see* p. 147 *above*), in order to find the non-periodic parts of certain integrals, as "une véritable *supercherie analytique*." Now this "supercherie" only consists in taking account of the variability of $\frac{de'}{d\nu}$, by putting for it the identical quantity $\frac{de'}{dt}\cdot\frac{dt}{d\nu}$.

M. Plana, in equation [10], p. 12, of his Memoir, finds, for the terms thus objected to by M. de Pontécoulant, exactly the same values as I have done, though his process entirely differs from mine.

On this same point, in a note to p. 315, M. de Pontécoulant makes the objection that in the last step of the integrations referred to I make $d\nu=n\,dt$, contrary to the supposition I had previously employed. But my object was simply to find the non-periodic parts of the integrals concerned; and it is obvious that if I had put for $d\nu$ its complete value $n\,dt-\phi(\nu)\,d\nu$, where $\phi(\nu)$ is a periodic function of ν, this function would only introduce periodic terms into the integrals, and would cause no change whatever in the terms which I have found.

But one of the most remarkable objections in the whole course of M. de Pontécoulant's communication occurs in p. 316, where he says he is going to put his finger on the error I have committed. From an equation in my Memoir he deduces the following:—

$$e' = q + q'\left\{\nu - \frac{11}{8}m^2 \sin(2\nu - 2m\nu) - \frac{77}{16}m^2 e' \sin^2(2\nu - 2m\nu - c'm\nu) + \&c.\right\}$$

and then adds the remark,—

"C'est-à-dire, que l'excentricité de l'orbite terrestre, outre sa variation séculaire, serait soumise à toutes les inégalités du mouvement lunaire; c'est-à-dire, à des variations dont le période serait d'un mois, d'une année, &c. ce qui est contraire, quelque petitesse qu'on suppose au coefficient q', à tous les principes de la théorie."

Now it is astonishing that M. de Pontécoulant does not see that the quantity enclosed within brackets, in the above equation, is simply the expression of the Moon's mean longitude nt in terms of the true longitude ν, so that the equation is equivalent to

$$e' = q + q'nt;$$

that is, the eccentricity of the Earth's orbit is made to vary uniformly with the time, which agrees with the supposition with which we started.

On the other hand, M. de Pontécoulant, by making

$$e' = q + q'\nu,$$

that is, by supposing the change in e' to be proportional to the Moon's true motion in longitude, would evidently cause the eccentricity of the Earth's orbit to be affected by all the inequalities of the lunar motion.

All attempts to express e' in terms of ν, without introducing periodic terms, lead to this absurdity.

I have already alluded to the strange notion expressed at the end of M. de Pontécoulant's paper, that there may be two values of the secular acceleration, one applicable to the true longitude and the other to the mean longitude. The difference between the true and the mean longitudes consists wholly of periodic quantities, and cannot contain any term increasing continually with the time.

How M. de Pontécoulant could have so far deceived himself as to imagine that this paper settled the question of the secular acceleration, "sans contestation possible désormais," is, I confess, beyond my comprehension.

P.S.—In the *Compte Rendu* of April 9, 1860, which has appeared since the foregoing paper was read, M. de Pontécoulant gives the value of the secular acceleration of the Moon's mean motion, which he has obtained by taking the time as the independent variable, and which he considers to be "désormais à l'abri de toute objection."

This result, however, of M. de Pontécoulant's is the same as that which he formerly communicated to me, the error of which I have already pointed out.

M. de Pontécoulant thus describes his method, "En développant la formule qui donne l'expression de la longitude vraie en fonction de la longitude moyenne, et en n'ayant égard qu'au premier terme de ce développement, c'est-à-dire à sa partie non-périodique j'en ai conclu le rapport du moyen mouvement de la lune dans son orbite troublée au moyen mouvement relatif à son orbite elliptique, c'est-à-dire à l'orbite que cet astre décrirait autour de la terre sans l'action du soleil... En différentiant ensuite cette valeur par rapport à l'excentricité e' de l'orbite terrestre qu'elle renferme,... j'ai obtenu une expression de cette forme:

$$\frac{\delta n}{n} = H\delta \, . \, e'^2.\text{"}$$

The value of H thus obtained is

$$H = -\frac{3}{2}m^2 + \frac{5337}{128}m^4$$

which, as I have shewn in p. 9 (*see* p. 167 *above*), is the result that would be found by differentiating the relation between n and a, and then neglecting the variation of a. The fallacy of M. de Pontécoulant's reasoning consists in his treating the Moon's "orbite elliptique, c'est-à-dire, l'orbite que cet astre décrirait autour de la terre sans l'action du soleil," as if it were a real elliptic orbit with an unalterable semi-axis major, whereas the semi-axis major of the elliptic orbit spoken of by M. Pontécoulant, which is the same quantity as that above denoted by the symbol a, is really variable, and its variation must be found by means of the differential equations in the way which I have before described.

The numerical value of the coefficient of the secular equation which M. de Pontécoulant obtains in this paper, when reduced so as to correspond with the value $-1270''t^2$ of the integral $\int (e'^2 - E'^2)\, n dt$ is $7''\cdot 96$ which, as we see, differs widely from the similarly reduced values of the coefficient according to the theories of M. Plana and M. Hansen, given in p. 14, (*see* p. 173 *above*) as well as from the values obtained by M. Delaunay and myself.

After giving his formula for the secular equation, M. de Pontécoulant remarks, "En comparant ce résultat à celui que M. Plana a déduit de ses formules, on voit qu'il en diffère d'une manière notable, et que l'espèce de compensation qui devait s'établir, selon ce géomètre, entre les quantités du quatrième ordre et celles des ordres supérieurs, et qui semblait permettre de s'en tenir, comme l'avait fait Laplace, aux termes résultans de la première approximation, n'existe pas réellement. La considération des puissances supérieures de la force perturbatrice altère sensiblement, au contraire, la valeur du coefficient qu'on obtient en faisant abstraction des quantités qui en dépendent, et comme tous les termes de la formule, jusqu'aux termes du septième ordre, sont affectés d'un signe négatif, la grandeur du coefficient qu'on s'était habitué à supposer à l'équation séculaire d'après les indications de Laplace, doit être considérablement diminuée."

It is needless for me to point out how totally inconsistent these remarks of M. de Pontécoulant are with the conclusion at which he arrives in his paper in the *Monthly Notices*, "Il résulte, je pense, sans *contestation possible désormais*, de la discussion précédente, que les formules employées jusqu'ici pour déterminer l'équation séculaire de la lune, ont toute la correction nécessaire à cet important objet."

24.

ON THE MOTION OF THE MOON'S NODE IN THE CASE WHEN THE ORBITS OF THE SUN AND MOON ARE SUPPOSED TO HAVE NO ECCENTRICITIES, AND WHEN THEIR MUTUAL INCLINATION IS SUPPOSED TO BE INDEFINITELY SMALL.

[From the *Monthly Notices of the Royal Astronomical Society.* Vol. XXXVIII. (1877).]

A VERY able paper has recently been published by Mr G. W. Hill, assistant in the office of the *American Nautical Almanac*, on the part of the motion of the lunar perigee which is a function of the mean motions of the Sun and Moon.

Assuming that the values of the Moon's coordinates in the case of no eccentricities are already known, the author finds the differential equations which determine the inequalities which involve the first power of the eccentricity of the Moon's orbit, and, by a most ingenious and skilful process, he makes the solution of those differential equations depend on the solution of a single linear differential equation of the second order, which is of a very simple form. This equation is equivalent to an infinite number of algebraical linear equations, and the author, by a most elegant method, shews how to develop the infinite determinant corresponding to these equations in a series of powers and products of the small quantities forming their coefficients. The value of the multiplier of each of such powers and products as are required is obtained in a finite form. By equating this determinant to zero, an equation is obtained which gives directly, and without the need of successive approximations, the motion of the Moon from the perigee during half of a synodic month. The small quantities which enter into the value of the above determinant are of the fourth, eighth, twelfth, &c. orders, considering, as usual, the ratio of the mean motion of the Sun to that of the Moon as a small quantity of the first order; and the author has taken into account all the terms of lower orders than the sixteenth. The ratio of the motion of the perigee to that of the Moon thus obtained is true to twelve or thirteen significant figures. The author compares his numerical result with that deduced from Delaunay's analytical formula, which gives the ratio just mentioned developed in a series of powers of m, the ratio of the mean motions of the Sun and Moon. The numerical coefficients of the successive terms of this series increase so rapidly that the convergence of the series is slow, so that the terms calculated do not suffice to give the first four significant figures of the result correctly, although by induction, a rough approximation may be made to the sum of the remaining terms of the series.

I have been led to dwell thus particularly on Mr Hill's investigation because my own researches in the Lunar Theory have followed, in some respects, a parallel course, *sed longo intervallo.*

I have long been convinced that the most advantageous way of treating the Lunar Theory is, first, to determine with all desirable accuracy the inequalities which are independent of the eccentricities e and e', and the inclination $2\sin^{-1}\gamma$, and then, in succession, to find the inequalities which are of one dimension, two dimensions, and so on, with respect to those quantities.

Thus the coefficient of any inequality in the Moon's coordinates would be represented by a series arranged in powers and products of e, e', and γ, and each term in this series would involve a numerical coefficient which is a function of m alone and which may be calculated for any given value of m without the necessity of developing it in powers of m. The variations of these coefficients which would result from a very small change in m might be found either independently or by making the calculation for two values of m differing by a small quantity.

This method is particularly advantageous when we wish to compare our results with those of an analytical theory such as Delaunay's, in which the eccentricities and the inclination are left indeterminate, since each numerical coefficient so obtained could be compared separately with its analytical development in powers of m.

It is to be remarked that it is only the series proceeding by powers of m in Delaunay's Theory which have a slow rate of convergence, so that it is probable that all the sensible corrections required by Delaunay's coefficients would be found among the terms of low order in e, e', and γ.

The differential equations which would require solution in these successive operations after the determination of the inequalities independent of eccentricities and inclination would be all linear and of the same form.

It is many years since I obtained the values of these last-named inequalities to a great degree of approximation, the coefficients of the longitude expressed in circular measure, and those of the reciprocal of the radius vector, or of the logarithm of the radius vector, being found to ten or eleven places of decimals.

In the next place I proceeded to consider the inequalities of latitude, or rather the disturbed value of the Moon's coordinate perpendicular to the Ecliptic, omitting the eccentricities as before, and taking account only of the first power of γ.

In this case the differential equation for finding z presents itself naturally in the form to which Mr Hill reduces, with so much skill, the equations depending on the first power of the eccentricity of the Moon's orbit.

In solving this equation I fell upon the same infinite determinant as that considered by Mr Hill, and I developed it in a similar manner in a series of powers and products of small quantities, the coefficient of each such term being given in a finite form.

The terms of the fourth order in the determinant were thus obtained by me on the 26th December 1868. I then laid aside the further investigation of this subject for a considerable time, but resumed it in 1874 and 1875, and on the 2nd of December in the latter year I carried the approximation to the value of the determinant as far as terms of the twelfth order, or to the same extent as that which has been attained by Mr Hill. I have also succeeded in reducing the determination of the inequalities of longitude and radius vector which involve the first power of the lunar eccentricity to the solution of a differential equation of the second order, but my method is much less elegant than that of Mr Hill.

Immediately after Mr Hill's paper reached me, I wrote to him expressing my opinion of its merits, and telling him what I had done in the same direction, and I received from him a very cordial and friendly letter in reply.

The equation which I had obtained by equating the above-mentioned determinant to zero differed in form from Mr Hill's, and on making the reductions required to make the two results immediately comparable, I found that there was an agreement between them except in one term of the twelfth order. On examining my work I found that this arose from a simple error of transcription in a portion of my work, and that when this had been rectified my result was in entire accordance with Mr Hill's.

The calculations by which I have found the value of the determinant are very different in detail from those required by Mr Hill's method, and appear to be considerably more laborious. I have not yet had time to copy out and arrange the details of the calculations from my old papers, but I hope soon to do so, thinking that they may not be without interest for the Society. Meantime I now make known the result which I have obtained for the motion of the Moon's node on the suppositions stated in the title of this paper.

If nt and $n't$ represent the mean longitudes of the Moon and the Sun at time t, omitting, for the sake of brevity in writing, the constants which always accompany nt and $n't$, and if θ and r represent the Moon's longitude and radius vector, I find that, in the case of no eccentricities and inclination, if $m = \frac{n'}{n} = 0{\cdot}0748013$, which is the value used by Plana,

$$\begin{aligned}\theta = nt &+ 0{\cdot}01021{,}13629{,}5 \sin 2(n-n')t \\ &+ 0{\cdot}00004{,}23732{,}7 \sin 4(n-n')t \\ &+ 0{\cdot}00000{,}02375{,}7 \sin 6(n-n')t \\ &+ 0{\cdot}00000{,}00015{,}1 \sin 8(n-n')t \\ &+ 0{\cdot}00000{,}00000{,}1 \sin 10(n-n')t;\end{aligned}$$

$$\begin{aligned}\frac{1}{r} = \quad & 1{\cdot}00090{,}73880{,}5 \\ &+ 0{\cdot}00718{,}64751{,}6 \cos 2(n-n')t \\ &+ 0{\cdot}00004{,}58428{,}9 \cos 4(n-n')t \\ &+ 0{\cdot}00000{,}03268{,}6 \cos 6(n-n')t \\ &+ 0{\cdot}00000{,}00024{,}3 \cos 8(n-n')t \\ &- 0{\cdot}00000{,}00000{,}3 \cos 10(n-n')t;\end{aligned}$$

supposing that θ is expressed in the circular measure, and that the unit of distance is the mean distance in an undisturbed orbit which would be described by the Moon about the Earth in the same periodic time. In this case, if μ denote the sum of the masses of the Earth and Moon, we shall have

$$\mu = n^2.$$

The differential equation which determines z, the Moon's coordinate perpendicular to the Ecliptic, is

$$\frac{d^2z}{dt^2} + \left(\frac{\mu}{r^3} + \frac{\mu'}{r_1^{\,3}}\right) z = 0.$$

Now, the Sun's orbit being circular, we have $\frac{\mu'}{r_1^{\,3}} = n'^2$, and the only function of the Moon's coordinates which we require in order to form this equation is $\frac{1}{r^3}$.

I find that, with the above unit of distance,

$$\begin{aligned}
\frac{1}{r^3} = \ & 1{\cdot}00280{,}21783{,}115 \\
& + 0{\cdot}02159{,}98364{,}4 \cos 2(n-n')t \\
& + 0{\cdot}00021{,}53273{,}9 \cos 4(n-n')t \\
& + 0{\cdot}00000{,}20644{,}8 \cos 6(n-n')t \\
& + 0{\cdot}00000{,}00192{,}9 \cos 8(n-n')t \\
& + 0{\cdot}00000{,}00000{,}3 \cos 10(n-n')t.
\end{aligned}$$

Let

$$\frac{1}{(n-n')^2}\left(\frac{\mu}{r^3} + \frac{\mu'}{r_1^{\,3}}\right), \text{ or } \frac{1}{(n-n')^2}\left(\frac{n^2}{r^3} + n'^2\right), = \frac{1}{(1-m)^2}\left(\frac{1}{r^3} + m^2\right),$$

$$= q^2 + 2q_1 \cos 2(n-n')t + 2q_2 \cos 4(n-n')t + 2q_3 \cos 6(n-n')t + \&c.;$$

then we find, from the above value of $\frac{1}{r^3}$, that

$$\begin{aligned}
& q^2 = 1{\cdot}17804{,}44973{,}149, \quad \text{and} \quad q = 1{\cdot}08537{,}75828{,}323, \\
& q_1 = 0{\cdot}01261{,}68354{,}6, \\
& q_2 = 0{\cdot}00012{,}57764{,}3, \\
& q_3 = 0{\cdot}00000{,}12059{,}0.
\end{aligned}$$

These are all the quantities necessary for finding the motion of the Moon's node, to the order which we require.

If $g\pi$ denote the angular motion of the Moon from its node in half a synodic period of the Moon, the equation so often referred to above gives

$$\cos g\pi = \cos q\pi \left\{ 1 - \frac{\pi^2 q_1^4}{32q^2(q^2-1)^2} - \frac{15q^4 - 35q^2 + 8}{256q^4(q^2-1)^4(q^2-4)}\pi^2 q_1^6 + \frac{3\pi^2 q_1^4 q_2}{32q^2(q^2-1)^2(q^2-4)} - \frac{\pi^2 q_1^2 q_2^2}{16q^2(q^2-1)(q^2-4)} \right\}$$

$$+ \sin q\pi \left\{ \frac{\pi q_1^2}{4q(q^2-1)} + \frac{15q^4 - 35q^2 + 8}{64q^3(q^2-1)^3(q^2-4)}\pi q_1^4 - \frac{\pi^3 q_1^6}{384q^3(q^2-1)^3} \right.$$

$$+ \frac{105q^{10} - 1155q^8 + 3815q^6 - 4705q^4 + 1652q^2 - 288}{256q^5(q^2-1)^5(q^2-4)^2(q^2-9)}\pi q_1^6$$

$$- \frac{3\pi q_1^2 q_2}{8q(q^2-1)(q^2-4)} - \frac{35q^6 - 280q^4 + 497q^2 - 108}{32q^3(q^2-1)^3(q^2-4)^2(q^2-9)}\pi q_1^4 q_2$$

$$+ \frac{\pi q_2^2}{4q(q^2-4)} + \frac{15q^6 - 110q^4 + 179q^2 - 36}{16q^3(q^2-1)^2(q^2-4)^2(q^2-9)}\pi q_1^2 q_2^2$$

$$\left. + \frac{\pi q_3^2}{4q(q^2-9)} - \frac{(3q^2-7)\pi q_1 q_2 q_3}{4q(q^2-1)(q^2-4)(q^2-9)} + \frac{5\pi q_1^3 q_3}{16q(q^2-1)(q^2-4)(q^2-9)} \right\}.$$

Now, if the coefficients of $\cos q\pi$ and $\sin q\pi$ in this formula be converted into numbers, employing the above values of q, q_1, &c., we find

$$\cos g\pi = \cos q\pi\,[0{\cdot}99999{,}97902{,}01654] + \sin q\pi\,[0{\cdot}00064{,}77652{,}06681].$$

But, with the above value of q, we find, from Briggs' Tables,

$$\cos q\pi = -0{\cdot}96424{,}37306{,}84295$$
$$\sin q\pi = -0{\cdot}26501{,}70331{,}05484.$$

Hence $\qquad \cos g\pi = -0{\cdot}96441{,}51972{,}00779.$

Whence, by the same Tables, we find that

$$g = 1{\cdot}08517{,}13927{,}46869,$$

and therefore the ratio of the Moon's motion from the node to its sidereal motion is

$$g(1-m) = 1{\cdot}00399{,}91618{,}46592.$$

This is the quantity ordinarily denoted by g in the Lunar Theory.

Delaunay's value of g, which agrees with that of Plana, is

$$g = 1 + \frac{3}{4}m^2 - \frac{9}{32}m^3 - \frac{273}{128}m^4 - \frac{9797}{2048}m^5 - \frac{199273}{24576}m^6 - \frac{6657733}{589824}m^7.$$

If this be converted into numbers by substituting the value of $m = 0{\cdot}0748013$, we find

$$g = 1{\cdot}00399{,}91722{,}8,$$

which differs from the true value in the eighth place of decimals.

If we take $\mathrm{m} = \dfrac{m}{1-m}$ and develop the value of g in powers of m, we find

$$g = 1 + \frac{3}{4}\mathrm{m}^2 - \frac{57}{32}\mathrm{m}^3 + \frac{123}{128}\mathrm{m}^4 - \frac{1925}{2048}\mathrm{m}^5 + \frac{25667}{24576}\mathrm{m}^6 - \frac{268309}{589824}\mathrm{m}^7;$$

and substituting the value of

$$\mathrm{m} = 0{\cdot}08084{,}89030{,}52,$$

we find
$$g = 1{\cdot}00399{,}91591{,}1,$$

which is considerably nearer the truth than the value found from the series in powers of m.

The numerical values of the successive terms of the series for $g-1$, in terms of powers of m and of m respectively, are given in the following comparative table:

	In powers of m.		In powers of m.
m^2	·00419,64258,6	m^2	·00490,24088,4
m^3 −	11,77117,9	m^3 −	94,13416,4
m^4 −	6,67712,1	m^4	4,10574,2
m^5 −	1,12023,4	m^5 −	,32469,2
m^6 −	,14203,4	m^6	,02916,8
m^7 −	,01479,0	m^7 −	,00102,7
	·00399,91722,8		·00399,91591,1

This shews that the development in powers of m is much more advantageous than that in powers of m.

The same thing likewise holds good with respect to the value of c, which determines the motion of the perigee.

The following is a similar table, shewing the numerical values of the successive terms of Delaunay's series for $1-c$ in powers of m and of the terms of the corresponding series in powers of m:—

In powers of m.		In powers of m.	
m^2	·00419,64258,6	m^2	·00490,24088
m^3	294,27947,8	m^3	292,31135
m^4	99,56981,8	m^4	55,37745
m^5	30,35769,9	m^5	14,37162
m^6	9,13946,6	m^6	3,49278
m^7	2,82999,6	m^7	,99062
m^8	,98356,5	m^8	,42111
m^9	,34684,2	m^9	,08515
	·00857,14945,0		·00857,29096

The true value reduced from Mr Hill's, so as to correspond to the value of m which we have employed, is

·00857,25645.

Hence, as in the former case, the advantage of developing in powers of m is very evident.

I have found that a similar advantage results from the employment of m instead of $\grave{m}$ in the development of the coefficients of the Moon's periodic inequalities.

25.

NOTE ON A REMARKABLE PROPERTY OF THE ANALYTICAL EXPRESSION FOR THE CONSTANT TERM IN THE RECIPROCAL OF THE MOON'S RADIUS VECTOR.

[From the *Monthly Notices of the Royal Astronomical Society.* Vol. XXXVIII. (1878).]

LET $nt+\epsilon$ denote the mean longitude of the Moon at the time t; $n't+\epsilon'$ that of the Sun.

$\xi = nt+\epsilon-n't-\epsilon'$, the mean elongation of the Moon from the Sun.

ϕ, the Moon's mean anomaly.

ϕ', that of the Sun.

η, the Moon's mean distance from the ascending node.

$c=\dfrac{d\phi}{ndt}$ and $g=\dfrac{d\eta}{ndt}$, so that $(1-c)n$ denotes the mean motion of the Moon's perigee, and $(g-1)n$ denotes the mean retrograde motion of the Moon's node, in a unit of time.

Also let e denote the mean eccentricity of the Moon's orbit.

e', the eccentricity of the Sun's orbit.

γ, the sine of half the mean inclination of the Moon's orbit to the ecliptic.

$m=\dfrac{n'}{n}$, the ratio of the mean motion of the Sun to that of the Moon.

μ, the sum of the masses of the Earth and Moon.

$a = \left(\frac{\mu}{n^2}\right)^{\frac{1}{3}}$, the mean distance in the purely elliptic orbit which the Moon if undisturbed would describe about the Earth in its actual periodic time.

To fix the ideas, we will suppose the quantities e and γ to be defined as in Delaunay's Theory of the Moon.

If r denote the Moon's radius vector, and if we omit terms depending on the Sun's parallax, then, as is well known, the value of $\frac{a}{r}$ may be expanded in an infinite series involving cosines of angles of the form

$$2i\xi \pm j\phi \pm j'\phi' \pm 2k\eta,$$

where i, j, j', k denote any positive integers, including zero, and the coefficient of the term with this argument contains $e^j e'^{j'} \gamma^{2k}$ as a factor, the remaining factor being a function of m, e^2, e'^2, and γ^2.

In particular, there is a constant term in $\frac{a}{r}$, corresponding to the case in which i, j, j', and k are all zero, and this term has the form

$$A + Be^2 + C\gamma^2 + Ee^4 + 2Fe^2\gamma^2 + G\gamma^4 + \&c.,$$

where

$$A = A_0 + A_1 e'^2 + A_2 e'^4 + \&c.$$

$$B = B_0 + B_1 e'^2 + B_2 e'^4 + \&c.$$

$$C = C_0 + C_1 e'^2 + C_2 e'^4 + \&c.$$

$$\&c. \quad \&c. \quad \&c.$$

and A_0, A_1 &c., B_0, B_1 &c., C_0, C_1 &c. are all functions of m.

Plana and, after him, Lubbock, Pontécoulant, and Delaunay have developed the functions of m which occur in the coefficients of the several terms of $\frac{a}{r}$ and of the other coordinates of the Moon, in series of ascending powers of m, and have severally determined, by different methods, the numerical coefficients of the leading terms in these developments.

With respect to the constant term in $\frac{a}{r}$, Plana shewed that the quantities denoted above by B_0 and C_0, viz. the coefficients of e^2 and γ^2 in the above constant, both vanish when account is taken of the terms involving m^2 and m^3. Pontécoulant carried the development of the quantities B_0 and C_0 two orders higher, viz. to terms involving m^5, and found that these terms likewise vanish.

These investigations of Plana and Pontécoulant, however, while they shew that the coefficients of the above mentioned powers of m vanish by the mutual destruction of the parts of which each of the coefficients is composed, supply no reason why this mutual destruction should take place, and throw no light whatever on the values of the succeeding coefficients in the series.

Thinking it probable that these cases in which the coefficients had been found to vanish were merely particular cases of some more general property, I was led to consider the subject from a new point of view, and on February 22, 1859, I succeeded in proving, not only that the coefficients B_0 and C_0 vanish identically, but that the same thing holds good of the more general coefficients B and C, so that the coefficients of

$$e^2,\ e^2e'^2,\ e^2e'^4,\ \&c.$$
$$\gamma^2,\ \gamma^2e'^2,\ \gamma^2e'^4,\ \&c.$$

in the constant term of $\frac{a}{r}$ are all identically equal to zero.

Further reflection on the subject led me, several years later, to a simpler and more elegant proof of the property above mentioned.

This new proof was found on February 27, 1868, and I now venture to lay it before the Society. The resulting theorem is remarkable for a degree of simplicity and generality of which the lunar theory affords very few examples.

There are also two remarkable relations between the coefficients of e^4, $e^2\gamma^2$, and γ^4 in the constant term of $\frac{a}{r}$, which we before denoted by E, F, and G. These relations may be thus stated:

If the terms of the quantity c or $\frac{d\phi}{ndt}$ which involve e^2 and γ^2 be denoted by

$$He^2 + K\gamma^2,$$

and similarly if the terms of g or $\frac{d\eta}{ndt}$ which involve e^2 and γ^2 be denoted by

$$Me^2 + N\gamma^2,$$

where H, K, M, and N are functions of m and e'^2, then we shall have

$$\frac{E}{F} = \frac{H}{K} \text{ and } \frac{F}{G} = \frac{M}{N}.$$

These relations are established by means of the same principle which was employed to prove the theorem above mentioned, viz. that $B=0$ and $C=0$.

They were, however, arrived at much later, namely on August 14, 1877.

Analysis.

Let x, y, z denote the rectangular coordinates of an imaginary Moon at any time t, the plane of xy being that of the ecliptic, and the axis of x the origin of longitudes.

Also let x', y' be the rectangular coordinates of the Sun, r' its radius vector, and μ' its mass.

Then if we neglect the terms which involve the Sun's parallax, the equations of motion are

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \frac{\mu' x}{r'^3} = \frac{3\mu' x'}{r'^5}(xx' + yy'),$$

$$\frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \frac{\mu' y}{r'^3} = \frac{3\mu' y'}{r'^5}(xx' + yy'),$$

$$\frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{\mu' z}{r'^3} = 0.$$

Now let x_1, y_1, z_1 be the rectangular coordinates, and r_1 the radius vector, of another imaginary Moon at the same time t as before, so that the same equations of motion hold good, and μ, μ', x', y', and r' are unaltered.

Hence
$$\frac{d^2x_1}{dt^2} + \frac{\mu x_1}{r_1^3} + \frac{\mu' x_1}{r'^3} = \frac{3\mu' x'}{r'^5}(x_1x' + y_1y'),$$

$$\frac{d^2y_1}{dt^2} + \frac{\mu y_1}{r_1^3} + \frac{\mu' y_1}{r'^3} = \frac{3\mu' y'}{r'^5}(x_1x' + y_1y'),$$

$$\frac{d^2z_1}{dt^2} + \frac{\mu z_1}{r_1^3} + \frac{\mu' z_1}{r'^3} = 0.$$

Multiply the first set of equations by x_1, y_1, z_1 respectively, and subtract their sum from the sum of the similar equations in x_1, y_1, z_1 multiplied by x, y, z respectively.

Thus we find

$$\left(x\frac{d^2x_1}{dt^2}-x_1\frac{d^2x}{dt^2}\right)+\left(y\frac{d^2y_1}{dt^2}-y_1\frac{d^2y}{dt^2}\right)+\left(z\frac{d^2z_1}{dt^2}-z_1\frac{d^2z}{dt^2}\right)$$
$$+\mu\,(xx_1+yy_1+zz_1)\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right)=0;$$

or

$$\frac{d}{dt}\left(x\frac{dx_1}{dt}-x_1\frac{dx}{dt}\right)+\frac{d}{dt}\left(y\frac{dy_1}{dt}-y_1\frac{dy}{dt}\right)+\frac{d}{dt}\left(z\frac{dz_1}{dt}-z_1\frac{dz}{dt}\right)$$
$$+\mu\,(xx_1+yy_1+zz_1)\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right)=0.$$

Hence the quantity

$$(xx_1+yy_1+zz_1)\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right)$$

is a complete differential coefficient with respect to t, and therefore when developed in cosines of angles which increase proportionally to the time it cannot contain any constant term*.

Now

$$xx_1+yy_1+zz_1=\frac{1}{2}\{2rr_1+(r-r_1)^2-(x-x_1)^2-(y-y_1)^2-(z-z_1)^2\}$$

and

$$\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right)=\left(\frac{1}{r_1}-\frac{1}{r}\right)\left\{\frac{3}{rr_1}+\left(\frac{1}{r_1}-\frac{1}{r}\right)^2\right\}.$$

Hence, if $x-x_1$, $y-y_1$, $z-z_1$, and therefore also $r-r_1$, and $\frac{1}{r_1}-\frac{1}{r}$ be quantities of the first order with respect to any symbol, then

$$(xx_1+yy_1+zz_1)\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right)$$

will differ from $3\left(\frac{1}{r_1}-\frac{1}{r}\right)$ by a quantity of the third order only.

* We may remark here that neither of the quantities

$$(xx_1+yy_1)\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right),$$

$$\text{or } zz_1\left(\frac{1}{r_1^3}-\frac{1}{r^3}\right),$$

can contain any constant term, but no use is made of this in what follows.

Hence, in the case supposed, the quantity $\frac{1}{r_1}-\frac{1}{r}$ cannot contain any constant term of lower order than the third.

More generally, the constant part of $\frac{1}{r_1}-\frac{1}{r}$ cannot be of a lower order than the constant part of the product of the quantity $\frac{1}{r_1}-\frac{1}{r}$ multiplied by one or other of the quantities

$$\left(\frac{1}{r_1}-\frac{1}{r}\right)^2, \text{ or } (x-x_1)^2+(y-y_1)^2+(r-r_1)^2.$$

Now, as the two systems x, y, z and x_1, y_1, z_1 satisfy the same differential equations, the solutions can only differ from each other by involving different values of the arbitrary constants.

By applying the principle just stated to four different cases of variation of the arbitrary constants, we shall be able to prove the properties already enunciated, viz.

$$B=0,\ \ C=0,\ \ \frac{E}{F}=\frac{H}{K},\ \text{and}\ \frac{F}{G}=\frac{M}{N}.$$

Let

$$x=u\cos(nt+\epsilon)-v\sin(nt+\epsilon),$$

$$y=u\sin(nt+\epsilon)+v\cos(nt+\epsilon);$$

and similarly

$$x_1=u_1\cos(nt+\epsilon)-v_1\sin(nt+\epsilon),$$

$$y_1=u_1\sin(nt+\epsilon)+v_1\cos(nt+\epsilon),$$

where $nt+\epsilon$ is supposed to retain the same value as before.

Then

$$(x-x_1)^2+(y-y_1)^2=(u-u_1)^2+(v-v_1)^2.$$

Hence, in the statement of our principle, we may replace

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2-(r-r_1)^2$$

by

$$(u-u_1)^2+(v-v_1)^2+(z-z_1)^2-(r-r_1)^2.$$

For the sake of simplicity, we will take the quantity which was before denoted by a as our unit of length, so that, instead of the quantity formerly designated by $\frac{a}{r}$, we shall write simply $\frac{1}{r}$.

Now it is known, *a priori*, that the values of r and u, as well as that of $\frac{1}{r}$, may be developed in an infinite series involving *cosines* of angles in the form

$$2i\xi \pm j\phi \pm j'\phi' \pm 2k\eta,$$

where i, j, j', and k denote any positive integers whatever, including zero, and that the value of v may be developed in a similar series involving *sines* of the same angles.

Also we know that the coefficient of the term with the above argument occurring in any of these series contains $e^j e'^{j'} \gamma^{2k}$ as a factor, the remaining factor being a function of m, e^2, e'^2 and γ^2.

Similarly we know that the value of z may be developed in an infinite series involving *sines* of angles of the form

$$2i\xi \pm j\phi \pm j'\phi' \pm (2k+1)\eta,$$

and that the coefficient of the term with this argument contains $e^j e'^{j'} \gamma^{2k+1}$ as a factor, the remaining factor being a function of m, e^2, e'^2 and γ^2 as in the former case.

It is essential to observe that $\frac{1}{r}$, r, u, and v involve only even powers of γ, while z involves only odd powers of the same quantity.

Having made these preliminary observations, we are now in a position to apply our principle to the four cases already alluded to.

Case I.

First, suppose that the values of x, y, z are those belonging to the solution in which e and γ vanish, therefore all the arguments in the values of $\frac{1}{r}$, r, u, and v will be of the form $2i\xi \pm j'\phi'$ and z will vanish.

Also let the values of x_1, y_1, z_1 belong to the solution in which e has a finite value, but γ is still $=0$, while $nt+\epsilon$, and therefore also n, retains the same value as before.

Hence z_1 also vanishes, and therefore $z - z_1 = 0$.

Then all the arguments which occur in the values of $\frac{1}{r}$, r, u, and v will also occur in those of $\frac{1}{r_1}$, r_1, u_1, and v_1, but the coefficients of the corresponding terms will differ by a quantity which contains e^2 as a factor.

Let the terms with these arguments be called terms of the *first class.*

Also there will be additional terms in the values of $\frac{1}{r_1}$, r_1, u_1, and v_1, with arguments of the form

$$2i\xi \pm j\phi \pm j'\phi',$$

where j does not vanish, and the coefficients of these terms will contain e as a factor.

Let the terms with these arguments be called terms of the *second class.*

Now, in the formation of the quantities

$$\left(\frac{1}{r_1} - \frac{1}{r}\right)^3 \text{ and } \left(\frac{1}{r_1} - \frac{1}{r}\right)\{(u-u_1)^2 + (v-v_1)^2 - (r-r_1)^2\}$$

terms with the argument zero can only arise by multiplying together three terms of the first class, one term of the first and two of the second class, or three terms of the second class, one of which at least involves e^2 as a factor. Such a term formed in the first of these ways would be of the order of e^6 at least, while one formed in the second or third of these ways would be of the order of e^4 at least. Hence, by the principle before proved, the value of $\frac{1}{r_1} - \frac{1}{r}$ can contain no constant term of the order of e^2.

Hence $B = 0$ generally, and as this holds good for every value of e', we must have

$$B_0 = 0, \quad B_1 = 0, \quad B_2 = 0, \text{ \&c.}$$

Case II.

In the next place, let the values x, y, z, as before, belong to the solution in which e and γ vanish, and let the values x_1, y_1, z_1 belong to the solution in which e is still equal to 0, but γ has a finite value, while $nt + \epsilon$, and therefore also n, retains the same value as before.

Then all the arguments which occur in the values of $\frac{1}{r}$, r, u, and v likewise occur in those of $\frac{1}{r_1}$, r_1, u_1, and v_1, but the coefficients of the corresponding terms will differ by a quantity which contains γ^2 as a factor.

Also there will be additional terms in the value of $\frac{1}{r_1}$, r_1, u_1, and v_1, with arguments of the form

$$2i\xi \pm j'\phi' \pm 2k\eta,$$

where k does not vanish, and these will also contain γ^2 as a factor in every term.

Hence $\frac{1}{r_1} - \frac{1}{r}$, $r - r_1$, $u - u_1$, and $v - v_1$ will contain γ^2 as a factor in every term.

Also $z = 0$, and therefore $(z - z_1)^2 = z_1^2$, which will also contain γ^2 as a factor in every term.

Hence $\left(\frac{1}{r_1} - \frac{1}{r}\right)^3$ will be of the order of γ^6 at least, while

$$\left(\frac{1}{r_1} - \frac{1}{r}\right)\{(u - u_1)^2 + (v - v_1)^2 + (z - z_1)^2 - (r - r_1)^2\}$$

will be of the order of γ^4 at least.

Therefore, by the same principle as before, the value of $\frac{1}{r_1} - \frac{1}{r}$ can contain no constant term of the order of γ^2.

That is, $C = 0$ generally; and as this holds good for every value of e' we must have

$$C_0 = 0, \quad C_1 = 0, \quad C_2 = 0, \text{ \&c.}$$

Case III.

Next, let the values x, y, z belong to the solution in which γ vanishes and e is finite, while x_1, y_1, z_1 belong to the general case in which e_1 and γ are both finite, the value of e being now changed to e_1 while $nt + \epsilon$, and therefore also n, retains the same value as before.

Then all the arguments which occur in the values of $\frac{1}{r}$, r, u, and v, and which are of the form

$$2i\xi \pm j\phi \pm j'\phi',$$

will occur unchanged in the values of $\frac{1}{r_1}$, r_1, u_1, and v_1, provided that ϕ, and therefore also $\frac{d\phi}{ndt}$ or c, remains unchanged, but the coefficients of the corresponding terms will differ by quantities which involve either $e-e_1$ or γ^2 as a factor.

Let the terms with these arguments be called terms of the *first* class.

Also there will be additional terms in the values of $\frac{1}{r_1}$, r_1, u_1, and v_1, the arguments of which are of the form

$$2i\xi \pm j\phi \pm j'\phi' \pm 2k\eta,$$

where k does not vanish. The coefficients of these terms will all contain γ^2 as a factor.

Call the terms with these arguments terms of the *second class.*

And $(z-z_1)^2 = z_1^2$, which contains γ^2 as a factor in every term.

Now the condition that c remains unchanged gives us the following relation between e^2, e_1^2, and γ^2:

$$He^2 = He_1^2 + K\gamma^2,$$

taking into account only the terms of lowest order in e^2, e_1^2, and γ^2.

Hence, ultimately,

$$\gamma^2 = \frac{H}{K}(e^2 - e_1^2).$$

If this value of γ^2 be substituted for it, we see that every term in the values of $\frac{1}{r_1} - \frac{1}{r}$, $r-r_1$, $u-u_1$, $v-v_1$, and $(z-z_1)^2$ will be divisible by $e-e_1$.

Hence the constant part of $\frac{1}{r_1} - \frac{1}{r}$ will be divisible by $(e-e_1)^2$, and therefore also by $(e^2-e_1^2)^2$, since this constant part involves only even powers of e^2 and e_1^2.

That is,
$$E(e_1^4-e^4)+2Fe_1^2\gamma^2$$
is divisible by $(e^2-e_1^2)^2$; or
$$E(e_1^4-e^4)+2Fe_1^2\frac{H}{K}(e^2-e_1^2)$$
is divisible by $(e^2-e_1^2)^2$.

Divide by $e^2-e_1^2$ and then put $e_1^2=e^2$,

therefore
$$-2Ee^2+2F\frac{H}{K}e^2=0,$$
or
$$\frac{E}{F}=\frac{H}{K}.$$

Case IV.

Lastly, let the values of x, y, z belong to the solution in which e vanishes and γ is finite, while x_1, y_1, z_1 belong to the general case in which e and γ_1 are both finite, the value of γ being changed to γ_1 while $nt+\epsilon$, and therefore also n, retains the same value as before.

Then all the arguments which occur in the values of $\frac{1}{r}$, r, u, and v, and which are of the form
$$2i\xi\pm j'\phi'\pm 2k\eta,$$
will occur unchanged in the values of $\frac{1}{r_1}$, r_1, u_1, and v_1, provided that η, and therefore also $\frac{d\eta}{ndt}$ or g, remains unchanged, but the coefficients of the corresponding terms will differ by quantities which involve either e^2 or $\gamma^2-\gamma_1^2$ as a factor.

Let the terms with these arguments be called terms of the *first class*.

Also there will be additional terms in the values of $\frac{1}{r_1}$, r_1, u_1, and v_1, the arguments of which are of the form
$$2i\xi\pm j\phi\pm j'\phi'\pm 2k\eta,$$
where j does not vanish. The coefficients of these terms will all involve e as a factor.

Call the terms with these arguments terms of the *second class.*

Moreover, all the arguments which occur in the value of z, and which are of the form

$$2i\xi \pm j'\phi' \pm (2k+1)\eta,$$

will occur unchanged in the value of z_1, but the coefficients of the corresponding terms will differ by quantities which involve either e^2 or $\gamma - \gamma_1$ as a factor.

Let the terms with these arguments be called terms of the *first class.*

Also there will be additional terms in the value of z_1, the arguments of which are of the form

$$2i\xi \pm j\phi \pm j'\phi' \pm (2k+1)\eta,$$

where j does not vanish. The coefficients of these terms will all involve $e\gamma_1$ as a factor.

Call the terms with these arguments terms of the *second class.*

Now the condition that g remains unchanged gives us the following relation between e^2, γ^2, and γ_1^2:

$$N\gamma^2 = Me^2 + N\gamma_1^2,$$

taking into account only the terms of lowest order in e^2, γ^2, and γ_1^2.

Hence, ultimately, $$e^2 = \frac{N}{M}(\gamma^2 - \gamma_1^2).$$

If this value of e^2 be substituted for it, we see that every term of the first class in the values of

$$\frac{1}{r_1} - \frac{1}{r}, \quad r - r_1, \quad u - u_1, \text{ and } v - v_1$$

will be divisible by $\gamma^2 - \gamma_1^2$, and that every term of the second class in the values of the same quantities will be divisible by e. Also every term of the first class in the value of $z - z_1$ will be divisible by $\gamma - \gamma_1$; and every term of the second class in the value of the same quantity will be divisible by $e\gamma_1$.

Now in the formation of the quantities

$$\left(\frac{1}{r_1} - \frac{1}{r}\right)^3, \quad \left(\frac{1}{r_1} - \frac{1}{r}\right)\{(u-u_1)^2 + (v-v_1)^2 - (r-r_1)^2\}, \text{ and } \left(\frac{1}{r_1} - \frac{1}{r}\right)(z-z_1)^2,$$

terms with the argument zero can only arise by multiplying together either

(1) Three terms of the first class;

(2) One term of the first and two of the second class;

or (3) Three terms of the second class, one of which at least involves e^2 as a factor.

Such a term formed in the first of these ways would be divisible by $(\gamma-\gamma_1)^3$ and therefore by $(\gamma^2-\gamma_1^2)^3$, since it can only involve even powers of γ and γ_1.

Such a term formed in the second of these ways would be divisible by $e^2(\gamma-\gamma_1)$ and therefore by $e^2(\gamma^2-\gamma_1^2)$ or by $(\gamma^2-\gamma_1^2)^2$.

Also such a term formed in the third of these ways would be divisible by e^4 or by $(\gamma^2-\gamma_1^2)^2$.

Hence, by the same principle as before, the value of $\frac{1}{r_1}-\frac{1}{r}$ must be divisible by $(\gamma^2-\gamma_1^2)^2$.

That is
$$2Fe^2\gamma_1^2+G(\gamma_1^4-\gamma^4)$$
is divisible by $(\gamma^2-\gamma_1^2)^2$; or
$$2F\frac{N}{M}(\gamma^2-\gamma_1^2)\gamma_1^2-G(\gamma^4-\gamma_1^4)$$
is divisible by $(\gamma^2-\gamma_1^2)^2$.

Now divide by $\gamma^2-\gamma_1^2$, and then put $\gamma_1^2=\gamma^2$;

therefore
$$2F\frac{N}{M}\gamma^2-2G\gamma^2=0,$$
or
$$\frac{F}{G}=\frac{M}{N},$$
which is the last of the relations announced above.

The results obtained in Cases III. and IV. may be rendered more general in the following manner:—

Let P denote the constant term in the reciprocal of the Moon's radius vector, considered as a function of e^2 and γ^2.

Then, taking e^2, e_1^2, and γ^2 to be related as in Case III., we have, by the same reasoning as before,
$$0=\frac{dP}{d(e^2)}(e_1^2-e^2)+\frac{dP}{d(\gamma^2)}\cdot\gamma^2+\text{terms of higher dimensions in } e_1^2-e^2 \text{ and } \gamma^2.$$

Also

$$0=\frac{dc}{d(e^2)}(e_1^2-e^2)+\frac{dc}{d(\gamma^2)}\cdot\gamma^2+\text{terms of higher dimensions in } e_1^2-e^2 \text{ and } \gamma^2.$$

Hence, we have ultimately, when $e_1^2=e^2$, and $\gamma^2=0$,

$$\text{Limit of } \frac{\gamma^2}{e^2-e_1^2}=\frac{\dfrac{dP}{d(e^2)}}{\dfrac{dP}{d(\gamma^2)}}=\frac{\dfrac{dc}{d(e^2)}}{\dfrac{dc}{d(\gamma^2)}}.$$

in which γ^2 is to be put $=0$ after the differentiations. The relation thus deduced holds good for all values of e^2. By equating the coefficients of e^2 on the two sides of the equation

$$\frac{dP}{d(e^2)}\cdot\frac{dc}{d(\gamma^2)}=\frac{dP}{d(\gamma^2)}\cdot\frac{dc}{d(e^2)},$$

we find $\dfrac{E}{F}=\dfrac{H}{K}$, as before.

Also, by equating the coefficients of higher powers of e^2, we obtain other relations between the coefficients of terms of higher orders in the value of P.

Similarly, taking e^2, γ^2, and γ_1^2 to be related as in Case IV., we have, by the same reasoning as before,

$$0=\frac{dP}{d(e^2)}\cdot e^2+\frac{dP}{d(\gamma^2)}(\gamma_1^2-\gamma^2)+\text{terms of higher dimensions in } e^2 \text{ and } \gamma_1^2-\gamma^2.$$

Also

$$0=\frac{dg}{d(e^2)}\cdot e^2+\frac{dg}{d(\gamma^2)}(\gamma_1^2-\gamma^2)+\text{terms of higher dimensions in } e^2 \text{ and } \gamma_1^2-\gamma^2.$$

Hence, we have ultimately, when $e^2=0$ and $\gamma_1^2=\gamma^2$,

$$\text{Limit of } \frac{\gamma^2-\gamma_1^2}{e^2}=\frac{\dfrac{dP}{d(e^2)}}{\dfrac{dP}{d(\gamma^2)}}=\frac{\dfrac{dg}{d(e^2)}}{\dfrac{dg}{d(\gamma^2)}},$$

in which e^2 is to be put $=0$ after the differentiations. The result thus deduced holds good for all values of γ^2. By equating the coefficients of γ^2

on the two sides of the equation

$$\frac{d(P)}{d(e^2)} \cdot \frac{dg}{d(\gamma^2)} = \frac{dP}{d(\gamma^2)} \cdot \frac{dg}{d(e^2)},$$

we find $\dfrac{F}{G} = \dfrac{M}{N}$, as before.

Similarly, by equating the coefficients of higher powers of γ^2, we obtain other relations between the coefficients of terms of higher orders in the value of P.

It may not be without interest to give here the result which I have obtained for the development of the constant term in the reciprocal of the Moon's radius vector.

The expression includes, besides the terms spoken of in the foregoing paper, an additional term depending on the square of the Sun's parallax. Reintroducing the symbol a to denote the length before defined, which in the paper has been taken as the unit of length, I find

The constant term in $\dfrac{a}{r}$

$$\begin{aligned}
&= 1 + \frac{1}{6}m^2 - \frac{179}{288}m^4 - \frac{97}{48}m^5 - \frac{757}{162}m^6 - \frac{4039}{432}m^7 - \frac{34751189}{1990656}m^8 - \frac{31013527}{995328}m^9 \\
&\quad + e'^2\left[\frac{1}{4}m^2 - \frac{799}{192}m^4 - \frac{873}{32}m^5 - \frac{287849}{2304}m^6 - \frac{268607}{576}m^7\right] \\
&\quad + e'^4\left[\frac{5}{16}m^2 - \frac{5401}{384}m^4 - \frac{18527}{128}m^5\right] \\
&\quad + \frac{a^2}{a'^2}\left[\frac{3}{16}m^2 + \frac{75}{128}m^3\right] \\
&\quad + e^4\left[\frac{1}{16}m^2 + \frac{225}{128}m^3\right] \\
&\quad + e^2\gamma^2\left[2m^2 + \frac{63}{8}m^3\right] \\
&\quad + \gamma^4\left[-m^2 + \frac{9}{8}m^3\right],
\end{aligned}$$

where e and γ have the same significations as in Delaunay's Theory.

The method which I employed in obtaining this expression is closely related to my first method, above alluded to, of proving the evanescence of the coefficients B and C.

The coefficients of e^4 and γ^4 were found independently, and from each of these, by means of the relations proved above, was derived a value of the coefficient of $e^2\gamma^2$. The perfect coincidence of these values supplied a test of the correctness of the calculations.

The terms of c and g which are required for this verification are the following:

$$c = \ldots\ldots + e^2\left(\frac{3}{8}m^2 + \frac{675}{64}m^3\right) + \gamma^2\left(6m^2 + \frac{189}{8}m^3\right) + \ldots$$

$$g = \ldots\ldots + e^2\left(\frac{3}{2}m^2 + \frac{189}{32}m^3\right) - \gamma^2\left(\frac{3}{2}m^2 - \frac{27}{16}m^3\right) + \ldots$$

I hope to lay the details of these calculations before the Society on some future occasion.

26.

NOTE ON SIR GEORGE AIRY'S INVESTIGATION OF THE THEORETICAL VALUE OF THE ACCELERATION OF THE MOON'S MEAN MOTION.

[From the *Monthly Notices of the Royal Astronomical Society* (1880), Vol. XL.]

I LOSE no time in pointing out briefly the reason why the Astronomer Royal, in the investigation which he communicated to the Society at the last Meeting, has failed to find my value of the coefficient of the Lunar Acceleration.

It may be useful, in the first place, to recall to mind that, according to my theory, the secular changes of

n, the Moon's mean motion,

and e', the eccentricity of the Earth's orbit,

are connected by the following relation:—

$$\frac{dn}{ndt} = \frac{e'de'}{dt}\left\{-3m^2 + \frac{3771}{32}m^4 + \frac{34047}{42}m^5 + \dots\right\},$$

where m denotes, as usual, the ratio of the Sun's mean motion to that of the Moon.

If we stop at the first term of the series within the brackets the result is identical with that found by Laplace.

We do not know why Laplace did not carry his investigations further than this first term; but he probably thought that the succeeding terms would prove to be inconsiderable.

It is seen, however, that these terms have very large numerical coefficients and that their sign is contrary to that of the first term, and on calculation it is found that the sum of the series is less than its first term nearly in the ratio of 3 to 5.

Hence the secular acceleration will be diminished in the same ratio, and its amount in a century, instead of being about 10″, will be reduced to nearly 6″.

No investigation of the Moon's secular acceleration can be satisfactory which does not take into account terms of the nature of those which give rise to the terms involving m^4, m^5, &c., above referred to.

There is nothing to object to in the general principles of the method adopted by the Astronomer Royal, but in the practical application of the method I notice very grave defects.

In the first place, the only periodic terms which are included in the Astronomer Royal's expressions for $T\frac{r}{a}$ and $P\frac{r}{a}$ and for the factors multiplying

$$\delta\frac{a}{r},\quad \frac{d}{dt}\left(\delta\frac{a}{r}\right),\quad \delta v,\quad \frac{d}{dt}(\delta v),\ \&c.,$$

on the right-hand side of the equations, are those which involve the angle $2D$ or F; whereas it will be seen by a reference to my paper in the *Philosophical Transactions* for 1853, that a great part of the coefficient of m^4 in the value of $\frac{dn}{ndt}$ there obtained arises from the combination of terms involving the angles S, $F-S$ and $F+S$ in the expressions for the Moon's coordinates with similar terms in

$$\delta\left(\frac{a}{r}\right),\quad \delta v,\ \&c.$$

In the present investigation terms of the forms last mentioned are simply ignored.

In the next place, it is to be noted that, although periodic terms depending on the angle F are introduced into the assumed values of $\delta\frac{a}{r}$ and δv, yet in Art. 12, the value of h which is the coefficient of t^2 in the value of δv, is found equal to $-Bb$, quite independently of the values of the coefficients e, f, g, k, and l, which occur in the terms thus introduced.

The result of this is to reduce the secular acceleration practically to its first term only; which accounts for the coincidence of the Astronomer Royal's value with that of Laplace.

It may also be remarked in reference to Art. 11, that although terms involving the argument $2F$ or $4D$ may be properly omitted, we must put

$$\sin^2 F = \frac{1}{2} - \frac{1}{2}\cos 2F,$$

and

$$\cos^2 F = \frac{1}{2} + \frac{1}{2}\cos 2F,$$

and the constant terms in these latter quantities should be taken into account.

After these general remarks, we will enter a little more closely on the consideration of one or two points in the investigation which are important.

Adopting the Astronomer Royal's notation, let

σ denote the Sun's mass,

A the semiaxis major of the Sun's (or Earth's) orbit,

E the eccentricity of the orbit,

R the radius vector at any time.

Then it may be shewn, as in the paper before us, that the mean value of

$$\frac{\sigma}{R^3} \text{ is } \frac{\sigma}{A^3}\frac{1}{(1-E^2)^3}, \quad = \frac{\sigma}{A^3}\left(1+\frac{3}{2}E^2\right) \text{ nearly.}$$

Hence if E receive the variation δE in the time t, this quantity will be increased in the ratio of $1+3E\delta E$ to 1 nearly, or in the ratio of $1+bt$ to 1, calling

$$3\frac{E\delta E}{t} = b.$$

Having arrived at this point, the Astronomer Royal assumes that the variation of the disturbing forces due to the variation δE in the eccentricity of the Sun's orbit will be represented by supposing

T to be replaced by $T(1+bt)$,

and similarly P to be replaced by $P(1+bt)$, and therefore that the new forces, the effects of which are to be found by the present method, are Tbt and Pbt respectively.

On consideration, however, it will appear that this is only true for the non-periodic term in P, and that the periodic terms, whether in P or T, will be changed by any given variation of E in very different ratios.

For instance, the periodic terms in both T and P which depend on the angle $2D$ or F will vary nearly in the same ratio as $1-\frac{5}{2}E^2$ does, instead of in the ratio in which $1+\frac{3}{2}E^2$ varies as in the above case.

Hence these terms will be changed by the above-mentioned variation of E in the ratio of $1+b't$ to 1, where

$$b' = -5\frac{E\delta E}{t} \text{ nearly.}$$

Again, the periodic terms in T and P which depend on the angles S, $F-S$ and $F+S$ will vary nearly in the same ratio as E does, so that these terms will be changed in the ratio of $1+b''t$ to 1, where

$$b'' = \frac{\delta E}{Et} \text{ nearly.}$$

Hence we see that the values of b' and b'' are quite different from that of b which belongs to the non-periodic term, and that b'' is much larger than the other two quantities.

The correct way of finding δT and δP, the changes of the disturbing forces T and P due to change in the eccentricity of the Sun's orbit, is to express T and P in terms of the Moon's coordinates v and r, the Sun's mean longitude L and its mean anomaly S, and the eccentricity E.

Hence δT and δP may be at once expressed in terms of δv, δr, and δE.

Thus calling V the Sun's longitude, and employing the other symbols in the sense before explained, we have

$$P = \frac{1}{2}\frac{\sigma r}{R^3} + \frac{3}{2}\frac{\sigma r}{R^3}\cos(2\nu - 2V),$$

$$T = \qquad -\frac{3}{2}\frac{\sigma r}{R^3}\sin(2\nu - 2V).$$

Or,
$$Pr = \frac{1}{2}\frac{\sigma r^2}{R^3} + \frac{3}{2}\frac{\sigma r^2}{R^3}\cos(2\nu - 2V),$$
$$Tr = \qquad -\frac{3}{2}\frac{\sigma r^2}{R^3}\sin(2\nu - 2V).$$

Now, by the formulæ of elliptic motion, we may find
$$\frac{1}{R^3} = \frac{1}{A^3}\left[1 + \frac{3}{2}E^2 + 3E\cos S\right],$$
$$\frac{1}{R^3}\cos(2\nu - 2V)$$
$$= \frac{1}{A^3}\left\{\left(1 - \frac{5}{2}E^2\right)\cos(2\nu - 2L) + \frac{7}{2}E\cos(2\nu - 2L - S) - \frac{1}{2}E\cos(2\nu - 2L + S)\right\},$$
$$\frac{1}{R^3}\sin(2\nu - 2V)$$
$$= \frac{1}{A^3}\left\{\left(1 - \frac{5}{2}E^2\right)\sin(2\nu - 2L) + \frac{7}{2}E\sin(2\nu - 2L - S) - \frac{1}{2}E\sin(2\nu - 2L + S)\right\},$$
neglecting terms involving $2S$, and powers of E above the second.

Substituting, and then taking the variation, we have
$$\begin{aligned}\delta(Pr) = {} & \frac{\sigma}{R^3}r\delta r + 3\frac{\sigma}{R^3}r\delta r\cos(2\nu - 2V) - 3\frac{\sigma}{R^3}r^2\delta\nu\sin(2\nu - 2V)\\ & + \frac{1}{2}\frac{\sigma r^2}{A^3}\left[3E\delta E + 3\delta E\cos S\right]\\ & + \frac{3}{2}\frac{\sigma r^2}{A^3}\Big[-5E\delta E\cos(2\nu - 2L) + \frac{7}{2}\delta E\cos(2\nu - 2L - S)\\ & \qquad\qquad - \frac{1}{2}\delta E\cos(2\nu - 2L + S)\Big]\end{aligned}$$
$$\begin{aligned}\delta(Tr) = {} & \qquad -3\frac{\sigma}{R^3}r\delta r\sin(2\nu - 2V) - 3\frac{\sigma}{R^3}r^2\delta\nu\cos(2\nu - 2V)\\ & - \frac{3}{2}\frac{\sigma r^2}{A^3}\Big[-5E\,\delta E\sin(2\nu - 2L) + \frac{7}{2}\delta E\sin(2\nu - 2L - S)\\ & \qquad\qquad - \frac{1}{2}\delta E\sin(2\nu - 2L + S)\Big]\end{aligned}$$
in which $-r^3\delta\left(\frac{1}{r}\right)$ may be written for $r\delta r$, and the expressions given by

the ordinary lunar theory in the case of unvaried eccentricity are to be substituted for v and r.

Hence, the expressions for $\delta\left(T\frac{r}{a}\right)$ and $\delta\left(P\frac{r}{a}\right)$, which are employed in the paper, are wholly incorrect, except in the case of the non-periodic term, which gives rise to the principal term of the secular acceleration or that found by Laplace.

The remark made near the close of the paper, viz. that the magnitudes of the quantities A, B, C, and therefore also that of the secular acceleration are proportional to the inverse cube of the Sun's distance, or to the cube of the Sun's parallax, can only be the result of inadvertence, as the Astronomer Royal himself will be the first to acknowledge.

In fact, the quantities A, B, C involve the factor $\frac{\sigma}{A^3}$ and this is equal to n'^2, where n' is the Sun's mean motion and is known. The Sun's mass σ is determined by means of the parallax from this equation; or conversely, if the Sun's mass be known the parallax is thereby determined.

The values of A, B, C are approximately as follows

$$A=\frac{3}{2}m^2,\quad B=\frac{1}{2}m^2,\quad C=\frac{3}{2}m^2,$$

where m denotes, as before, the ratio of the Sun's mean motion to that of the Moon.

27.

INVESTIGATION OF THE SECULAR ACCELERATION OF THE MOON'S MEAN MOTION, CAUSED BY THE SECULAR CHANGE IN THE ECCENTRICITY OF THE EARTH'S ORBIT.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. XL. (1880).]

As the question of the Moon's secular acceleration has lately been again brought before the Society, I have thought that it might not be useless or without interest to communicate an investigation of the two leading terms of that acceleration which I gave many years ago in my lectures on the lunar theory.

1. Let r, θ be the polar coordinates of the Moon at time t, $u = \dfrac{1}{r}$, $H = r^2 \dfrac{d\theta}{dt}$, μ the sum of the masses of the Earth and Moon; also let m' be the mass of the Sun, r', θ' its polar coordinates, a' the Sun's mean distance, n' its mean motion, and e' the eccentricity of its orbit, $\lambda' = n't + \epsilon'$ its mean longitude, and $\phi' = n't + \epsilon' - \omega'$ its mean anomaly.

Then the equations to be satisfied are

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{H^2} - \frac{1}{2}\frac{m'}{H^2u^3r'^3} - \frac{3}{2}\frac{m'}{H^2u^3r'^3}\cos 2(\theta - \theta')$$
$$+ \frac{3}{2}\frac{m'}{H^2u^4r'^3}\frac{du}{d\theta}\sin 2(\theta - \theta'),$$

and
$$\frac{d(H^2)}{d\theta} = -\frac{3m'}{u^4r'^3}\sin 2(\theta - \theta').$$

Also, by the formulæ of elliptic motion

$$\frac{m'}{r'^3} = \frac{m'}{a'^3}\left(\frac{a'}{r'}\right)^3 = n'^2\left\{1 + \frac{3}{2}e'^2 + 3e'\cos\phi' + \frac{9}{2}e'^2\cos 2\phi'\right\},$$

$$\begin{aligned}\frac{m'}{r'^3}\cos 2(\theta-\theta') &= \frac{m'}{a'^3}\left(\frac{a'}{r'}\right)^3\cos 2(\theta-\theta') \\ &= n'^2\left\{\left(1-\frac{5}{2}e'^2\right)\cos 2(\theta-\lambda') + \frac{7}{2}e'\cos(2\theta-2\lambda'-\phi')\right. \\ &\qquad -\frac{1}{2}e'\cos(2\theta-2\lambda'+\phi') \\ &\qquad \left.+\frac{17}{2}e'^2\cos(2\theta-2\lambda'-2\phi')\right\},\end{aligned}$$

and

$$\begin{aligned}\frac{m'}{r'^3}\sin 2(\theta-\theta') &= \frac{m'}{a'^3}\left(\frac{a'}{r'}\right)^3\sin 2(\theta-\theta') \\ &= n'^2\left\{\left(1-\frac{5}{2}e'^2\right)\sin 2(\theta-\lambda') + \frac{7}{2}e'\sin(2\theta-2\lambda'-\phi')\right. \\ &\qquad -\frac{1}{2}e'\sin(2\theta-2\lambda'+\phi') \\ &\qquad \left.+\frac{17}{2}e'^2\sin(2\theta-2\lambda'-2\phi')\right\}.\end{aligned}$$

The angles involved in these expressions are formed by combining the angle $2\theta - 2\lambda'$ with multiples of ϕ'.

For our present purpose we may omit the terms which involve $2\phi'$. Also, for the sake of brevity we may write

$$\begin{aligned} &n't \text{ instead of } n't+\epsilon'-\omega' \text{ or } \phi', \\ &2\theta-2n't \text{ instead of } 2\theta-2(n't+\epsilon') \text{ or } 2\theta-2\lambda', \\ &2\theta-3n't \text{ instead of } 2\theta-2\lambda'-\phi', \\ &2\theta-\ \ n't \text{ instead of } 2\theta-2\lambda'+\phi', \end{aligned}$$

since no ambiguity can arise from this abbreviation.

Hence our equations become

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{H^2} - \frac{1}{2}\frac{n'^2}{H^2u^3}\left\{1 + \frac{3}{2}e'^2 + 3e'\cos n't\right\}$$

$$-\frac{3}{2}\frac{n'^2}{H^2u^3}\left\{\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't) + \frac{7}{2}e'\cos(2\theta - 3n't)\right.$$

$$\left. - \frac{1}{2}e'\cos(2\theta - n't)\right\}$$

$$+\frac{3}{2}\frac{n'^2}{H^2u^4}\frac{du}{d\theta}\left\{\left(1 - \frac{5}{2}e'^2\right)\sin(2\theta - 2n't) + \frac{7}{2}e'\sin(2\theta - 3n't)\right.$$

$$\left. - \frac{1}{2}e'\sin(2\theta - n't)\right\},$$

and

$$\frac{d(H^2)}{d\theta} = -\frac{3n'^2}{u^4}\left\{\left(1 - \frac{5}{2}e'^2\right)\sin(2\theta - 2n't) + \frac{7}{2}e'\sin(2\theta - 3n't)\right.$$

$$\left. - \frac{1}{2}e'\sin(2\theta - n't)\right\}.$$

2. After these preliminaries, it will be convenient to begin by finding the relations between the actual mean motion n of the Moon and the constant parts of u and H^2 when these quantities are developed in the form we have adopted, carrying the approximation as far as terms involving $m^4e'^2$, on the supposition that e' and therefore also that n is constant.

For this purpose it is sufficient to take

$$nt + \epsilon = \theta + 3me'\sin n't - \frac{11}{8}m^2\left(1 - \frac{5}{2}e'^2\right)\sin(2\theta - 2n't)$$

$$-\frac{77}{16}m^2e'\sin(2\theta - 3n't) + \frac{11}{16}m^2e'\sin(2\theta - n't),$$

$$u = \frac{1}{a}\left\{1 - \frac{3}{2}m^2e'\cos n't + m^2\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't)\right.$$

$$\left. + \frac{7}{2}m^2e'\cos(2\theta - 3n't) - \frac{1}{2}m^2e'\cos(2\theta - n't)\right\},$$

which are readily derived from the equations of motion.

Differentiate the first of these equations and put

$$\frac{n'}{n} = m,$$

$$\therefore\ \frac{ndt}{d\theta}\left\{1 - 3m^2e'\cos n't - \frac{11}{4}m^3\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't) - \frac{231}{16}m^3e'\cos(2\theta - 3n't)\right.$$
$$\left. + \frac{11}{16}m^3e'\cos(2\theta - n't)\right\}$$
$$= 1 - \frac{11}{4}m^2\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't) - \frac{77}{8}m^2e'\cos(2\theta - 3n't) + \frac{11}{8}m^2e'\cos(2\theta - n't),$$

or
$$\frac{ndt}{d\theta} = 1 + \frac{9}{2}m^4e'^2 + 3m^2e'\cos n't - \frac{11}{4}m^2\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't)$$
$$- \frac{77}{8}m^2e'\cos(2\theta - 3n't) + \frac{11}{8}m^2e'\cos(2\theta - n't),$$

since the other terms only give rise to terms of higher orders than we have here taken into account.

Hence
$$H^2 = \left(\frac{d\theta}{u^2dt}\right)^2 = u^{-4}\left(\frac{dt}{d\theta}\right)^{-2}$$
$$= n^2a^4\left\{1 + 5m^4(1 - 5e'^2) + \frac{45}{4}m^4e'^2 + \frac{245}{4}m^4e'^2 + \frac{5}{4}m^4e'^2 + 6m^2e'\cos n't\right.$$
$$\left. - 4m^2\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't) - 14m^2e'\cos(2\theta - 3n't) + 2m^2e'\cos(2\theta - n't)\right\}$$
$$\times\left\{1 - 9m^4e'^2 + \frac{27}{2}m^4e'^2 + \frac{363}{32}m^4(1 - 5e'^2) + \frac{17787}{128}m^4e'^2 + \frac{363}{128}m^4e'^2\right.$$
$$- 6m^2e'\cos n't + \frac{11}{2}m^2\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't) + \frac{77}{4}m^2e'\cos(2\theta - 3n't)$$
$$\left. - \frac{11}{4}m^2e'\cos(2\theta - n't)\right\};$$

or, by actual multiplication,

$$H^2 = n^2a^4\left\{1 + \frac{523}{32}m^4(1 - 5e'^2) + \frac{14083}{64}m^4e'^2 - 18m^4e'^2 - 11m^4(1 - 5e'^2)\right.$$
$$- \frac{539}{4}m^4e'^2 - \frac{11}{4}m^4e'^2 + \frac{3}{2}m^2\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't)$$
$$\left. + \frac{21}{4}m^2e'\cos(2\theta - 3n't) - \frac{3}{4}m^2e'\cos(2\theta - n't)\right\}$$

$$= n^2a^4\left\{1 + \frac{171}{32}m^4(1-5e'^2) + \frac{4131}{64}m^4e'^2 + \frac{3}{2}m^2\left(1-\frac{5}{2}e'^2\right)\cos(2\theta-2n't)\right.$$

$$\left. + \frac{21}{4}m^2e'\cos(2\theta-3n't) - \frac{3}{4}m^2e'\cos(2\theta-n't)\right\}.$$

Hence the constant part of H^2 is

$$n^2a^4\left\{1 + \frac{171}{32}m^4 + \frac{2421}{64}m^4e'^2\right\},$$

n being the actual mean motion.

Hence

$$\frac{\mu}{H^2} = \frac{\mu}{n^2a^4}\left\{1 - \frac{171}{32}m^4(1-5e'^2) - \frac{4131}{64}m^4e'^2 + \frac{9}{8}m^4(1-5e'^2) + \frac{441}{32}m^4e'^2 + \frac{9}{32}m^4e'^2\right.$$

$$\left. - \frac{3}{2}m^2\left(1-\frac{5}{2}e'^2\right)\cos(2\theta-2n't) - \frac{21}{4}m^2e'\cos(2\theta-3n't) + \frac{3}{4}m^2e'\cos(2\theta-n't)\right\}$$

$$= \frac{\mu}{n^2a^4}\left\{1 - \frac{135}{32}m^4(1-5e'^2) - \frac{3231}{64}m^4e'^2 - \frac{3}{2}m^2\left(1-\frac{5}{2}e'^2\right)\cos(2\theta-2n't)\right.$$

$$\left. - \frac{21}{4}m^2e'\cos(2\theta-3n't) + \frac{3}{4}m^2e'\cos(2\theta-n't)\right\},$$

and therefore the constant part of $\frac{\mu}{H^2}$ is

$$\frac{\mu}{n^2a^4}\left\{1 - \frac{135}{32}m^4 - \frac{1881}{64}m^4e'^2\right\}.$$

3. Also

$$\frac{du}{d\theta} = \frac{1}{a}\left\{\frac{3}{2}m^3e'\sin n't - 2m^2\left(1-\frac{5}{2}e'^2\right)\sin(2\theta-2n't)\right.$$

$$\left. - 7m^2e'\sin(2\theta-3n't) + m^2e'\sin(2\theta-n't)\right\},$$

and

$$\frac{d^2u}{d\theta^2} = \frac{1}{a}\left\{-4m^2\left(1-\frac{5}{2}e'^2\right)\cos(2\theta-2n't) - 14m^2e'\cos(2\theta-3n't)\right.$$

$$\left. + 2m^2e'\cos(2\theta-n't)\right\};$$

also

$$\frac{n'^2}{H^2u^3}=\frac{m^2}{a}\left\{1+\frac{9}{2}m^2e'\cos n't-\frac{9}{2}m^2\left(1-\frac{5}{2}e'^2\right)\cos(2\theta-2n't)\right.$$
$$\left.-\frac{63}{4}m^2e'\cos(2\theta-3n't)+\frac{9}{4}m^2e'\cos(2\theta-n't)\right\},$$

and

$$\frac{n'^2}{H^2u^4}\frac{du}{d\theta}=\frac{m^2}{a}\left\{-2m^2\left(1-\frac{5}{2}e'^2\right)\sin(2\theta-2n't)-7m^2e'\sin(2\theta-3n't)\right.$$
$$\left.+m^2e'\sin(2\theta-n't)\right\}.$$

Hence, substituting in the first differential equation and transposing, we find the quantity which is to be equated to $\frac{\mu}{H^2}$ to be

$$\frac{1}{a}\left\{1+\left(\frac{1}{2}+\frac{3}{4}e'^2\right)m^2+\frac{27}{8}m^4e'^2-\frac{27}{8}m^4(1-5e'^2)-\frac{1323}{32}m^4e'^2-\frac{27}{32}m^4e'^2\right.$$
$$+\frac{3}{2}m^4(1-5e'^2)+\frac{147}{8}m^4e'^2+\frac{3}{8}m^4e'^2$$
$$\left.-\frac{3}{2}m^2\left(1-\frac{5}{2}e'^2\right)\cos(2\theta-2n't)-\frac{21}{4}m^2e'\cos(2\theta-3n't)+\frac{3}{4}m^2e'\cos(2\theta-n't)\right\}$$
$$=\frac{1}{a}\left\{1+\frac{1}{2}m^2\left(1+\frac{3}{2}e'^2\right)-\frac{15}{8}m^4(1-5e'^2)-\frac{321}{16}m^4e'^2\right.$$
$$-\frac{3}{2}m^2\left(1-\frac{5}{2}e'^2\right)\cos(2\theta-2n't)-\frac{21}{4}m^2e'\cos(2\theta-3n't)$$
$$\left.+\frac{3}{4}m^2e'\cos(2\theta-n't)\right\}.$$

Comparing this with the former expression and observing that $\frac{\mu}{n^2a^3}$ is nearly $=1$, we see that the periodic terms agree, and by equating the non-periodic parts, we have

$$\frac{\mu}{n^2a^3}\left\{1-\frac{135}{32}m^4(1-5e'^2)-\frac{3231}{64}m^4e'^2\right\}$$
$$=1+\frac{1}{2}m^2\left(1+\frac{3}{2}e'^2\right)-\frac{15}{8}m^4(1-5e'^2)-\frac{321}{16}m^4e'^2,$$

or $$\frac{\mu}{n^2a^3} = 1 + \frac{1}{2}m^2\left(1 + \frac{3}{2}e'^2\right) + \frac{75}{32}m^4(1 - 5e'^2) + \frac{1947}{64}m^4e'^2$$

$$= 1 + \frac{1}{2}m^2\left(1 + \frac{3}{2}e'^2\right) + \frac{75}{32}m^4 + \frac{1197}{64}m^4e'^2,$$

which gives the relation between n and a.

4. In the above, e' is considered constant throughout; if now we consider e' to be variable, we may choose n and a so that the constant (or rather the non-periodic) parts of u and of H^2 may have the same forms as before, and in this case we shall find the same relation between n and a as that which has just been found, and n will continue to signify the *actual mean motion* at the time to which θ belongs, but n and a will now become variable quantities, and, in order to satisfy our equations, it will be necessary to add certain periodic terms to u and H^2 which would not exist if e' were constant.

Suppose then that

$$u = \frac{1}{a}\left\{1 + \delta v - \frac{3}{2}m^2e'\cos n't + m^2\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't) + \frac{7}{2}m^2e'\cos(2\theta - 3n't)\right.$$
$$\left. - \frac{1}{2}m^2e'\cos(2\theta - n't)\right\},$$

and

$$H^2 = n^2a^4\left\{1 + 2\delta\eta + \frac{171}{32}m^4 + \frac{2421}{64}m^4e'^2 + \frac{3}{2}m^2\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't)\right.$$
$$\left. + \frac{21}{4}m^2e'^2\cos(2\theta - 3n't) - \frac{3}{4}m^2e'\cos(2\theta - n't)\right\}.$$

We will suppose e' to vary uniformly with the time, and very slowly, or, in other words, we will suppose

$$\frac{de'}{dt} \text{ to be constant, so that } \frac{d^2e'}{dt^2} = 0,$$

and we will neglect $$\left(\frac{de'}{dt}\right)^2.$$

We must therefore recollect that $\frac{de'}{d\theta}$ is not constant, but is equal to

$$\frac{de'}{dt}\cdot\frac{dt}{d\theta}=\frac{1}{Hu^2}\cdot\frac{de'}{dt}$$

$$=\frac{de'}{ndt}\left\{1+3m^2e'\cos n't-\frac{11}{4}m^2\cos(2\theta-2n't)-\frac{77}{8}m^2e'\cos(2\theta-3n't)\right.$$
$$\left.+\frac{11}{8}m^2e'\cos(2\theta-n't)\right\}.$$

5. In consequence of the variability of e', $\frac{du}{d\theta}$ will contain the additional terms

$$\frac{1}{Hu^2}\cdot\frac{1}{a}\left\{-\frac{da}{adt}-\frac{3}{2}m^2\frac{de'}{dt}\cos n't-5m^2e'\frac{de'}{dt}\cos(2\theta-2n't)\right.$$
$$\left.+\frac{7}{2}m^2\frac{de'}{dt}\cos(2\theta-3n't)-\frac{1}{2}m^2\frac{de'}{dt}\cos(2\theta-n't)\right\}$$
$$+\frac{1}{a}\cdot\frac{d\,.\,\delta v}{d\theta},$$

or

$$\frac{1}{an}\left\{-\frac{da}{adt}-\frac{3}{2}m^2\frac{de'}{dt}\cos n't-5m^2e'\frac{de'}{dt}\cos(2\theta-2n't)+\frac{7}{2}m^2\frac{de'}{dt}\cos(2\theta-3n't)\right.$$
$$\left.-\frac{1}{2}m^2\frac{de'}{dt}\cos(2\theta-n't)\right\}$$
$$+\frac{1}{a}\cdot\frac{d\,.\,\delta v}{d\theta},$$

to the order of approximation required.

Therefore also $\frac{d^2u}{d\theta^2}$ will contain the additional terms

$$\frac{1}{an}\left\{10m^2e'\frac{de'}{dt}\sin(2\theta-2n't)-7m^2\frac{de'}{dt}\sin(2\theta-3n't)+m^2\frac{de'}{dt}\sin(2\theta-n't)\right.$$
$$\left.+10m^2e'\frac{de'}{dt}\sin(2\theta-2n't)-7m^2\frac{de'}{dt}\sin(2\theta-3n't)+m^2\frac{de'}{dt}\sin(2\theta-n't)\right\}$$
$$+\frac{1}{a}\frac{d^2\,.\,\delta v}{d\theta^2},$$

neglecting $$\frac{d^2a}{dt^2},\ \left(\frac{da}{dt}\right)^2 \text{ and also } m^2\frac{de'}{dt}$$

in the coefficients of the periodic terms.

Hence $$\frac{d^2u}{d\theta^2}+u$$

contains the additional terms

$$\frac{1}{a}\left\{\frac{d^2 . \delta v}{d\theta}+\delta v\right\}$$

$$+\frac{1}{an}\left\{20m^2e'\frac{de'}{dt}\sin(2\theta-2n't)-14m^2\frac{de'}{dt}\sin(2\theta-3n't)+2m^2\frac{de'}{dt}\sin(2\theta-n't)\right\}.$$

Also $\frac{\mu}{H^2}$ contains the additional term $\frac{\mu}{n^2a^4}[-2\delta\eta]$.

The other terms which enter into the first differential equation receive no additional terms of the order to which we restrict ourselves.

6. Also differentiating the expression for H^2, and including terms of the order $m^4e'\frac{de'}{dt}$ in the non-periodic part, but only those of the orders

$$m^2\frac{de'}{dt} \text{ and } m^2e'\frac{de'}{dt}$$

in the periodic part, we have the following additional terms in $\frac{d(H^2)}{d\theta}$, viz.

$$n^2a^4\frac{1}{Hu^2}\left\{\frac{2dn}{ndt}+\frac{4da}{adt}+\frac{2421}{32}m^4e'\frac{de'}{dt}-\frac{15}{2}m^2\frac{e'de'}{dt}\cos(2\theta-2n't)\right.$$
$$\left.+\frac{21}{4}m^2\frac{de'}{dt}\cos(2\theta-3n't)-\frac{3}{4}m^2\frac{de'}{dt}\cos(2\theta-n't)\right\}$$
$$+n^2a^4\left(2\frac{d . \delta\eta}{d\theta}\right\}.$$

Also the right-hand side of the second differential equation contains the following additional quantity :—

$$m^2n^2a^4[4\delta v]\left\{3\sin(2\theta-2n't)+\frac{21}{2}e'\sin(2\theta-3n't)-\frac{3}{2}e'\sin(2\theta-n't)\right\},$$

which, as we shall immediately find, contains non-periodic terms of the order

$$m^4e'\frac{de'}{dt}.$$

Hence, taking the periodic parts of this equation, we have

$$2\frac{d\,.\,\delta\eta}{d\theta}=\frac{1}{n}\left\{\frac{15}{2}m^2e'\frac{de'}{dt}\cos(2\theta-2n't)-\frac{21}{4}m^2\frac{de'}{dt}\cos(2\theta-3n't)\right.$$
$$\left.+\frac{3}{4}m^2\frac{de'}{dt}\cos(2\theta-n't)\right\};$$

$$\therefore\ 2(\delta\eta)=\frac{1}{n}\left\{\frac{15}{4}m^2e'\frac{de'}{dt}\sin(2\theta-2n't)-\frac{21}{8}m^2\frac{de'}{dt}\sin(2\theta-3n't)\right.$$
$$\left.+\frac{3}{8}m^2\frac{de'}{dt}\sin(2\theta-n't)\right\}.$$

7. Substitute this in the first equation, putting $\frac{\mu}{n^2a^3}=1$ in the coefficients of the periodic terms, as these are only required to the order of m^2, and we obtain

$$\frac{d^2\,.\,\delta v}{d\theta^2}+\delta v=-\frac{1}{n}\left\{20m^2e'\frac{de'}{dt}\sin(2\theta-2n't)-14m^2\frac{de'}{dt}\sin(2\theta-3n't)\right.$$
$$+2m^2\frac{de'}{dt}\sin(2\theta-n't)+\frac{15}{4}m^2e'\frac{de'}{dt}\sin(2\theta-2n't)$$
$$\left.-\frac{21}{8}m^2\frac{de'}{dt}\sin(2\theta-3n't)+\frac{3}{8}m^2\frac{de'}{dt}\sin(2\theta-n't)\right\}$$
$$=-\frac{1}{n}\left\{\frac{95}{4}m^2e'\frac{de'}{dt}\sin(2\theta-2n't)-\frac{133}{8}m^2\frac{de'}{dt}\sin(2\theta-3n't)\right.$$
$$\left.+\frac{19}{8}m^2\frac{de'}{dt}\sin(2\theta-n't)\right\}.$$

$$\therefore\ \delta v=\frac{1}{n}\left\{\frac{95}{12}m^2e'\frac{de'}{dt}\sin(2\theta-2n't)-\frac{133}{24}m^2\frac{de'}{dt}\sin(2\theta-3n't)\right.$$
$$\left.+\frac{19}{24}m^2\frac{de'}{dt}\sin(2\theta-n't)\right\}.$$

Substitute this value of δv, and also the value of $\frac{1}{Hu^2}$, viz.

$$\frac{1}{n}\left\{1+3m^2e'\cos n't-\frac{11}{4}m^2\cos(2\theta-2n't)-\frac{77}{8}m^2e'\cos(2\theta-3n't)\right.$$
$$\left.+\frac{11}{8}m^2e'\cos(2\theta-n't)\right\},$$

for that quantity in the second differential equation, and equate the non-periodic parts which result from this substitution,

$$\therefore\ \frac{2dn}{ndt}+\frac{4da}{adt}+\frac{2421}{32}m^4e'\frac{de'}{dt}+\frac{165}{16}m^4e'\frac{de'}{dt}-\frac{1617}{64}m^4e'\frac{de'}{dt}-\frac{33}{64}m^4e'\frac{de'}{dt}$$

$$=\frac{95}{2}m^4e'\frac{de'}{dt}-\frac{931}{8}m^4e'\frac{de'}{dt}-\frac{19}{8}m^4e'\frac{de'}{dt},$$

or

$$\frac{2dn}{ndt}+4\frac{da}{adt}+\frac{963}{16}m^4e'\frac{de'}{dt}=-\frac{285}{4}m^4e'\frac{de'}{dt},$$

$$\therefore\ \frac{dn}{ndt}+2\frac{da}{adt}=-\frac{2103}{32}m^4e'\frac{de'}{dt}.$$

8. The substitution of the values of δv and $\delta\eta$ in the first differential equation introduces no non-periodic terms depending on $\frac{de'}{dt}$; consequently the value of $\frac{\mu}{n^2a^3}$ remains of the same form as before.

Hence

$$\log\left(\frac{\mu}{n^2a^3}\right)=\frac{1}{2}m^2\left(1+\frac{3}{2}e'^2\right)+\frac{75}{32}m^4+\frac{1197}{64}m^4e'^2-\frac{1}{8}m^4(1+3e'^2)$$

$$=\frac{1}{2}m^2\left(1+\frac{3}{2}e'^2\right)+\frac{71}{32}m^4+\frac{1173}{64}m^4e'^2;$$

$$\therefore\ \frac{2dn}{ndt}+3\frac{da}{adt}=-\frac{3}{2}m^2e'\frac{de'}{dt}-\frac{1173}{32}m^4e'\frac{de'}{dt}-m^2\left(\frac{dm}{mdt}\right)$$

$$=-\left(\frac{3}{2}m^2+\frac{1173}{32}m^4\right)e'\frac{de'}{dt}+m^2\left(\frac{dn}{ndt}\right),$$

since

$$m=\frac{n'}{n},\ \text{and}\ \therefore\ \frac{dm}{mdt}=-\frac{dn}{ndt},$$

n' being constant.

Hence

$$(4-2m^2)\frac{dn}{ndt}+6\frac{da}{adt}=-\left(3m^2+\frac{1173}{16}m^4\right)e'\frac{de'}{dt},$$

also from above

$$3\frac{dn}{ndt}+6\frac{da}{adt}=\qquad -\frac{6309}{32}m^4e'\frac{de'}{dt};$$

$$\therefore\ (1-2m^2)\frac{dn}{ndt}=-\left(3m^2-\frac{3963}{32}m^4\right)e'\frac{de'}{dt},$$

and $$\frac{dn}{ndt} = -\left(3m^2 - \frac{3771}{32}m^4\right)e'\frac{de'}{dt};$$

$$\therefore\quad 2\frac{da}{adt} = \left(3m^2 - \frac{3771}{32}m^4\right)e'\frac{de'}{dt} - \frac{2103}{32}m^4e'\frac{de'}{dt}$$

$$= \left(3m^2 - \frac{2937}{16}m^4\right)e'\frac{de'}{dt},$$

or $$\frac{da}{adt} = \left(\frac{3}{2}m^2 - \frac{2937}{32}m^4\right)e'\frac{de'}{dt}.$$

9. These equations give the rate of variation of the quantities n and a. We will now shew that n denotes the actual mean motion, as it did when e' was constant.

From the values of u and H^2 we find

$$\frac{dt}{d\theta} = \frac{1}{Hu^2} = \frac{1}{n}\left\{1 - 2\delta v - \delta\eta + \frac{9}{2}m^4e'^2 + 3m^2e'\cos n't - \frac{11}{4}m^2\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't)\right.$$
$$\left. - \frac{77}{8}m^2e'\cos(2\theta - 3n't) + \frac{11}{8}m^2e'\cos(2\theta - n't)\right\},$$

or

$$\frac{ndt}{d\theta} = 1 + \frac{9}{2}m^4e'^2 + 3m^2e'\cos n't - \frac{11}{4}m^2\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't)$$
$$- \frac{77}{8}m^2e'\cos(2\theta - 3n't) + \frac{11}{8}m^2e'\cos(2\theta - n't)$$
$$- \frac{425}{24}m^2e'\frac{de'}{ndt}\sin(2\theta - 2n't) + \frac{595}{48}m^2\frac{de'}{ndt}\sin(2\theta - 3n't)$$
$$- \frac{85}{48}m^2\frac{de'}{ndt}\sin(2\theta - n't).$$

Divide by

$$1 + \frac{9}{2}m^4e'^2 + 3m^2e'\cos n't$$

and take into account $m^4e'^2$ in the non-periodic term,

$$\therefore\quad \frac{ndt}{d\theta}\{1 - 3m^2e'\cos n't\} = 1 - \frac{11}{4}m^2\left(1 - \frac{5}{2}e'^2\right)\cos(2\theta - 2n't)$$

$$-\frac{77}{8}m^2e'\cos(2\theta-3n't)+\frac{11}{8}m^2e'\cos(2\theta-n't)$$

$$-\frac{425}{24}m^2e'\frac{de'}{ndt}\sin(2\theta-2n't)$$

$$+\frac{595}{48}m^2\frac{de'}{ndt}\sin(2\theta-3n't)-\frac{85}{48}m^2\frac{de'}{ndt}\sin(2\theta-n't),$$

and therefore

$$\int ndt=\theta+3me'\sin n't-\frac{11}{8}m^2\left(1-\frac{5}{2}e'^2\right)\sin(2\theta-2n't)$$

$$-\frac{77}{16}m^2e'\sin(2\theta-3n't)+\frac{11}{16}m^2e'\sin(2\theta-n't)$$

$$+3\frac{de'}{ndt}\cos n't+\frac{295}{24}m^2e'\frac{de'}{ndt}\cos(2\theta-2n't)-\frac{413}{48}m^2\frac{de'}{ndt}\cos(2\theta-3n't)$$

$$+\frac{59}{48}m^2\frac{de'}{ndt}\cos(2\theta-n't).$$

Hence θ differs from $\int ndt$ by periodic terms only, which proves the proposition.

The value of $\frac{dn}{ndt}$ above found agrees with that found in my paper published in the *Philosophical Transactions* for 1853.

28.

NOTE ON THE CONSTANT OF LUNAR PARALLAX.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. XL. (1880).]

FROM the report of a discussion which took place at a late meeting of the Society, I have reason to believe that an explanation of the apparent discrepancy between the value of the constant of parallax given by me in the Appendix to the *Nautical Almanac* for 1856, and in the *Monthly Notices*, vol. xiii. p. 263, and the value of the constant found by Hansen in the Introduction to his Lunar Tables, may not be unacceptable to some of our members.

It will be proper to begin this explanation by recalling to mind that my formula, in the article of the *Monthly Notices* above referred to, does not represent the parallax itself, but rather the sine of that quantity converted into seconds of arc by dividing by $\sin 1''$ or, which is the same thing, by multiplying by the number of seconds in the arc equal to the radius. The employment of the sine of the parallax instead of the parallax itself appears to be desirable both on theoretical as well as practical grounds.

In the first place, the sine of the parallax, being proportional to the reciprocal of the radius vector, is the quantity given directly by the lunar theory, and, in the next place, it is the same quantity which is wanted in the reduction of lunar observations.

What I have called the constant of parallax in the papers above referred to is, then, the constant term in the expression for the converted sine of the parallax, supposing the periodic terms to be expressed in cosines of angles which increase in proportion to the time. The value found for this constant was $3422''\cdot325$.

This quantity may also be called very appropriately the mean sine of the parallax, although I do not use the term in the papers referred to.

The value of the corresponding constant in the expression of the parallax itself is $0''\cdot157$ greater than this, or $3422''\cdot48$, which may appropriately be called the mean parallax.

The formula in the Introduction to Hansen's Lunar Tables does not give the sine of the parallax, but the *logarithm* of the sine of the parallax, and the constant which Hansen calls C is a quantity such that the constant term in his expression for the logarithm of the sine of the parallax is $\log \sin C$.

Now, it is plain that the constant term in the development of log sin parallax is a different quantity from the logarithm of the constant term of the sine of the parallax, and hence my constant of parallax differs from Hansen's quantity $\dfrac{\sin C}{\sin 1''}$.

We may readily express the relation between these two constants in the case in which the orbit is supposed to be an undisturbed ellipse.

In this case, if the reciprocal of the radius vector, which is proportional to the sine of the parallax, be developed in terms of cosines of multiples of the mean anomaly,

then, a being the semi-axis major,

and e the eccentricity of the orbit,

the constant term in the development will be $\dfrac{1}{a}$.

In the same case, the constant term in the development of the logarithm of the reciprocal of the radius vector, expressed in terms of the same form as before, will be

$$\log \frac{1}{a}\left(1 - \frac{1}{4}e^2\right)$$

very nearly, instead of $\log \dfrac{1}{a}$; so that if c denote the constant term in the former development, and $\log c'$ the constant term in the latter, we shall have

$$\frac{c'}{c} = 1 - \frac{1}{4}e^2 \text{ very nearly.}$$

This relation will still be approximately though not exactly satisfied when the Moon's perturbations are taken into account.

Hansen himself, in a paper in the 17th volume of the *Astronomische Nachrichten*, p. 299, in which he gives the results which he had obtained in a preliminary investigation of the lunar perturbations, finds that the number corresponding to the constant term in the logarithm of the sine of the parallax requires to be augmented by $2''\cdot71$ in order to reduce it to the constant term in the sine of the parallax itself.

Calling the parallax p, Hansen finds that the value of the constant term in $\log\left(\frac{\sin p}{\sin 1''}\right)$ is

$$\log(3419''\cdot35),$$

and hence he concludes that the constant term in $\left(\frac{\sin p}{\sin 1''}\right)$ is $3422''\cdot06$.

By repeating Hansen's calculation and taking into account some small terms omitted by him, I find the amount of the reduction to be slightly less than the above, viz. $2''\cdot67$, so that the constant term in $\frac{\sin p}{\sin 1''}$ according to Hansen's preliminary theory would be $3422''\cdot02$.

This value, however, is not immediately comparable with my own, being founded on different elements.

Both values are purely theoretical, depending on the ratio of the Moon's mass to that of the Earth, the ratio of the Earth's equatorial and polar axes, and the ratio of the Earth's radius to the length of the seconds' pendulum in a given latitude.

If M denote the mass of the Earth,

m that of the Moon,

A the Earth's equatorial radius,

R the Earth's radius at a point of which the sine of the latitude is

$$\frac{1}{\sqrt{3}},$$

P the length of the seconds' pendulum at the same point;

then the constant term of the sine of the horizontal parallax corresponding to the latitude just specified may be represented by

$$\left(\frac{M}{M+m} \cdot \frac{R}{P}\right)^{\frac{1}{3}} F,$$

and therefore the constant term of the sine of the equatorial horizontal parallax may be represented by

$$\frac{A}{R}\left(\frac{M}{M+m} \cdot \frac{R}{P}\right)^{\frac{1}{3}} F = \left(\frac{M}{M+m} \cdot \frac{A^3}{R^2 P}\right)^{\frac{1}{3}} F,$$

where F is a factor which may be found by theory from elements which may be considered as known with all desirable accuracy.

The values of $\frac{M}{m}$, A, R and P employed in finding my constant are the following:—

$$\frac{M}{m} = 81{\cdot}5,$$

which corresponds very nearly to Dr Peters' constant of Nutation;

$$A = 20923505 \text{ English feet,}$$
$$R = 20900320 \quad ,,$$
$$P = 3{\cdot}256989 \quad ,,$$

R and P belong to a point the sine of the *geographical* latitude of which is $\frac{1}{\sqrt{3}}$.

A and R are the quantities found from Bessel's latest determination of the figure and dimensions of the Earth as given in *Astron. Nachr.*, Vol. XIX., p. 216, supposing that

$$1 \text{ Toise} = 6{\cdot}394564 \text{ English feet.}$$

P is found thus: according to the formula given in p. 94 of Baily's Report on Foster's Pendulum experiments, (*Mem. of the Roy. Astr. Soc.*, Vol. VII.), the square of the number of vibrations made in a mean solar day, at a point the sine of whose geographical latitude is $\frac{1}{\sqrt{3}}$, by a pendulum which vibrates seconds in London is

$$7441625711 + \frac{1}{3}(38286335) = 7454387823.$$

Also Captain Kater's determination of the length of the seconds' pendulum in London is

$$39{\cdot}13929 \text{ inches} = 3{\cdot}2616075 \text{ feet}.$$

Hence as the square of the number of vibrations made at a given place in a given time varies inversely as the length of the pendulum, we derive the value above given for P.

The values of the fundamental elements employed by Hansen are the following:—

$$\frac{M}{m} = 80,$$

$$A = 6377157 \text{ metres},$$

$$R_1 = 6370063 \quad ,,$$

$$P_1 = 0{\cdot}992666 \quad ,,$$

and R_1 and P_1 belong to a point the sine of the *geocentric* latitude of which is $\frac{1}{\sqrt{3}}$.

The corresponding values of R and P for a point the sine of whose geographical latitude is $\frac{1}{\sqrt{3}}$ are the following:—

$$R = 6370126 \text{ metres},$$

$$P = 0{\cdot}992651 \quad ,,$$

And the constant term of the sine of the equatorial horizontal parallax may be represented either by

$$\left(\frac{M}{M+m}\,\frac{A^3}{R^2P}\right)^{\frac{1}{3}} F, \text{ or by } \left(\frac{M}{M+m}\,\frac{A^3}{R_1^2P_1}\right)^{\frac{1}{3}} F_1.$$

In my calculation of the factor F, I took into account terms of the order of the square of the Earth's compression. It would otherwise have been useless to distinguish between R^2P and $R_1^2P_1$ or between F and F_1.

At the time when Hansen's paper appeared in the *Astron. Nachr.* Bessel's latest determination of the figure and dimensions of the Earth was not available. Hansen employed an earlier determination given by Bessel in *Astron. Nachr.*, Vol. XIV., p. 344, in which the results were affected by an error in the calculation of the French arc of the meridian which was discovered later.

Hence the corrections to be applied to the logarithms employed by Hansen in order to make them agree with those employed by me are the following, expressed in units of the 7th decimal:—

	Correction.
$\log\left(\frac{M}{M+m}\right)$	$+987$
$\log\left(\frac{A}{R}\right)$	$+\ 25$
$\log\left(\frac{R}{P}\right)$	-150

The correction to be applied to Hansen's value of the logarithm of the constant term in the sine of the parallax is therefore

$$25+\frac{1}{3}(987-150)=304 \text{ of the same units.}$$

And the corresponding correction of the constant term of the sine of the parallax will be $0''\cdot24$, and therefore according to Hansen's preliminary theory, employing my system of fundamental data, the value of this constant term will be $3422''\cdot26$.

In my independent transformation of Hansen's expression I found the rather more precise value $3422''\cdot264$.

This is less than my own value of the same constant by $0''\cdot06$ nearly, as stated in my paper in the Appendix to the *Nautical Almanac* for 1856.

I there intimated my belief that Hansen's definitive theory would probably be found to introduce a correction to his former value of the constant term in question, and this turns out to be the case.

In *Astron. Nachr.*, Vol. XVII., p. 298, the constant term in $-w$ which denotes the perturbations of the natural logarithm of the reciprocal of the radius vector, divided by $\sin 1''$, is given as $1345''\cdot281$, but in the Introduction to Hansen's Lunar Tables this same quantity is given as $1348''\cdot840$. Hence, the correction to the former value is $3''\cdot559$, and multiplying this by $\sin 1''$ and by $3422''$ we find the corresponding correction of the constant of parallax to be $0''\cdot059$, so that this constant becomes $3422''\cdot323$, a result which agrees perfectly with my own.

In this connection it may be worth mentioning that the only periodic term in which I found any difference much exceeding $0''\cdot01$ between my coefficients of parallax and those obtained by a transformation of the results of Hansen's preliminary theory was that which has the argument denoted by $t+z$ in Damoiseau's notation.

The corresponding term in $-w$ is in Hansen's preliminary theory

$$10''\cdot 92 \cos(t+z),$$

whereas in the Introduction to the Lunar Tables this term is

$$8''\cdot 73 \cos(t+z);$$

the correction to the coefficient is $-2''\cdot 19$, and multiplying this as before by $\sin 1''$ and by $3422''$ we find the correction to the corresponding term of the sine of the parallax to be

$$-0''\cdot 036 \cos(t+z),$$

and if this be applied to the value of this term in the preliminary theory, viz. $0''\cdot 181 \cos(t+z)$,

the result is $0''\cdot 145 \cos(t+z)$,

which agrees perfectly with my own.

It should be remarked that, in the Introduction to his Lunar Tables, Hansen still continues to use the same fundamental data as he had done in his earlier paper, so that the value of the constant term in the sine of the parallax according to the data adopted in the Tables is $3422''\cdot 08$.

Note added June 17, 1880.

In Professor Newcomb's valuable transformation of Hansen's Lunar Theory, which I have just received, it is wrongly assumed that I employed the same data as Hansen for the figure and dimensions of the Earth, and that my value of P, viz. 3·256989 feet, relates, like Hansen's, to a point the sine of whose *geocentric* latitude is $\frac{1}{\sqrt{3}}$, whereas it should be the *geographical* latitude, as that is the latitude which enters into Baily's formula from which my value of P is deduced.

In consequence of this, Professor Newcomb finds a discrepancy of $0''\cdot 03$ between Hansen's value of the constant of parallax and mine when both are derived from the same system of fundamental data; but it has been shewn above that no such discrepancy exists.

By a typographical error, the value of P which Professor Newcomb quotes from me is printed as 3·256 89 feet, instead of 3·256989 feet.

29.

NOTE ON THE INEQUALITY IN THE MOON'S LATITUDE WHICH IS DUE TO THE SECULAR CHANGE OF THE PLANE OF THE ECLIPTIC.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. XLI. (1881).]

THE first theoretical explanation of this inequality was given by Hansen in the year 1849, in No. 685 of the *Astronomische Nachrichten*, just a year after the Astronomer Royal had pointed out, in a letter published in the same journal—*Beilage zu No.* 648—that such an inequality was clearly indicated by the observations. In the same paper Hansen shews that there is a small term in the Moon's longitude depending on the same cause, the coefficient of which amounts to about $0''\cdot 5$, the inequality being proportional to the cosine of the longitude of the Moon's node. The existence of this inequality also had been indicated by the Astronomer Royal from the observations, though he assigns to it a somewhat larger coefficient.

The calculation of both these inequalities is given by Hansen somewhat more fully in p. 491, Art. 176 of his *Darlegung*.

In 1853 I communicated to Mr Godfray a simple theoretical explanation of the inequality in latitude, which he inserted in his *Elementary Treatise on the Lunar Theory*. This explanation is there given in rather too compendious a form, and I propose in the course of this paper to present to the Society the same investigation, with some slight modification, together with some additional remarks, which will, I hope, render it clearer than before.

At the Meeting of the Society in March last, the Astronomer Royal gave an investigation of the inequality in latitude based upon the equations supplied by the "Factorial Tables" of his "Numerical Lunar Theory." About one portion of this investigation I wish to make a remark which seems to be important.

The Astronomer Royal forms his equations with reference to the *fixed* ecliptic, and, by integrating them, derives the value of the disturbed latitude above the *fixed* ecliptic, whence the latitude above the variable ecliptic is immediately deduced.

The latitude so found contains not only the inequality in latitude required, but also the small residual terms

$$Bt\{\cdot 003 \sin \overline{nt - C} + \cdot 005 \sin \overline{nt - 2Nt + C}\},$$

which the Astronomer Royal rejects, attributing them to accidental errors in the last places of the decimals employed.

I shall presently attempt to shew that these terms must indeed be rejected, though not for the reason here supposed, but because they are destroyed by other terms which would be found by a more complete investigation.

It should be remarked that if terms of the above form really existed, they would, notwithstanding the smallness of their numerical coefficients, ultimately become much more important than the other terms in which t does not occur in the coefficients.

I propose to prove that in the complete solution of the differential equations no terms of the above-mentioned form can occur, supposing the displacements of the plane of the ecliptic to be proportional to the first power of t. The method which I employ for this purpose is the following.

Instead of solving the differential equations of motion with reference to the *fixed ecliptic* and then transforming the results so as to make them apply to the *variable ecliptic*, I first transform the differential equations of motion, so as to make them refer to the *variable ecliptic*, and when this is done, it is found that the terms which contain t in their coefficients disappear completely from the differential equations, so that the solution may be effected by the ordinary methods without any difficulty.

Employing the same data and notation as the Astronomer Royal, and taking into account only the terms which are independent of the Moon's eccentricity and inclination, I find

$$\delta s = -1''\cdot 424 \cos(nt - C) + 0''\cdot 048 \cos(-nt + 2Nt - C) - 0''\cdot 007 \cos(3nt - 2Nt - C).$$

The reason why, in the result found by the Astronomer Royal, the terms which are multiplied by t do not completely destroy each other, as they ought to do, appears to be the following.

It is at once seen, from the form of the periodic terms to which the Astronomer Royal confines his attention, that his investigation is only complete with respect to the terms which are independent of the eccentricity and inclination of the Moon's orbit. In order to take the eccentricity and inclination into account, other periodic terms must be included, the arguments of which involve the Moon's mean anomaly and its mean distance from the node. From the combination of these terms with each other will arise terms with the same *arguments* as those which are independent of the eccentricity and inclination, while each of their *coefficients* contains the square of one of these elements as a factor. Hence it is clear that terms of this order are omitted in the investigation.

On the other hand, a slight examination shews that the coefficients in the Astronomer Royal's expressions for

$$\frac{r}{a}\cos l \text{ and } v,$$

as well as in the quantities taken from his Factorial Table, include very sensible portions depending on the squares of the eccentricity and inclination.

In fact, it is plain that this must necessarily be the case since the quantities in question are functions of the Moon's *actual* coordinates, in which the numerical values of those elements are essentially involved.

Now, if terms depending on the squares of the eccentricity and inclination were either wholly neglected, or completely taken into account, the terms which are multiplied by t would be found identically to destroy each other; but if, as in the present case, such terms are taken into account in one part of the investigation, and omitted in another part, it will follow that some of the terms multiplied by t will remain outstanding.

A curious circumstance relating to this inequality of latitude remains to be noticed.

In the *Mécanique Céleste*, tome III. p. 185, Laplace proves that the plane of the Earth's orbit in its secular motion carries the plane of the

Moon's orbit with it, so that the inclination of the Moon's orbit to the variable ecliptic is not liable to any secular variation.

In the same place he finds an analytical expression for the perturbation of latitude in reference to the variable ecliptic which is caused by the secular change in that plane.

Now the point to be noticed is that this analytical expression given by Laplace requires only the very slightest possible development to furnish for the inequality in question a result which is identical with the value given by the formula of Hansen, in which displacements of the ecliptic varying not only as the first but also as the second power of the time are taken into account. It is true that Laplace imagined that this inequality would turn out to be insensible, but this was only because he had not attempted to turn his formula into numbers.

Analysis.

I. Investigation of the inequality in the Moon's latitude which is due to the secular motion of the plane of the ecliptic, making the same suppositions and employing the same data as the Astronomer Royal.

At the time t let x, y, z be the rectangular coordinates of the Moon, and x', y' those of the Sun, referred to the Earth's centre as origin, the variable plane of the ecliptic at the same time being taken as the plane of xy.

Also at the time t let ξ, η, ζ be the rectangular coordinates of the Moon, and ξ', η', ζ' those of the Sun, taking the fixed plane of the ecliptic corresponding to $t=0$ as the plane of $\xi\eta$.

For greater simplicity we will suppose, with the Astronomer Royal, that the variable ecliptic intersects the fixed ecliptic in a fixed line, and that the angle between these two planes is proportional to the time.

Let this fixed line be taken as the axis of x and also as the axis of ξ, and let ωt be the angle between the variable and the fixed ecliptic, then the relations between the coordinates belonging to the two systems will be

$$\begin{aligned}\xi &= x,\\ \eta &= y\cos\omega t - z\sin\omega t,\\ \zeta &= z\cos\omega t + y\sin\omega t,\end{aligned}$$

and similarly
$$\xi' = x',$$
$$\eta' = y' \cos \omega t,$$
$$\zeta' = y' \sin \omega t.$$

Let r be the Moon's radius vector at time t, r' that of the Sun, m' the Sun's mass, μ the sum of the masses of the Earth and Moon, and R the disturbing function, then we have

$$R = -\frac{1}{2}\frac{m'r^2}{r'^3} + \frac{3}{2}\frac{m'(\xi\xi' + \eta\eta' + \zeta\zeta')^2}{r'^5},$$

and the equations of motion, with reference to the fixed ecliptic, will be

$$\frac{d^2\xi}{dt^2} + \frac{\mu\xi}{r^3} = \frac{dR}{d\xi},$$

$$\frac{d^2\eta}{dt^2} + \frac{\mu\eta}{r^3} = \frac{dR}{d\eta},$$

$$\frac{d^2\zeta}{dt^2} + \frac{\mu\zeta}{r^3} = \frac{dR}{d\zeta},$$

or, substituting the values of

$$\frac{dR}{d\xi},\ \frac{dR}{d\eta} \text{ and } \frac{dR}{d\zeta},$$

$$(1)\quad \frac{d^2\xi}{dt^2} + \frac{\mu\xi}{r^3} = -\frac{m'\xi}{r'^3} + \frac{3m'\xi'}{r'^5}(\xi\xi' + \eta\eta' + \zeta\zeta'),$$

$$(2)\quad \frac{d^2\eta}{dt^2} + \frac{\mu\eta}{r^3} = -\frac{m'\eta}{r'^3} + \frac{3m'\eta'}{r'^5}(\xi\xi' + \eta\eta' + \zeta\zeta'),$$

$$(3)\quad \frac{d^2\zeta}{dt^2} + \frac{\mu\zeta}{r^3} = -\frac{m'\zeta}{r'^3} + \frac{3m'\zeta'}{r'^5}(\xi\xi' + \eta\eta' + \zeta\zeta').$$

Now we have, from the values of η and ζ above given,

$$\frac{d^2\eta}{dt^2} = \left(\frac{d^2y}{dt^2} - 2\omega\frac{dz}{dt} - \omega^2 y\right)\cos\omega t - \left(\frac{d^2z}{dt^2} + 2\omega\frac{dy}{dt} - \omega^2 z\right)\sin\omega t,$$

and

$$\frac{d^2\zeta}{dt^2} = \left(\frac{d^2z}{dt^2} + 2\omega\frac{dy}{dt} - \omega^2 z\right)\cos\omega t + \left(\frac{d^2y}{dt^2} - 2\omega\frac{dz}{dt} - \omega^2 y\right)\sin\omega t,$$

and therefore

$$\frac{d^2\eta}{dt^2}\cos\omega t + \frac{d^2\zeta}{dt^2}\sin\omega t = \frac{d^2y}{dt^2} - 2\omega\frac{dz}{dt} - \omega^2 y,$$

$$\frac{d^2\zeta}{dt^2}\cos\omega t - \frac{d^2\eta}{dt^2}\sin\omega t = \frac{d^2z}{dt^2} + 2\omega\frac{dy}{dt} - \omega^2 z.$$

Now substitute for ξ, η, ζ and ξ', η', ζ' their values in terms of x, y, z and x', y' respectively, in

(1),

(2) $\cos\omega t +$ (3) $\sin\omega t$,

(3) $\cos\omega t -$ (2) $\sin\omega t$,

bearing in mind that

$$\xi\xi' + \eta\eta' + \zeta\zeta' = xx' + yy',$$

since each of these quantities represents $rr'\cos(r, r')$, and we have

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} = -\frac{m'x}{r'^3} + \frac{3m'x'}{r'^5}(xx' + yy'),$$

$$\frac{d^2y}{dt^2} - 2\omega\frac{dz}{dt} - \omega^2 y + \frac{\mu y}{r^3} = -\frac{m'y}{r'^3} + \frac{3m'y'}{r'^5}(xx' + yy'),$$

$$\frac{d^2z}{dt^2} + 2\omega\frac{dy}{dt} - \omega^2 z + \frac{\mu z}{r^3} = -\frac{m'z}{r'^3},$$

which are the equations of the Moon's motion, with reference to the variable ecliptic.

The motion of the ecliptic is so slow (that is, ω is so small) that the terms involving ω^2 may be neglected.

We will now change the notation by writing for the Moon's coordinates $x + \delta x$, $y + \delta y$, and $z + \delta z$, instead of x, y, z respectively, in which expressions the new quantities x, y, z are taken so as to satisfy the equations of motion

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} = -\frac{m'x}{r'^3} + \frac{3m'x'}{r'^5}(xx' + yy'),$$

$$\frac{d^2y}{dt^2} + \frac{\mu y}{r^3} = -\frac{m'y}{r'^3} + \frac{3m'y'}{r'^5}(xx' + yy'),$$

$$\frac{d^2z}{dt^2} + \frac{\mu z}{r^3} = -\frac{m'z}{r'^3},$$

which are those of the ordinary lunar theory, in which the motion of the ecliptic is not taken into account, so that x, y, z may be supposed to be known functions of t.

Hence the equations for determining the small increments δx, δy, δz of the coordinates, which are due to the motion of the ecliptic, are the following:

$$\frac{d^2\delta x}{dt^2} + \left(\frac{\mu}{r^3} + \frac{m'}{r'^3}\right)\delta x - \frac{3\mu x}{r^5}(x\delta x + y\delta y + z\delta z) \qquad = \frac{3m'x'}{r'^5}(x'\delta x + y'\delta y),$$

$$\frac{d^2\delta y}{dt^2} + \left(\frac{\mu}{r^3} + \frac{m'}{r'^3}\right)\delta y - \frac{3\mu y}{r^5}(x\delta x + y\delta y + z\delta z) - 2\omega\frac{dz}{dt} = \frac{3m'y'}{r'^5}(x'\delta x + y'\delta y),$$

$$\frac{d^2\delta z}{dt^2} + \left(\frac{\mu}{r^3} + \frac{m'}{r'^3}\right)\delta z - \frac{3\mu z}{r^5}(x\delta x + y\delta y + z\delta z) + 2\omega\frac{dy}{dt} = 0.$$

We may remark that no terms involving arbitrary constants need be added to the values of δx, δy, δz, since these may be supposed to be already included in the values of x, y, z.

Hence we may choose for δx, δy, δz any particular values which satisfy these differential equations, and we may consider these values to contain ω as a factor throughout.

If γ denote the sine of the mean inclination of the Moon's orbit, the value of z, and therefore that of $\frac{dz}{dt}$, will contain γ as a factor throughout. Hence the form of the first two of these differential equations shews that the values of δx, δy, found under the above conditions, will contain $\gamma\omega$ as a factor throughout, and therefore that the term

$$\frac{3\mu z}{r^5}(x\delta x + y\delta y + z\delta z),$$

which occurs in the third differential equation, will contain the factor $\gamma^2\omega$ throughout.

If, therefore, we neglect the square of γ, the equation for δz takes the simple form

$$\frac{d^2\delta z}{dt^2} + \left(\frac{\mu}{r^3} + \frac{m'}{r'^3}\right)\delta z + 2\omega\frac{dy}{dt} = 0.$$

Now let θ be the Moon's longitude at time t measured from the axis of x, that is from the line of intersection of the variable and of the fixed ecliptic.

Also let nt and $n't$ be the mean longitudes of the Moon and the Sun, omitting, for the sake of brevity in writing, the constants which always accompany nt and $n't$ respectively.

For the sake of simplicity, we will now neglect the eccentricities of the two orbits as well as their mutual inclination.

In this case we have, with abundant accuracy for our present purpose,

$$r = 0{\cdot}99911{,}92 - 0{\cdot}00717{,}34 \cos 2(nt - n't) - 0{\cdot}00002{,}00 \cos 4(nt - n't),$$

$$\theta = nt \qquad + 0{\cdot}01021{,}14 \sin 2(nt - n't) + 0{\cdot}00004{,}24 \sin 4(nt - n't),$$

where, as in my paper in the *Monthly Notices*, Vol. XXXVIII. p. 46, the angles are expressed in the circular measure, and the unit of distance is the mean distance in an undisturbed orbit which would be described by the Moon about the Earth in its actual periodic time.

Hence we have, as in the paper referred to—

$$\mu = n^2, \text{ and } \frac{m'}{r'^3} = n'^2.$$

Now choose the unit of time such that $n - n' = 1$;
therefore, since in the case of the Moon

$$\frac{n'}{n} = 0{\cdot}07480{,}13,$$

we have $$n' = 0{\cdot}08084{,}89,$$

and $$n = 1{\cdot}08084{,}89.$$

From the values of r and θ above given, it is readily found that

$$\begin{aligned} y = r \sin\theta = & -0{\cdot}00868{,}79 \sin(-nt + 2n't) \\ & + 0{\cdot}99909{,}31 \sin nt \\ & + 0{\cdot}00151{,}43 \sin(3nt - 2n't) \\ & + 0{\cdot}00000{,}59 \sin(5nt - 4n't), \end{aligned}$$

and hence that

$$\frac{dy}{dt} = +0{\cdot}00798{,}55\cos(-nt+2n't)$$
$$+1{\cdot}07986{,}87\cos nt$$
$$+0{\cdot}00466{,}54\cos(3nt-2n't)$$
$$+0{\cdot}00002{,}98\cos(5nt-4n't),$$

and also, as in the paper referred to above,

$$\frac{\mu}{r^3}+\frac{m'}{r'^3} = 1{\cdot}17804{,}45 + 0{\cdot}02523{,}37\cos 2(nt-n't) + 0{\cdot}00025{,}16\cos 4(nt-n't).$$

Hence the equation to be solved becomes

$$\frac{d^2\delta z}{dt^2} + \delta z\left[1{\cdot}17804{,}45 + 0{\cdot}02523{,}37\cos 2(nt-n't) + 0{\cdot}00025{,}16\cos 4(nt-n't)\right]$$
$$+\omega\left[0{\cdot}01597{,}1\cos(-nt+2n't) + 2{\cdot}15973{,}7\cos nt\right.$$
$$\left.+0{\cdot}00933{,}1\cos(3nt-2n't) + 0{\cdot}00006{,}0\times\cos(5nt-4n't)\right]=0.$$

Assume

$$\delta z = \omega\left[c_{-3}\cos(-3nt+4n't) + c_{-1}\cos(-nt+2n't) + c_1\cos nt\right.$$
$$\left.+c_3\cos(3nt-2n't) + c_5\cos(5nt-4n't)\right],$$

then, by substituting for δz and equating coefficients of similar terms, we have

$$-7{\cdot}34339{,}8c_{-3} + 0{\cdot}01261{,}7c_{-1} + 0{\cdot}00012{,}6c_1 = 0,$$
$$0{\cdot}01261{,}7c_{-3} + 0{\cdot}33320{,}6c_{-1} + 0{\cdot}01261{,}7c_1 + 0{\cdot}00012{,}6c_3 + 0{\cdot}01597{,}1\omega = 0,$$
$$0{\cdot}00012{,}6c_{-3} + 0{\cdot}01261{,}7c_{-1} + 0{\cdot}00981{,}0c_1 + 0{\cdot}01261{,}7c_3 + 0{\cdot}00012{,}6c_5$$
$$+2{\cdot}15973{,}7\omega = 0,$$
$$+0{\cdot}00012{,}6c_{-1} + 0{\cdot}01261{,}7c_1 - 8{\cdot}31358{,}5c_3 + 0{\cdot}01261{,}7c_5 + 0{\cdot}00933{,}1\omega = 0,$$
$$+0{\cdot}00012{,}6c_1 + 0{\cdot}01261{,}7c_3 - 24{\cdot}63698{,}1c_5 + 0{\cdot}00006{,}0\omega = 0.$$

If we find the values of c_{-3}, c_3, and c_5 from these equations in terms of the two remaining coefficients c_{-1} and c_1, which can be advantageously done, since c_{-3} has a large coefficient in the first equation, c_3 in the fourth and c_5 in the fifth equation, we find

$$c_{-3} = 0{\cdot}00171{,}8c_{-1} + 0{\cdot}00001{,}72\ c_1,$$
$$c_3 = 0{\cdot}00001{,}5c_{-1} + 0{\cdot}00151{,}76\ c_1 + 0{\cdot}00112{,}2\omega,$$
$$c_5 = \qquad +0{\cdot}00000{,}589c_1 + 0{\cdot}00000{,}3\omega,$$

and substituting these values in the 2nd and 3rd equations, they become

$$0{\cdot}33322{,}76c_{-1}+0{\cdot}01261{,}74\ c_1+0{\cdot}01597{,}1\omega=0,$$
$$0{\cdot}01261{,}74c_{-1}+0{\cdot}00982{,}925c_1+2{\cdot}15975{,}1\omega=0.$$

Whence again, we find

$$c_{-1}=\quad 8{\cdot}69441\omega,$$
$$c_1\ =-230{\cdot}8866\ \omega,$$

from which by substitution we obtain

$$c_{-3}=\quad 0{\cdot}01097\omega,$$
$$c_3\ =-\quad 0{\cdot}34915\omega,$$
$$c_5\ =-\quad 0{\cdot}00136\omega.$$

Hence the solution of the differential equation for δz is

$$\delta z=\omega\{0{\cdot}01097\cos(-3nt+4n't)+8{\cdot}69441\cos(-nt+2n't)-230{\cdot}8866\cos nt$$
$$-0{\cdot}34915\cos(3nt-2n't)-0{\cdot}00136\cos(5nt-4n't)\}.$$

Here ω is expressed in terms of the circular measure, and δz in terms of the unit of length defined before.

If s denote the sine of the Moon's latitude,

$$s=\frac{z}{r};$$

and if δs be the change in s due to the secular change in the plane of the ecliptic, we have

$$\delta s=\frac{\delta x}{r},$$

since $$\delta r=0,$$
according to the suppositions made above.

Also

$$\frac{1}{r}=1{\cdot}00090{,}74+0{\cdot}00718{,}65\cos 2(nt-n't)+0{\cdot}00004{,}58\cos 4(nt-n't).$$

Hence by substitution

$$\delta z=\omega\{0{\cdot}0369\cos(-3nt+4n't)+7{\cdot}8727\cos(-nt+2n't)-231{\cdot}0661\cos nt$$
$$-1{\cdot}1789\cos(3nt-2n't)-0{\cdot}0079\cos(5nt-4n't)\}.$$

Also s being supposed very small, δs is equal to the circular measure of the change of the Moon's latitude due to the secular change in the plane of the ecliptic, and if we divide δs by $\sin 1''$ we shall find the change of the latitude in *seconds*

$$= \frac{\omega}{\sin 1''}\{0{\cdot}0369 \cos(-3nt + 4n't) + 7{\cdot}8727 \cos(-nt + 2n't) - 231{\cdot}0661 \cos nt$$
$$- 1{\cdot}1789 \cos(3nt - 2n't) - 0{\cdot}0079 \cos(5nt - 4n't)\}.$$

Now, according to the data adopted by the Astronomer Royal, the circular measure of the angular motion of the plane of the ecliptic in 1 year is $0{\cdot}479 \sin 1''$.

Also 1 year is represented in our notation by the time $\frac{2\pi}{n'}$.

Hence
$$\frac{2\pi}{n'}\omega = 0{\cdot}479 \sin 1'',$$

and
$$\frac{\omega}{\sin 1''} = 0{\cdot}479 \frac{n'}{2\pi} = 0{\cdot}00616{,}354.$$

Therefore the inequality of latitude expressed in seconds is

$$0''{\cdot}0002 \cos(-3nt + 4n't) + 0''{\cdot}0485 \cos(-nt + 2n't) - 1''{\cdot}4242 \cos nt$$
$$- 0''{\cdot}0073 \cos(3nt - 2n't).$$

In this expression the mean longitudes nt and $n't$ are reckoned from the node of the variable ecliptic upon the fixed ecliptic. If the mean longitudes are reckoned from the equinox in the ordinary way, and if C be the longitude of the above-mentioned node, we must replace nt and $n't$ in the above by $nt - C$ and $n't - C$ respectively, and the expression for the inequality in latitude becomes

$$0''{\cdot}0002 \cos(-3nt + 4n't - C) + 0''{\cdot}0485 \cos(-nt + 2n't - C)$$
$$- 1''{\cdot}4242 \cos(nt - C) - 0''{\cdot}0073 \cos(3nt - 2n't - C).$$

In the above investigation the quantities ω and C are supposed to be constant. If these be subject to small secular variations, the differential equations become a little less simple, but are easily formed, and the above solution will require the following modifications, viz.—

(1) Instead of the constant value of ω we must employ the variable value which is of the form

$$\omega_0 + \omega' t;$$

(2) The coefficients of the above expression will be very slightly changed by quantities which are proportional to

$$\omega \frac{dC}{dt};$$

(3) The expression for the inequality of latitude will contain extremely small additional terms of the form

$$\frac{d\omega}{dt}\{g_{-3}\sin(-3nt+4n't-C)+g_{-1}\sin(-nt+2n't-C)+g_1\sin(nt-C)$$
$$+g_3\sin(3nt-2n't-C)\};$$

that is to say, these terms will involve the *sines* instead of the *cosines* of the same arguments as before, and the coefficients of these new terms are proportional to

$$\frac{d\omega}{dt}.$$

II. Theoretical explanation of the same inequality, which was originally given, in substance, in Godfray's *Elementary Treatise on the Lunar Theory.*

The general principle of this explanation may be very simply stated.

If, for a moment, we suppose the plane of the Moon's orbit to remain fixed, and imagine the plane of the ecliptic to turn through a very small given angle about a line in its own plane, this will give rise to corresponding small changes in the longitude of the Moon's node and in the inclination of the orbit to the ecliptic, and the magnitude of these changes will depend on the angular distance of the Moon's node from the line about which the ecliptic is supposed to be turning.

If now the planes of both orbits be supposed to vary continuously, the total changes in the longitude of the node and inclination of the orbit produced in an indefinitely small time will be found by adding together the changes respectively due to the motion of the plane of the ecliptic, and to the motion of the plane of the Moon's orbit with respect to the ecliptic when the latter is supposed to remain fixed during that small time. The motion last mentioned is given by the formulæ of the ordinary Lunar Theory, in terms of the disturbing force of the Sun. In consequence of the action of this force, the Moon's node gradually makes complete revolutions with respect to the line about which the ecliptic is turning, and the summation of all the momentary changes of node and inclination due to the motion of the ecliptic will produce periodic changes in those elements, the magnitudes of which, at any given time, like the momentary changes themselves, will depend on the angular distance, at that time, between the Moon's node and the line about which the ecliptic is turning.

The combined effect of these periodic changes in the position of the node and in the inclination is to produce the inequality in latitude which is now under consideration.

The motion of the Moon's node is not uniform, but the principal inequalities by which that motion is affected have periods which are short compared with the time of revolution of the node.

Hence the periodic changes of node and inclination above described, will be accompanied by others which are due to the same cause, but which in consequence of the shortness of their periods will be comparatively unimportant, and the combined effect of these changes in the elements will be to add other terms which are equally unimportant to the expression of the inequality in latitude.

We proceed to find the analytical expressions for the changes in the longitude of the Moon's node and in the inclination of the orbit, due to the motion of the plane of the ecliptic, supposing the Moon's orbit itself to remain fixed.

Take C the longitude of the instantaneous axis about which the ecliptic is rotating at the time t,

ω the angular velocity of the ecliptic,

N the longitude of the Moon's node,

and i the inclination of the orbit, at the same instant.

Then, in the indefinitely small time δt, a point of the ecliptic situated in any arbitrary longitude L will move through an angular space

$$\omega \delta t \sin (L - C)$$

in a direction perpendicular to the ecliptic.

Hence the point of the ecliptic originally coincident with the node N will move through the space

$$\omega \delta t \sin (N - C)$$

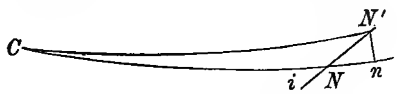
perpendicular to the ecliptic.

And if δN be the consequent increase of the longitude of the node we have evidently from the figure,

$$\delta N = \omega \delta t \sin (N - C) \cot i$$

or

$$\frac{dN}{dt} = \omega \sin (N - C) \cot i.$$



Again, the point of the ecliptic 90° in advance of N will move through the space

$$\omega \delta t \sin (90^\circ + N - C),$$

or

$$\omega \delta t \cos (N - C),$$

perpendicular to the ecliptic, and this quantity will measure the diminution in the inclination of the Moon's orbit.

Hence we have

$$\delta i = -\omega \delta t \cos (N - C),$$

or

$$\frac{di}{dt} = -\omega \cos (N - C).$$

Thus we have found the rates of change of the longitude of the Moon's node and of the inclination which are due to the motion of the ecliptic.

Now, suppose the formulæ which give the rates of change of the same two elements, with respect to a fixed ecliptic, which are due to the Sun's disturbing force, to be represented by

$$\frac{dN}{dt} = -c \cos i + F(\theta, \theta'),$$

and

$$\frac{di}{dt} = f(\theta, \theta'),$$

where $-c \cos i$ denotes the non-periodic term in $\frac{dN}{dt}$, c being approximately equal to $\frac{3}{4}\frac{n'^2}{n}$, and $F(\theta, \theta')$, $f(\theta, \theta')$ consist wholly of periodic terms which involve the longitudes θ, θ' of the Moon and Sun respectively, as well as the elements N and i.

Hence by what has been before said if N', i' denote the longitude of the node and the inclination at the time t, with respect to the variable ecliptic, $\frac{dN'}{dt}$ and $\frac{di'}{dt}$ will be given by the following formulæ:—

$$\frac{dN'}{dt} = -c \cos i' + F(\theta, \theta') + \omega \sin (N' - C) \cot i',$$

$$\frac{di'}{dt} = f(\theta, \theta') - \omega \cos (N' - C),$$

in which $F(\theta, \theta')$, $f(\theta, \theta')$ now involve the elements N' and i', instead of N and i.

Now let N be the longitude of the node, and i the inclination at the time t, on the supposition that the ecliptic remains fixed, all the other circumstances of the Moon's motion remaining unaltered; then we have as before

$$\frac{dN}{dt} = -c \cos i + F(\theta, \theta'),$$

$$\frac{di}{dt} = f(\theta, \theta').$$

Let $$N' = N + \delta N,$$

and $$i' = i + \delta i,$$

where δN and δi are entirely due to the motion of the ecliptic and therefore vanish with ω*.

Then neglecting the square of ω and supposing the value of θ, or the Moon's longitude, to remain unchanged, we have

$$\frac{d\delta N}{dt} = c \sin i \delta i + \left(\frac{dF}{dN}\right) \delta N + \left(\frac{dF}{di}\right) \delta i + \omega \sin (N - C) \cot i,$$

$$\frac{d\delta i}{dt} = \left(\frac{df}{dN}\right) \delta N + \left(\frac{df}{di}\right) \delta i - \omega \cos (N - C).$$

Now $$\left(\frac{dF}{dN}\right), \left(\frac{dF}{di}\right),$$

and $$\left(\frac{df}{dN}\right), \left(\frac{df}{di}\right),$$

* It is hardly necessary to mention that δN and δi are here employed in a wholly different sense from that in which the same symbols were used, for a temporary purpose, in the earlier part of this investigation.

are composed of periodic terms which have short periods compared with the time of revolution of the Moon's node—that is, with the period of the terms

$$\sin(N-C) \text{ and } \cos(N-C).$$

Hence in integrating we may at first neglect the terms

$$\left(\frac{dF}{dN}\right)\delta N,\ \left(\frac{dF}{di}\right)\delta i \text{ and } \left(\frac{df}{dN}\right)\delta N,\ \left(\frac{df}{di}\right)\delta i,$$

leaving them to be taken into account, if necessary, in a subsequent approximation.

For the same reason we may suppose $\cot i$ to be constant in integrating, and we may take

$$\frac{dN}{dt} = -c\cos i,$$

omitting the periodic term $F(\theta, \theta')$; and we may also suppose that ω and C are constants.

With these simplifications, we have

$$\frac{d\delta N}{dt} = c\sin i\,\delta i + \omega\cot i\sin(N-C),$$

$$\frac{d\delta i}{dt} = \frac{\omega}{c\cos i}\cos(N-C)\frac{dN}{dt}.$$

From the latter of these equations

$$\delta i = \frac{\omega}{c\cos i}\sin(N-C),$$

and substituting this value of δi in the former, we find

$$\frac{d\delta N}{dt} = \omega\tan i\sin(N-C) + \omega\cot i\sin(N-C),$$

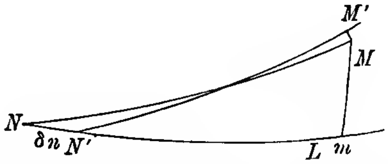
$$= \frac{\omega}{\sin i\cos i}\sin(N-C),$$

$$= -\frac{\omega}{c\sin i\cos^2 i}\sin(N-C)\frac{dN}{dt},$$

and therefore

$$\delta N = \frac{\omega}{c\sin i\cos^2 i}\cos(N-C).$$

Now let NM be the Moon's orbit and M the place of the Moon as found from formulæ in which the plane of the ecliptic is supposed to be



fixed, and let $N'M'$ be the Moon's orbit and M' the place of the Moon at the same time taking into account the motion of the ecliptic.

Let $NM = \psi$, and $N'M' = \psi + \delta\psi$.

Also let s denote the sine of the Moon's latitude, and β the latitude itself, in the case when the ecliptic is supposed fixed;

And let $s + \delta s$ denote the sine of the latitude, and $\beta + \delta\beta$ the latitude itself, when the ecliptic is supposed to be variable.

Then
$$s = \sin i \sin \psi,$$
and
$$\delta s = \cos i \sin \psi \, \delta i + \sin i \cos \psi \, \delta\psi.$$

Now let us assume that MM' is perpendicular to NM, in which case we shall have
$$\delta\psi = -\cos i \, \delta N,$$
and therefore
$$\delta s = \cos i \sin \psi \, \delta i - \sin i \cos i \cos \psi \, \delta N,$$
or substituting the values above found for δi and δN,
$$\delta s = \frac{\omega}{c} \sin \psi \sin (N - C) - \frac{\omega}{c \cos i} \cos \psi \cos (N - C).$$

But if θ denote the Moon's longitude, we have
$$\cos i \sin \psi = \cos \beta \sin (\theta - N),$$
and
$$\cos \psi = \cos \beta \cos (\theta - N).$$

Hence
$$\delta s = \frac{\omega}{c \cos i} \cos \beta \left[\sin (\theta - N) \sin (N - C) - \cos (\theta - N) \cos (N - C)\right],$$
or
$$\cos \beta \, \delta\beta = -\frac{\omega}{c \cos i} \cos \beta \cos (\theta - C),$$

and therefore $$\delta\beta = -\frac{\omega}{c\cos i}\cos(\theta - C),$$

which is the inequality in latitude due to the motion of the ecliptic, expressed in the circular measure.

This value of $\delta\beta$ agrees exactly with that found in my article inserted in Godfray's *Lunar Theory*, since $c\cos i$ in this formula has the same signification as $\frac{1}{c}$ in Godfray, viz. the mean angular velocity of the Moon's node.

The steps, however, by which this result is arrived at, are slightly different in the two investigations. In the earlier one, the variation of $\cos i$ was neglected, and $\delta\psi$ was taken $= -\frac{\delta N}{\cos i}$, whereas in the present investigation the variation of $\cos i$ is taken into account, and $\delta\psi$ is taken $= -\cos i\,\delta N$, on the assumption that MM' is perpendicular to NM.

It should be remarked that in both forms of this investigation, the neglect to take account of any variation of the Moon's radius vector and orbital longitude, due to the motion of the ecliptic, may produce errors in the coefficient of the inequality in latitude which are of the order of the small quantity $\frac{\omega}{c}\sin^2 i$, so that the investigation is incompetent to decide such a question, for instance, as whether $\frac{\omega}{c\cos i}$ or $\frac{\omega}{c}$ is the more correct value of this coefficient.

The coefficient above found, expressed in seconds, is

$$\frac{\omega}{c\cos i\sin 1''}.$$

In order to evaluate this quantity numerically, we observe that $\frac{\omega}{c\cos i}$ is the ratio of two angular velocities: viz. the velocity of rotation of the plane of the ecliptic, and the mean angular velocity of the Moon's node; and in comparing these it is indifferent what unit of time is employed. According to the data adopted before, taking 1 year as the unit of time,

$$\omega = 0{\cdot}479\sin 1'', \text{ or } \frac{\omega}{\sin 1''} = 0{\cdot}479.$$

Also since the Moon's node takes about 18·6 years to perform a complete revolution

$$c \cos i = \frac{2\pi}{18 \cdot 6} \text{ nearly.}$$

Hence $$\frac{\omega}{c \cos i \sin 1''} = \frac{0 \cdot 479 \times 18 \cdot 6}{2\pi}, \text{ expressed in seconds,}$$
$$= 1'' \cdot 42,$$

which agrees with the value of the coefficient of the principal term found in the former investigation.

The form above found for $\delta\beta$ suggests a very simple geometrical interpretation of this inequality in latitude.

If we suppose a fictitious ecliptic to be inclined to the true ecliptic at the angle $1''\cdot 42$, the circular measure of which is $\frac{\omega}{c \cos i}$, and if we also suppose that the longitude of its ascending node on the true ecliptic is $90° + C$, then the elevation of the fictitious above the true ecliptic corresponding to the longitude θ will be

$$= \frac{\omega}{c \cos i} \sin (\theta - \overline{90° + C}),$$

$$= -\frac{\omega}{c \cos i} \cos (\theta - C),$$

$$= \delta\beta.$$

Hence the latitude above the fictitious ecliptic will be equal to β, that is, the expression for the Moon's latitude with respect to the fictitious ecliptic is the same as the expression found for the latitude in the case when the ecliptic is taken to be a fixed plane.

This geometrical interpretation of the inequality was first given by Hansen.

III. Note on the *Mécanique Céleste*, tome III. p. 185 (edition of 1802).

At any arbitrary point whose longitude is λ, Laplace takes the elevation of the variable ecliptic above the fixed plane of reference to be represented by

$$\Sigma k \sin (\lambda + it + \epsilon),$$

and he shews that if s_1 denotes the perturbation of the Moon's latitude with respect to the variable ecliptic which is due to the motion of that plane.

Then $$s_1 = \Sigma \frac{(2i + i^2)\, k \sin(\nu + i\nu + \epsilon)}{\frac{3}{2}m^2 - 2i - i^2},$$

where ν denotes the Moon's longitude;

or $$s_1 = \Sigma \left[\frac{2ki}{\frac{3}{2}m^2} + \frac{4ki^2}{(\frac{3}{2}m^2)^2}\right] \sin(\nu + i\nu + \epsilon)$$

very nearly, neglecting i^2 compared with i except when it is divided by an additional power of $\frac{3}{2}m^2$.

Or, replacing $i\nu$ by it

$$s_1 = \sin\nu\, \Sigma \left[\frac{2ki}{\frac{3}{2}m^2} + \frac{4ki^2}{(\frac{3}{2}m^2)^2}\right] \cos(it + \epsilon)$$
$$+ \cos\nu\, \Sigma \left[\frac{2ki}{\frac{3}{2}m^2} + \frac{4ki^2}{(\frac{3}{2}m^2)^2}\right] \sin(it + \epsilon).$$

Now, Hansen's expression for the elevation of the variable above the fixed ecliptic at any point whose longitude is λ is of the form

$$-p \cos\lambda + q \sin\lambda,$$

where p and q are functions of t, expressed in series of powers of t.

Comparing this with Laplace's expression for the same quantity, we have

$$-p = \Sigma k \sin(it + \epsilon),$$

hence $$-\frac{dp}{dt} = \Sigma ki \cos(it + \epsilon),$$

and $$\frac{d^2p}{dt^2} = \Sigma ki^2 \sin(it + \epsilon);$$

similarly $$q = \Sigma k \cos(it + \epsilon),$$

$$-\frac{dq}{dt} = \Sigma ki \sin(it + \epsilon),$$

$$-\frac{d^2q}{dt^2} = \Sigma ki^2 \cos(it + \epsilon).$$

Hence, by substituting for $k \sin (it + \epsilon)$, $k \cos (it + \epsilon)$, &c. in Laplace's expression for s_1, their values in terms of p, q and their differential coefficients, we find

$$s_1 = \sin \nu \left[-\frac{1}{\frac{3}{4}m^2} \frac{dp}{dt} - \frac{1}{(\frac{3}{4}m^2)^2} \frac{d^2q}{dt^2} \right] + \cos \nu \left[-\frac{1}{\frac{3}{4}m^2} \frac{dq}{dt} + \frac{1}{(\frac{3}{4}m^2)^2} \frac{d^2p}{dt^2} \right],$$

which exactly agrees with Hansen's expression in his *Darlegung*, p. 490*, except that Hansen's argument $f + \omega - \theta_1$ represents the longitude on the orbit, whereas Laplace's argument ν is the longitude on the ecliptic; but these two longitudes may be employed indifferently in terms of the order of small quantities to which the approximation is restricted.

Laplace remarks that $\frac{3}{2}m^2$ is at least 4,000 times greater than $2i$, and he therefore infers that the above value of s_1 may be neglected as insensible. If, however, the numerical values of the quantities denoted by k had been known to Laplace, he would have seen that some of those values are very considerable, exceeding one degree, and therefore that $\frac{1}{4000}$ of this amount is by no means to be neglected.

Finally, we will reduce Laplace's transformed expression to a form immediately comparable with our former results.

The velocity perpendicular to the ecliptic of a point in any arbitrary longitude L is represented in one system by

$$-\frac{dp}{dt} \cos L + \frac{dq}{dt} \sin L,$$

and in the other system by

$$\omega \sin (L - C).$$

Hence $$\frac{dp}{dt} = \omega \sin C,$$

* In this expression $\frac{dp}{dt}$ is equivalent to $b + b't$ in Hansen, and $\frac{dq}{dt}$ is equivalent to $c + c't$.

Also Hansen's expression $n(\alpha + \eta)$, which denotes the mean motion of the Moon's node, is equivalent to $\frac{3}{4}m^2$ in Laplace, as the latter takes n, the Moon's mean motion, to be equal to unity.

and $$\frac{dq}{dt} = \omega \cos C;$$

$$\frac{d^2p}{dt^2} = \frac{d\omega}{dt} \sin C + \omega \frac{dC}{dt} \cos C,$$

and $$\frac{d^2q}{dt^2} = \frac{d\omega}{dt} \cos C - \omega \frac{dC}{dt} \sin C.$$

Hence, putting c for $\frac{3}{4}m^2$, and denoting the Moon's longitude by θ as before, instead of Laplace's ν, we have

$$s_1 = \quad \sin\theta \left[-\frac{1}{c}\omega \sin C - \frac{1}{c^2}\left(\frac{d\omega}{dt}\cos C - \omega \frac{dC}{dt}\sin C\right)\right]$$

$$+ \cos\theta \left[-\frac{1}{c}\omega \cos C + \frac{1}{c^2}\left(\frac{d\omega}{dt}\sin C + \omega \frac{dC}{dt}\cos C\right)\right],$$

or $$s_1 = -\frac{\omega}{c}\cos(\theta - C) - \frac{1}{c^2}\frac{d\omega}{dt}\sin(\theta - C) + \frac{\omega}{c^2}\frac{dC}{dt}\cos(\theta - C),$$

$$= -\left(\frac{\omega}{c} - \frac{\omega}{c^2}\frac{dC}{dt}\right)\cos(\theta - C) - \frac{1}{c^2}\frac{d\omega}{dt}\sin(\theta - C),$$

which is in accordance with the remark made at the close of investigation I.

30.

NOTE ON DELAUNAY'S EXPRESSION FOR THE MOON'S PARALLAX.

[From the *Monthly Notices of the Royal Astronomical Society.* Vol. XLIII. (1883).]

THE process employed in Delaunay's Theory of the Moon consists in making a great number of successive changes from one system of elements to another, these changes being so conducted that the equations which give the variations of the elements always retain their canonical form, until at length all the sensible periodic terms in the disturbing function are got rid of, and the elements are thus reduced to three constants and three angles which vary in proportion to the time.

After each such change of elements, the expressions for the three coordinates of the Moon, which are supposed to be known in terms of the old system of elements, must be transformed so as to be expressed in terms of the new.

These transformations being made independently, we may, if we choose, find some of the coordinates with a greater degree of precision than others.

Delaunay has, as is well known, followed the example of Plana in developing his coefficients in series of ascending powers of the small quantities m, e, e' and γ.

Now, two of the Moon's coordinates, viz. the longitude and latitude, can be directly compared with observation, whereas the third coordinate, viz.

the radius vector, can only be indirectly inferred from observation through the parallax, to the sine of which it is inversely proportional.

Hence the accuracy of the theoretical values of the longitude and latitude can be much more severely tested by observation than that of the radius vector.

Delaunay has, on account of this circumstance, found the analytical expressions for the longitude and latitude with a much greater degree of accuracy than that for the reciprocal of the radius vector.

In the two former coordinates he has taken into account generally the terms of the 7th order, and in cases where the convergence of the series is found to be slow, he has included terms of the 8th and 9th orders. In the reciprocal of the radius vector, however, he has confined his attention to terms of the 5th order. Consequently, while the coefficients of the inequalities in longitude and latitude as found by him are generally only a small fraction of a second in error, the inequalities in the reciprocal of the radius vector are not found with sufficient precision to give even the parallax itself with all the accuracy which is desirable.

The coefficients of the inequalities of the parallax given by me in Vol. XIII. of the *Monthly Notices*, p. 263 (*see p.* 109 *above*), are considerably more accurate than those of Delaunay.

In the paper just referred to, I have given the coefficients to hundredths of a second only, and, as I have there stated, terms with coefficients less than 0″·05 have been omitted except when they can be included in the same table with larger terms.

It may be worth while to give here a more complete view of the values of the coefficients of parallax which I obtained in 1853. These results are exhibited to thousandths of a second, as the calculation gave them, although the figures in the last place of decimals are not to be depended upon.

I add, for the sake of comparison, Delaunay's coefficients of the corresponding terms as given in the *Connaissance des Temps* for 1869, and also the coefficients of Hansen's theory as transformed by Professor Newcomb. The several arguments are expressed in Delaunay's notation*.

* In the following table the arguments are also given in Damoiseau's notation, which has been employed in paper 18 (see p. 109 above).

Table of Comparative Values of the Coefficients of $\dfrac{\sin . Parallax}{\sin . 1''}$.

Argument. Delaunay.	Argument. Damoiseau.	Delaunay.	Adams.	Hansen transformed by Newcomb.
		$''$	$''$	$''$
0	0	3422·7	3422·324	3422·09
l'	z	−0·4273	−0·400	−0·393
l	x	+186·5870	186·513	186·483
$2l$	$2x$	10·1984	10·170	10·161
$3l$	$3x$	0·6314	0·628	0·620
$4l$	$4x$	·0414	·041	·040
$l-l'$	$(x-z)$	1·0523	1·157	1·144
$l+l'$	$(x+z)$	−0·9118	−0·948	−0·961
$2l-l'$	$(2x-z)$	0·1030	0·123	0·149
$2l+l'$	$(2x+z)$	−0·0917	−0·100	−0·122
$2F-l$	$(2y-x)$	−0·7079	−0·710	−0·709
$2D$	$2t$	28·1788	28·232	28·225
$2D-l'$	$(2t-z)$	1·8764	1·915	1·920
$2D+l'$	$(2t+z)$	−0·3276	−0·306	−0·301
$2D-2l'$	$(2t-2z)$	0·0760	0·089	0·092
$2D+l$	$(2t+x)$	3·0636	3·090	3·084
$2D+l-l'$	$(2t+x-z)$	0·1967	0·222	0·229
$2D+l+l'$	$(2t+x+z)$	−0·0401	−0·047	−0·049
$2D-l$	$(2t-x)$	34·1662	34·304	34·309
$2D-l-l'$	$(2t-x-z)$	1·4523	1·449	1·447
$2D-l-2l'$	$(2t-x-2z)$	0·0454	0·050	0·049
$2D-l+l'$	$(2t-x+z)$	−0·3789	−0·231	−0·227
$2D+2l$	$(2t+2x)$	0·2707	0·281	0·283
$2D-2l$	$(2t-2x)$	−0·2770	−0·307	−0·302
$2D-3l$	$(2t-3x)$	−0·1012	−0·116	−0·121
$2D-2F$	$(2t-2y)$	−0·1092	−0·106	−0·105
$2D-2F+l$	$(2t-2y+x)$	−0·0501	−0·048	−0·048
$2D-2F-l$	$(2t-2y-x)$	−0·0816	−0·086	−0·083
$4D$	$4t$	0·1960	0·260	0·261
$4D-l$	$(4t-x)$	0·4991	0·600	0·599
$4D-2l$	$(4t-2x)$	0·3104	0·372	0·372
$4D-l-l'$	$(4t-x-z)$	0·0297	0·063	0·069
D	t	−0·9378	−0·949	−0·953
$D+l'$	$(t+z)$	0·1507	0·145	0·146

Argument. Delaunay.	Argument. Damoiseau.	Delaunay.	Adams.	Hansen transformed by Newcomb.
$D+l$	$(t+x)$	$-0''.0971$	$-0''.106$	$-0''.106$
$3D$	$3t$	0.0158	0.005	0.003
$3D-l$	$(3t-x)$	-0.0199	-0.036	-0.037
$3D+l$	$(3t+x)$	0.0025	0.002	
$D-l$	$(t-x)$	0.0076	0.014	$+0.011$
$2D-2l-l'$	$(2t-2x-z)$	-0.0127	-0.015	-0.019
$4D+l$	$(4t+x)$	0.0185	0.032	0.043
$4D-2l-l'$	$(4t-2x-z)$	0.0159	0.030	0.032
$4D-l'$	$(4t-z)$	0.0110	0.034	0.035

In the above many very small coefficients have been omitted.

As stated in my paper in the appendix to the *Nautical Almanac* for 1856, or in the *Monthly Notices*, Vol. XIII. p. 177, my coefficients of parallax were obtained by comparing the results of the theories of Damoiseau, Plana, and Pontécoulant, and tracing out the origin of the discordances in the cases where those results did not agree with each other. These coefficients were also compared with those which I obtained by a transformation of Hansen's preliminary results as given in a paper in Vol. XVII. of the *Astronomische Nachrichten*.

In Pontécoulant's method the expression for the reciprocal of the radius vector is first found, and then the expression for the longitude is derived from it. Hence the analytical values of the coefficients of parallax, given by Pontécoulant, Vol. IV. pp. 149—152, 281, 282, 336, 337, are at least as accurate as the values of his coefficients of longitude.

In his final expression, however, in pp. 568—572, in which the several terms of the reciprocal of the radius vector are collected together, he neglects all terms of orders higher than the 5th, and the same omission takes place in the conversion of his coefficients of parallax into numbers.

Accordingly these numerical values, which are calculated in pp. 599—601, and collected together in p. 635, nearly coincide with the values of Delaunay, but are on the whole still less accurate.

It is greatly to be desired that some intrepid and competent calculator would undertake to make the numerous substitutions which would be required in order to find, by Delaunay's method, the expression for the reciprocal of the radius vector to the same order of accuracy as that which Delaunay has already attained in the case of the corresponding expressions for the longitude and latitude. The work would be one of simple substitution, not requiring the solution of any new equations, and consequently its only difficulty would consist in its great length.

The fact that Delaunay's determination of the value of the reciprocal of the radius vector is a comparatively rough one, affords a ready explanation of a difficulty which Sir George Airy has recently met with in his *Numerical Lunar Theory*.

The first operation required in this method is the substitution in the differential equations of motion of the numerical values of the Moon's coordinates as obtained in Delaunay's theory. If the theory were exact, the result of the substitution in each equation would be identically zero, so that the coefficient of each separate term in the result of the substitution would vanish. In consequence of errors in the coefficients obtained by Delaunay, however, this mutual destruction of terms will not take place, and the result of the substitution will consist of a number of terms the coefficients of which will depend on the errors of the assumed coefficients.

If, as is actually the case, these latter errors be so small that their squares and products may be neglected, each of the residual coefficients may be represented by a linear function of the errors of the assumed coefficients, and the formation of the corresponding linear equations constitutes the second operation in Sir George Airy's method. The solution of these linear equations by successive approximations will finally give the corrections which must be applied to Delaunay's coefficients in order to satisfy the differential equations.

Now, since the proportionate errors of Delaunay's coefficients of parallax are considerable, and much greater than the errors affecting his coefficients of longitude and latitude, it will be readily understood that the result of the substitutions will be to leave considerable residual coefficients in the two equations which relate to motion parallel to the ecliptic, and much smaller residual coefficients in the third equation which relates to motion normal to the ecliptic, since in this last equation every error in the coefficients of the radius vector or of its reciprocal will be multiplied by the sine of the inclination of the Moon's orbit. This result, which might thus have been anticipated, is exactly what Sir George Airy has found to take place, according to a memorandum which he has recently addressed to the Board of Visitors of the Royal Observatory.

Since the errors affecting Delaunay's coefficients of parallax are comparatively large, it will be necessary to determine the factors by which these errors are multiplied in the equations of condition with a much greater degree of accuracy than is required in the case of the factors by which the errors of the coefficients of longitude and latitude are multiplied in the same equations. Otherwise, it will not be possible to deduce these last-mentioned errors from the equations with the requisite degree of precision. It will be necessary to take special precautions in order to determine with accuracy the corrections of the assumed coefficients in the inequalities of longitude which have long periods.

31.

REMARKS ON MR STONE'S EXPLANATION OF THE LARGE AND INCREASING ERRORS OF HANSEN'S LUNAR TABLES BY MEANS OF A SUPPOSED CHANGE IN THE UNIT OF MEAN SOLAR TIME.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. XLIV. (1883).]

In some recent communications to the Royal Astronomical Society Mr Stone contends that the mean solar day in use before 1864—when Le Verrier's Solar Tables were substituted for Bessel's in calculating the sidereal time at mean noon given in the *Nautical Almanac*—differs from the mean solar day adopted since that time.

In the *Monthly Notices*, Vol. XLIII. p. 403, Mr Stone states that the consequent error in our present reckoning in time is increasing at about the rate of $1^{s}\cdot 46$ per annum, and in the same volume, p. 335, he adduces this supposed error in explanation of the increasing errors of Hansen's Lunar Tables.

That this view of Mr Stone's is erroneous may, I think, be shewn by very simple considerations.

The only mean Sun known to astronomers is an imaginary body which moves uniformly in the equator at such a rate that the difference between its Right Ascension and that of the true Sun consists wholly of periodic quantities.

These periodic terms are due to the obliquity of the ecliptic, the eccentricity of the Earth's orbit, and also to the small perturbations of the Earth's motion about the Sun.

The difference between the Right Ascensions of the two bodies at any moment is called the Equation of Time.

The instant of Mean Noon is determined by the transit of this imaginary Mean Sun over the meridian of a given place just as the instant of Apparent Noon is determined by the transit of the true Sun over the same meridian.

Hence, the mean time, according to the definition of it above given, may be determined by observation of the transit of the true Sun over the meridian, subject only to the small error to which all transit observations are liable, and also to the extremely small error which is possible in the theoretical expression for the equation of time. When this mode of determining the mean time is employed, no accumulation of error in proportion to the interval of time from a given epoch is possible.

If, as it is frequently convenient to do, we wish to determine the mean solar time by means of the sidereal time supposed to be known, without having to make a transit observation of the Sun, we must employ the sidereal time at mean noon calculated from the proper formula or from the Solar Tables. This sidereal time at mean noon is equal to the Sun's mean longitude at mean noon corrected by the equation of the equinoxes in Right Ascension.

In order to find the mean time correctly in this way it is necessary to employ the correct value of the Sun's mean longitude, and any error in the assumed value of this quantity will produce an equivalent error in the mean time deduced.

Any such error can be at once checked and corrected by observation of the Sun's transit over the meridian.

If we wilfully refuse to check our results by solar observations, the error in the determination of the mean time by means of the sidereal time would, no doubt, increase in proportion to the interval of time from a certain epoch. Practically, however, it would be intolerable to use Solar Tables which were grossly erroneous, and long before the error of time became important the tables would be replaced by more accurate ones.

For many years previously to 1864 Bessel's formula had been employed in the *Nautical Almanac* for the calculation of the sidereal time at Greenwich mean noon.

In 1864 the error of Bessel's formula amounted to rather more than half a second of time, and accordingly in that and subsequent years the sidereal time at mean noon was deduced from Le Verrier's Solar Tables, which gave much more accurate results.

Now it is contended by Mr Stone that by the change thus introduced into the *Nautical Almanac* the unit of mean solar time was practically altered to such a degree that at the end of 1881 the difference in the count of mean solar time amounted to nearly 27 seconds, and that the difference is increasing at the rate of about 1·46 seconds per annum.

It is clear, therefore, that if no such change had been made in the *Nautical Almanac*—that is, if Bessel's formula had continued to be employed—no such change of the unit of time would have taken place.

Let us see then, what difference this would have made in the count of mean solar time as derived from sidereal time when compared with the count found by means of our present *Nautical Almanac*.

Bessel's formula for the sidereal time at Greenwich mean noon of Jan. 1 in any year is given in the prefaces to the *Nautical Almanacs* from 1834 to 1863 inclusive. In 1864 and subsequent years the sidereal time at Greenwich mean noon is derived from Le Verrier's tables.

The following little table shews the sidereal time at Greenwich mean noon of Jan. 1 as calculated for every fifth year from 1860 to 1885 by Bessel's formula, and as taken from the several *Nautical Almanacs*:—

	By Bessel's Formula. h. m. s.	From *Nautical Almanac*. h. m. s.		Diff. s.
1860	18 41 28·87	18 41 28·87	Bessel's formulæ employed	0·00
1865	18 44 35·36	18 44 35·92	Le Verrier's Tables employed	0·56
1870	18 43 43·87	18 43 44·44	,, ,,	0·57
1875	18 42 54·47	18 42 55·06	,, ,,	0·59
1880	18 42 5·95	18 42 6·56	,, ,,	0·61
1885	18 45 11·73	18 45 12·37	,, ,,	0·64

Hence we see that the difference of sidereal times at mean noon in consequence of the change from Bessel's formula to Le Verrier's Tables, which amounted to $0^s{\cdot}56$ in 1865, had increased to $0^s{\cdot}64$ in 1885. That is, the difference increases at the rate of $0^s{\cdot}08$ in twenty years, or of $0^s{\cdot}02$ in five years.

But according to Mr Stone's theory as shewn in his tabular comparisons of mean solar times computed from sidereal times by means of the *Nautical Almanac* and of those sidereal times "corrected to agree with Bessel's sidereal times," the differences would be as follows:—

1865	$2^{s}{\cdot}0$	1875	$16^{s}{\cdot}6$
1870	$9{\cdot}3$	1880	$23{\cdot}9$

and at the end of 1881 the difference would have increased to $26^{s}{\cdot}8$; so that the increase in five years would be $7^{s}{\cdot}3$ instead of $0^{s}{\cdot}02$ as above. In fact the difference according to Mr Stone's theory is just 365 times as great as it should be.

The origin of this enormous discrepancy between Mr Stone's theory and the fact is readily seen by considering that mean solar time is measured, not by the Sun's mean motion in *longitude*, as Mr Stone's theory supposes, but by the motion of the mean Sun in *hour angle*, which is about 365 times greater in amount. Hence any small error in the determination of the Sun's mean motion in longitude causes a proportionate error of only about a 365th part of the amount in the interval of mean solar time as inferred from the interval of sidereal time. In fact, if n denote the Sun's mean motion in longitude in a mean solar day, then the length of the mean solar day will be to the sidereal day in the ratio of

$$360^\circ + n : 360^\circ.$$

If now $n + dn$ denote another slightly different determination of the Sun's mean motion in longitude in a mean solar day, the ratio of the length of a mean solar to that of a sidereal day will become

$$360^\circ + n + dn : 360^\circ.$$

Hence the measure of a mean solar day when expressed in sidereal time will be increased in the ratio of

$$360^\circ + n + dn : 360^\circ + n,$$

or

$$1 + \frac{dn}{360^\circ + n} : 1.$$

Since 360° is nearly 365 times n, this ratio will be

$$1 + \frac{1}{366}\frac{dn}{n} : 1 \text{ nearly.}$$

Whereas, according to Mr Stone's theory, this ratio should be

$$1 + \frac{dn}{n} : 1.$$

It has been already remarked that it is convenient practically to determine the mean solar time from the sidereal time, but in order to do this correctly, it is of course necessary to employ the correct value of the Sun's mean longitude. At the present time Bessel's value of the Sun's mean longitude is about $0^s\cdot 6$ in error, and therefore the mean solar time inferred by means of it from the sidereal time would be in error to the same amount. The mean longitude found from Le Verrier's Tables is much nearer to the truth, and therefore the mean solar time found from the sidereal time by using this value would be much more nearly correct.

It must not be forgotten however that, as we have already stated, the mean solar time may be derived from observations of the transit of the Sun over the meridian, without employing the sidereal time at all. Apparent solar time, which is found directly from observation of the Sun is converted into *mean* solar time by applying the equation of time, which is known from the solar theory, without reference to the sidereal time.

32.

REMARKS ON SIR GEORGE AIRY'S NUMERICAL LUNAR THEORY.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. XLVIII. (1888).]

In the Report of the Council on the subject of Sir George Airy's *Numerical Lunar Theory*, it has been explained that the large discordances which have been found by the author to result from the substitution of the values of the Moon's coordinates, as found by Delaunay, in the differential equations of motion, are caused by the large errors of Delaunay's coefficients of parallax, which Sir George has employed. It may be useful and not uninteresting to give on this subject some additional details. In the first place it will be well to prevent a possible misapprehension. In speaking of the errors of Delaunay's coefficients it is not intended to imply that there is any mistake in Delaunay's *theory*. The terms of the analytical expression for the Moon's parallax which Delaunay gives are all correct, but they only extend to the fifth order of small quantities, and are therefore not nearly precise enough to be used for the purpose to which the expression for the parallax is applied by Sir George Airy. Delaunay intended this value of the parallax to be employed merely in reducing the apparent place of the Moon to its place as seen from the Earth's centre, and for this purpose the value is perhaps sufficiently accurate.

If the several transformations of the elements given by Delaunay in his great work had been applied to the analytical expression for the reciprocal of the radius vector, and if Delaunay had carried the developments to the same extent as he had done in the case of the Moon's longitude and latitude, the theory would have been quite competent to give the third coordinate with the same degree of precision as had been attained in the case of the two other coordinates.

The following table, which is reduced from the table given in pp. 398, 399, of Vol. XLIII. of the *Monthly Notices*, R. A. S., shews the proportional values of the coefficients of parallax as found by me, mainly after Ponté-coulant, when compared with those employed by Sir George Airy after Delaunay.

Argt.	My Coefficient.	Delaunay's.
o	10000000,	10000000,
l	544989, 3	545145, 6
$2D-l$	100236, 0	99822, 4
$2D$	82493, 6	82329, 2
$2l$	29716, 6	29796, 4
$2D+l$	9029, 0	8950, 8
$2D-S$	5595, 6	5482, 2
$2D-l-S$	4234, 0	4243, 1
$l-S$	3380, 7	3074, 5
D	− 2773, 0	− 2739, 9
$l+S$	− 2770, 0	− 2664, 0
$2f-l$	− 2074, 6	− 2068, 25
$3l$	1835, 0	1844, 7
$4D-l$	1753, 2	1458, 2
S	− 1168, 8	− 1248, 4
$2D-l+S$	− 675, 0	− 1107, 0
$2D+S$	− 894, 1	− 957, 1
$4D-2l$	1087, 0	906, 9
$2D-2l$	− 897, 05	− 809, 3
$2D+2l$	821, 0	790, 9
$2D+l-S$	648, 7	574, 7
$4D$	759, 7	572, 6
$D+S$	423, 7	440, 3
$2D-2f$	− 309, 7	− 319, 0
$2l-S$	359, 4	300, 9
$2D-3l$	− 338, 95	− 295, 7
$D+l$	− 309, 7	− 283, 7
$2l+S$	− 292, 2	− 267, 9
$2D-2f-l$	− 251, 3	− 238, 4
$2D-2S$	260, 05	222, 0

Argt.	My Coefficient.	Delaunay's.
$2D-2f+l$	$-$ 140, 25	$-$ 146, 4
$2D-l-2S$	146, 1	132, 6
$4l$	119, 8	121, 0
$2D+l+S$	$-$ 137, 3	$-$ 117, 2
$4D-l-S$	184, 1	86, 8
$3D-l$	$-$ 105, 2	$-$ 58, 1
$4D+l$	93, 5	54, 05
$2D+3l$	51, 2	51,
$4D-2l-S$	87, 7	46, 5
$3D$	14, 6	46, 2
$2D+2f-2l$	$-$ 42, 6	$-$ 43,
$D+l+S$	38, 9	38, 9
$l-2S$	38, 3	38, 3
$2D-l+2S$	$-$ 37, 4	$-$ 37, 4
$2D-2l-S$	$-$ 37, 1	$-$ 37, 1

This table shews at a glance how great the errors of Delaunay's coefficients of parallax, when reduced to the form in which they are employed by Sir George Airy, in many cases really are. Hence the discordances which he met with in the results of the substitutions should occasion no surprise. In the Introduction to the *Numerical Lunar Theory*, p. 4, line 20, it is stated through inadvertence that the factor which Sir George Airy calls M is a quantity "depending on the proportion of the masses of the Earth and Moon." This is not the case however, since M is simply the ratio of the sum of the actual masses of the Earth and Moon to the sum of the masses which would be required to make the Moon describe an undisturbed orbit about the Earth in which the periodic time and the mean parallax were the same as in the actual orbit.

The theoretical value of M is simply expressed as the *cube* of the constant term in Delaunay's value of $\frac{a}{r}$. This value is given analytically in p. 802 or p. 914 of the second volume of Delaunay's Theory, but only to the fifth order of small quantities, which is not accurate enough. The development of the constant term of $\frac{a}{r}$ has been carried by me to a much greater extent at p. 472 of Vol. XXXVIII. of the *Monthly Notices* (*see p.* 203 *above*). Turning this expression into numbers, and cubing it, we find the value of M to be 1·0027259, which agrees very closely with the value found by Sir George Airy by comparing the constant terms on the two sides of his equation (10).

The other two ways of finding M proposed by Sir George in p. 76 of his Theory, viz. by comparing the quantities on the two sides of the equations (10) and (12), corresponding to the arguments 2 and 301 respectively, are not satisfactory, as the results will be affected by errors in the theoretical determinations of the mean motions of the Moon's perigee and node respectively.

The multiplier M, representing the sum of the masses of the Earth and Moon, must be employed wherever the mutual attraction of these two bodies comes in question. In Sir George Airy's note at p. 254 of the March number of the *Monthly Notices*, he calls M the coefficient of the *solar* term, but this is plainly a mistake. I should mention that I have already communicated the substance of this paper to Sir George Airy himself.

33.

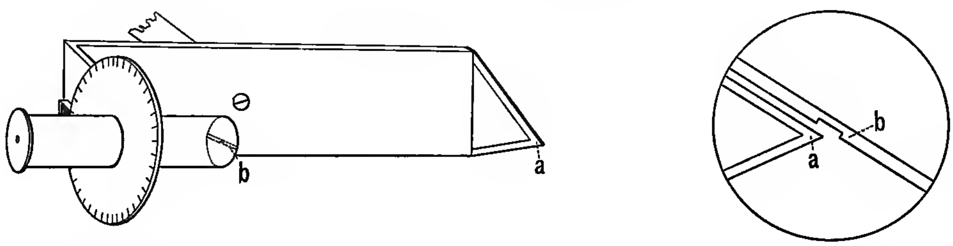
ON THE METEORIC SHOWER OF NOVEMBER, 1866.

[From the *Proceedings of the Cambridge Philosophical Society.* Vol. II.]

THE author described the instrument used in the observation of the Meteors, and mentioned the various hypotheses which have been advanced concerning the orbit of these bodies; he explained the calculations which he had made to determine this, and shewed that the attractions of the Earth, Jupiter, Saturn and Uranus were nearly sufficient to account for a hitherto unexplained change of about 29 minutes in the position of the nodes of the orbit in each period of 33 years. He called attention to the fact that the orbit calculated appeared to coincide very nearly with those of certain comets; and held that the latter were elongated ellipses with a periodic time of 33 years.

[The instrument consists of an axle which is mounted in all respects as the axle of a theodolite. To one end of the axle is fixed a graduated circle, as in the theodolite, which marks 0° when the line of sight of the instrument is horizontal.

To the other end of the axle and at right angles to it is a bar to which are attached a V-shaped piece of metal, *a*, and an eyepiece.



On the eyepiece, about 3 in. from the eye towards the V is a thin bar, *b*, with a notch at its middle point, which can turn about the line in which the instrument is pointing.

Attached to the thin bar is a circle divided to degrees, which marks 0° when the bar is exactly parallel to the upper edge of the V with the notch downwards.

The circle is provided with a vernier of 12 divisions, so that angles can be read to 5′. The point of the V is on the axis or line of sight about which the thin bar turns.

The altitude and azimuth of any point in the line of sight can be read off on the vertical and horizontal circles of the instrument.

When the instrument is directed to a meteor, the thin bar can be readily turned with its circle so as to coincide in direction with the apparent path of the meteor across the field of view.]

34.

ON THE ORBIT OF THE NOVEMBER METEORS.

[From the *Monthly Notices of the Royal Astronomical Society.* Vol. XXVII. (1867.)]

It is known to the President and to several members of the Society that I have been for some time past engaged in researches respecting the November meteors, and allusion is made to some of my earlier results in the last Annual Report. As my investigations are now in some measure complete, and the results which I have obtained appear to me important, I have thought that they may not be without interest for the Society.

In a memoir on the November Star Showers, by Professor H. A. Newton, contained in Nos. 111 and 112 of *The American Journal of Science and Arts*, the author has collected and discussed the original accounts of 13 displays of the above phenomenon in years ranging from A.D. 902 to 1833.

The following table exhibits the dates of these displays, and the Earth's longitude at each date, together with the same particulars for the shower of November last, which have been added for the sake of completeness.

No.	A.D.	Day and hour.	Earth's longitude.
		d. h.	° ′
1	902	Oct. 12 17	24 17
2	931	14 10	25 57
3	934	13 17	25 32
4	1002	14 10	26 45
5	1101	16 17	30 2
6	1202	18 14	32 25
7	1366	22 17	37 48
8	1533	24 14	41 12
9	1602	27 10 O.S.	44 19
10	1698	Nov. 8 17 N.S.	47 21
11	1799	11 21	50 2
12	1832	12 16	50 49
13	1833	12 22	50 49
14	1866	13 13	51 28

From these data Professor Newton infers that these displays recur in cycles of 33·25 years, and that during a period of two or three years at the end of each cycle a meteoric shower may be expected. He concludes that the most natural explanation of these phenomena is, that the November Meteors belong to a system of small bodies describing an elliptic orbit about the Sun, and extending in the form of a stream along an arc of that orbit which is of such a length that the whole stream occupies about one-tenth or one-fifteenth of the periodic time in passing any particular point. He shews that in one year the group must describe either

$$2 \pm \frac{1}{33{\cdot}25}, \text{ or } 1 \pm \frac{1}{33{\cdot}25}, \text{ or } \frac{1}{33{\cdot}25}$$

revolutions, or, in other words, that the periodic time must be either 180·0 days, 185·4 days, 354·6 days, 376·6 days, or 33·25 years.

It is seen that the time of the year at which the meteoric shower takes place becomes gradually later and later, and that accordingly the Earth's longitude at that time, or the longitude of the node of the orbit of the meteors, is gradually increasing. Professor Newton finds that the node has a mean motion of $102''{\cdot}6$ annually with respect to the Equinox, or of $52''{\cdot}4$ with respect to the fixed stars; and he remarks that since the periodic time is limited to five possible values, each capable of an accurate determination, and since therefore from the position of the radiant point the other elements of the orbit can be found, it seems possible to compute the secular motion of the node for each periodic time with considerable accuracy, and the actual motion of the node being known, we have thus an apparently simple method of deciding which of the five periods is the correct one.

Soon after the remarkable display of these meteors in November last, I undertook the examination of this question. From the position of the radiant point as observed by myself, I calculated the elements of the orbit of the meteors, starting with the supposition that the periodic time was 354·6 days, the value which Professor Newton considered to be the most probable one. The orbit which corresponds to this period is very nearly circular, and it readily follows from the ordinary theory that the action of *Venus* would produce an annual increase of about $5''$ in the longitude of the node, and that of *Jupiter* an annual increase of about $6''$. The calculation of the motion of the node due to the Earth's action, presented greater difficulty in consequence of the two orbits nearly intersecting each other. I succeeded, however, in obtaining an approximate solution, applicable to this case, from which it followed that the Earth's action would produce an annual increase of nearly 10″ in the longitude of the node. Thus the three planets above mentioned which alone, in the case supposed, sensibly affect the motion of the node, would cause a motion of about 21″ annually, or nearly 12′ in 33·25 years. It has been already mentioned that the observed motion of the node is 52″·4 annually, or about 29′ in 33·25 years. Hence the observed motion of the node is totally irreconcilable with the supposition that the periodic time of the meteors about the Sun is 354·6 days. If the periodic time were supposed to be about 377 days, the calculated motion of the node would differ very little from that in the case already considered, while, if the periodic time were a little greater or a little less than half a year, the calculated motion of the node would be still smaller. Hence, of the five possible periods indicated by Professor Newton, four are entirely incompatible with the observed motion of the node, and it only remains to examine whether the fifth period, viz. one of 33·25 years, will give a motion of the node in accordance with observation.

The calculations which have been above described were entirely founded on my own determination of the radiant point. In order to have as secure a basis as possible for the subsequent calculations, I adopted for the position of the radiant point the mean of my own and five other determinations, partly taken from published documents and partly privately communicated to me. These determinations are as follows, the several authorities being placed in alphabetical order:—

	R.A.	Decl.
	° ′	° ′
Adams	148 50	22 10 N.
Baxendell	149 33	22 57
Brünnow	150	22
Challis	149 39	23 12
Herschel	148 9	23 48
Herschel, A.	149	24
Mean	149 12	23 1 N.

Or with reference to the ecliptic,

Long. 143° 22′ Lat. 9° 51′ N.

Starting from this position of the radiant point, and the assumed period, and taking into account the action of the Earth on the meteors as they were approaching it, I obtained the following elliptic elements of their orbit:—

Period	33·25	years (assumed)
Mean distance	10·3402	
Eccentricity	0·9047	
Perihelion distance	0·9855	
Inclination.............................	16° 46′	
Longitude of Node	51 28	
Distance of Perihelion from Node	6 51	
Motion Retrograde		

In order to determine the secular motion of the node in this orbit, I employed the method given by Gauss in his beautiful investigation "*Determinatio attractionis, &c.*"

It may be proved that if two planets revolve about the Sun in periodic times which are incommensurable with each other, the secular variations which either of these bodies produces in the elements of the orbit of the other would be the same as if the whole mass of the disturbing body had been distributed over its orbit in such a manner that the portion of the mass distributed over any given arc should be always proportional to the time which the body takes to describe that arc. In the memoir just referred to, Gauss shews how to determine the attraction of such an elliptic ring on a point in any given position. When this attraction has been calculated for any point in the orbit of the meteors, we can at once deduce the changes which it would produce in the elements of the orbit, while the meteors are describing any given small arc contiguous to the given point. Hence, by dividing the orbit of the meteors into a number of small portions, and summing up the changes corresponding to these portions, we may find the total secular changes of the elements produced in a complete period of the meteors.

In this manner I have found that during a period of 33·25 years, the longitude of the node is increased 20′ by the action of *Jupiter*, nearly 7′ by the action of *Saturn*, and about 1′ by that of *Uranus*. The other planets produce scarcely any sensible effects, so that the entire calculated increase of the longitude of the node in the above-mentioned period is about 28′.

As already stated, the observed increase of longitude in the same time is 29′. This remarkable accordance between the results of theory and observation appears to me to leave no doubt as to the correctness of the period of 33·25 years.

In order to attain a sufficient degree of approximation it is requisite to break up the orbit of the meteors into a considerable number of portions, for each of which the attractions of the elliptic rings corresponding to the several disturbing planets have to be determined; hence the calculations are necessarily very long, although I have devised a modification of Gauss's formulæ which greatly facilitates their application to the present problem. In these numerical calculations I have been greatly aided by my assistants, more especially by Mr Graham. I am now engaged in obtaining a closer approximation by subdividing certain parts of the orbit of the meteors into still smaller portions, but the results which have been given above cannot be materially changed.

Since I entered upon the foregoing investigation other astronomers have been led, on totally independent grounds, to conclusions which strongly confirm, and are confirmed by, those at which I have myself arrived.

In the *Bullettino Meteorologico dell' Osservatorio del Collegio Romano*, Vol. v. Nos. 8, 10, 11, 12, are published four letters from Sig. Schiaparelli, Director of the Observatory of Milan, "*Intorno al corso ed all' origine probabile delle Stelle Meteoriche.*" In these letters the author arrives at the conclusion that the orbits which the Meteors describe about the Sun are very elongated, like those of comets, and that probably both these classes of bodies originally come into our system from very distant regions of space. In his last letter, dated 31st Dec. 1866, Sig. Schiaparelli shews that if the August Meteors be supposed to describe a parabola, or a very elongated ellipse, the elements of their orbit calculated from the observed position of their radiant point, agree very closely with those of the orbit of Comet II. 1862, calculated by Dr Oppolzer. The following table exhibits this agreement:—

	August Meteors.	Comet II. 1862.
Perihelion distance	0·9643	0·9626
Inclination	64° 3′	66° 25′
Longitude of Perihelion	343 28	344 41
Longitude of Node......	138 16	137 27
Direction of Motion ...	Retrograde	Retrograde

Hence it appears probable that the great Comet of 1862 is a part of the same current of matter as that to which the August Meteors belong.

In the letter which has just been referred to, Sig. Schiaparelli likewise gives approximate elements of the orbit of the November Meteors, calculated on the supposition that the period is 33·25 years; but as the calculations were founded on an imperfect determination of the radiant point, these elements were not sufficiently accurate, and Sig. Schiaparelli failed to find any cometary orbit which could be identified with that of the meteors.

Soon after this, on the 21st January, 1867, M. Le Verrier communicated to the Academy of Sciences a theory of the origin and nature of shooting stars, very similar in its main features to that of Sig. Schiaparelli, and at the same time gave more accurate elements of the orbit of the November Meteors, his calculations being based on a better determination of the radiant point than that employed by the astronomer of Milan.

In the *Astronomische Nachrichten*, of the 29th January, Mr C. F. W. Peters of Altona pointed out that the elements given by M. Le Verrier closely agreed with those of Tempel's Comet (I. 1866), calculated by Dr Oppolzer, and on the 2nd February, Sig. Schiaparelli, having recalculated the elements of the orbit of the meteors on better data than before, himself noticed the same agreement.

Dr Oppolzer's elements of Tempel's comet are as follows:—

Period	33·18 years
Mean distance	10·3248
Eccentricity	0·9054
Perihelion distance	0·9765
Inclination	17° 18′
Longitude of Node	51 26
Distance of Perihelion from Node	9 2
Direction of Motion	Retrograde

If these elements be compared with those of the November Meteors which I have given in a former part of this communication, it will be seen that their agreement is remarkably close.

The curious and unexpected resemblance which is thus shewn to exist between the orbits of known comets and those of the meteors, both of August and November, opens a wide field for speculation. It is difficult to believe that the coincidences which have been noticed are merely accidental; but whether or not we are disposed to adopt the ideas of Sig. Schiaparelli as to the intimate relations between meteors and comets, I cannot help thinking that my researches respecting the motion of the node of the November Meteors have settled the question as to the periodic time of these bodies beyond a doubt.

35.

NOTE ON THE ELLIPTICITY OF MARS, AND ITS EFFECT ON THE MOTION OF THE SATELLITES.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. XL. (1879).]

ONE of the results of Professor Asaph Hall's able discussion of his observations of the satellites of *Mars* is to shew that the orbits of both the satellites are at present inclined at small angles to the plane of the planet's equator. It becomes an interesting question to inquire whether this state of things is a permanent one. The plane of *Mars'* orbit is inclined to its equator at an angle of 27° or 28°. If then the planes of the orbits of the satellites retain constant inclinations to the orbit of the planet, as they would do if the Sun's disturbing force were the only force tending to alter those planes, their inclinations to the plane of *Mars'* equator, and still more their inclinations to each other, would in time become considerable.

In No. 2280 of the *Astronomische Nachrichten*, Mr Marth has found the motions of the nodes of the orbits of the satellites on the orbit of the planet due to the Sun's action, and he concludes that, if there is no force depending on the internal structure of *Mars* which counteracts or greatly modifies the Sun's action, the nodes of the orbits will be in opposition to each other a thousand years hence, when the mutual inclination of the satellites' orbits will amount to about 49°.

In this case the near approach to coincidence between the planet's equator and the planes of the orbits of the satellites, which is observed to exist at the present time, would be merely fortuitous; but this appears *à priori* to be very improbable.

It is well known that, if there were no external disturbing force, the ellipticity of a planet would cause the nodes of a satellite's orbit to retrograde on the plane of the planet's equator, while the orbit would preserve a constant inclination to that plane. Laplace has shewn that, when both the action of the Sun and the ellipticity of the planet are taken into account, the orbit of the satellite will move so as to preserve a nearly constant inclination to a fixed plane passing through the intersection of the planet's equator with the plane of the planet's orbit, and lying between those planes, and that the nodes of the satellite's orbit will have a nearly uniform retrograde motion on the fixed plane. The angles which this fixed plane makes with the planes of the planet's equator and its orbit respectively will depend on the ratio between the rates of the above-mentioned retrogradations of the nodes produced by the Sun's action and by the ellipticity of the planet. If the latter of these causes would produce a much slower motion of the nodes than the former, as in the case of our Moon, the fixed plane will nearly coincide with the planet's orbit; but if, as in the case of the inner satellites of *Jupiter*, the ellipticity of the planet would produce a much more rapid motion of the nodes than the Sun's action, then the fixed plane will nearly coincide with the planet's equator.

The ratio of the motion of a satellite's node to that of the satellite itself, when the Sun's action is the disturbing force, varies, *ceteris paribus*, as the square of the satellite's periodic time, that is as the cube of its mean distance from the planet. On the other hand, the ratio of the same two motions, when the ellipticity of the planet is the disturbing cause, varies inversely as the square of the mean distance. Hence, for different satellites of the same planet, the motion of the nodes caused by the ellipticity will bear to the motion caused by the Sun's action the ratio of the inverse fifth powers of the mean distances.

Now, the distance of the inner satellite of *Mars* from the planet's centre is only about $2\frac{3}{4}$ radii of the planet, a greater comparative proximity than is known to exist elsewhere in the Solar System, and the distance of the outer satellite from the same centre is only about 7 radii of the planet, while the periodic times of both are very small compared with the periodic time of *Mars*. Hence the effect of a given small ellipticity of *Mars* on the motion of the nodes of the satellites will be greatly magnified.

It is true that the ellipticity of *Mars* is still unknown, and is probably too small to be ever directly measureable; but we are not without means of determining, within not very wide limits, its probable amount, and we shall presently see that, in all probability, in the case of both the satellites the motion of the nodes produced by the ellipticity greatly exceeds the motion caused by the Sun's action, so that the fixed planes for both satellites are only slightly inclined to the planet's equator.

From measures of the planet's diameter and of the greatest elongations of the satellites, combined with the known time of rotation of *Mars* and the periodic times of the satellites, it is found that the ratio of the centrifugal force to gravity at *Mars'* equator is about $\frac{1}{220}$. Hence it follows that if the planet were homogeneous its ellipticity would be about $\frac{1}{176}$. If, instead of the planet being homogeneous, its internal density varied according to the same law as that of the Earth, so that the ellipticity would bear the same ratio to the above-mentioned ratio of centrifugal force to gravity at the equator as in the case of the Earth, then the ellipticity would be about $\frac{1}{228}$. In all probability the actual ellipticity of *Mars* lies between these limits.

The following Table shews the annual motions of the nodes of the two satellites, caused by the Sun's action and by the planet's ellipticity respectively, for the above values of that ellipticity, and also for the ellipticity $\frac{1}{118}$, which has been deduced from Professor Kaiser's observations, although I have no doubt that this value is too great. The Table likewise contains the corresponding inclinations of the fixed planes, so often mentioned above, to the planet's equator.

Satellite I.			*Satellite II.*		
Annual motion of the node due to the Sun's action, $0^\circ{\cdot}06$.			Annual motion of the node due to the Sun's action, $0^\circ{\cdot}24$.		
Supposing ellipticity =			Supposing ellipticity =		
$\frac{1}{118}$	$\frac{1}{176}$	$\frac{1}{228}$	$\frac{1}{118}$	$\frac{1}{176}$	$\frac{1}{228}$
the annual motion of the node due to that ellipticity will be			the annual motion of the node due to that ellipticity will be		
333°	182°	113°	$13^\circ{\cdot}4$	$7^\circ{\cdot}3$	$4^\circ{\cdot}5$
Corresponding inclinations of fixed plane to planet's equator:			Corresponding inclinations of fixed plane to planet's equator:		
$17''$	$31''$	$50''$	$27'$	$50'$	$1^\circ\,19'$

From this it may be inferred that the orbit of the 1st satellite preserves a constant inclination to a plane which is inclined less than $1'$ to the plane of *Mars'* equator, and that the orbit of the 2nd satellite preserves a constant inclination to a plane which is inclined about 1° to the plane of the same equator.

The ellipticity will also cause rapid motions in the apses of the orbits of the satellites, particularly in that of the first; and as this orbit appears from Professor Hall's determination to have a sensible eccentricity, it will be possible, by future observations, to determine the motion of the apse, and therefore the ellipticity of the planet. If further observations shew that the orbits of the satellites are sensibly inclined to their fixed planes, the motion of their nodes will supply another means of determining the ellipticity of the planet.

36.

NOTE ON WILLIAM BALL'S OBSERVATIONS OF SATURN.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. XLIII. (1883).]

IN No. 9 of Vol. I. of the *Philosophical Transactions*, a brief account is given of an observation of *Saturn* made on Oct. 13, 1665, at 6 o'clock, by William Ball, at Mamhead, near Exeter, and it is suggested that the appearance presented by the planet may perhaps be caused by its being surrounded by *two* rings instead of *one*.

This account has recently given rise to considerable discussion; and there are some difficulties connected with it which do not appear to have been satisfactorily cleared up. In a few copies of the volume this account is illustrated by a figure, in which the external boundary of the ring, instead of being of a regular elliptical form, has two blunt notches or indentations at the extremities of the minor axis. The plate containing this figure, however, is wanting in by far the larger number of the copies.

Now, I think, it may be safely asserted that no telescope, capable of shewing *Saturn's* ring at all, ever exhibited it in this extraordinary form, and therefore if the above figure faithfully represents William Ball's drawing, he was either a very inaccurate and careless observer, or he must have been provided with very inadequate instrumental means.

On the other hand, we have ample proof that he was a careful and assiduous observer, that in particular he made a long series of observations of *Saturn*, and that these were made with instruments not much inferior to those employed by Huyghens himself in similar observations.

It is well known that Huyghens's discovery of the true nature of the appendage to *Saturn*, which had so puzzled Galileo and others, was contested by Father Fabri at Rome, who wrote under the name of "Eustacius de Divinis."

Huyghens replied to Fabri's objections in a tract which appeared in 1660, entitled *Brevis Assertio Systematis Saturnii sui*, and which is contained in the third volume of his collected works.

In this tract he repeatedly appeals to Ball's observations in England in confirmation of his own. It is clear that Huyghens was in possession of drawings by Ball which represented the various appearances presented by the planet during the four years from 1656 to 1659 inclusive, and that he had carefully compared them with those which he had himself taken during the same interval. After mentioning the dark band which he had observed on the disk of *Saturn* at times when the remainder of the ring was invisible, he quotes a letter from Dr Wallis, dated Dec. 22, 1658, in which reference is made to an earlier letter dated May 29, 1656, wherein Dr Wallis had mentioned this band as having been observed by Ball, and had inquired whether his correspondent had likewise perceived it. Huyghens goes on to say that from Feb. 5, 1656, to July 2, when the planet appeared round and without ansæ, this band or dark shading was observed by Ball to cross the centre of the disk, as shewn in his drawing, exactly as in Huyghens's own figure.

Afterwards, when the ansæ had re-appeared, the band was seen with more difficulty, and its position was less accurately laid down in Ball's drawing. From Nov. 5, 1656, to July 9, 1657, when the oblong arms of *Saturn* were seen apparently united to the disk, Ball gives a figure quite similar to that of Huyghens, except that he makes the arms a little thicker.

Again, from Nov. 9, 1657, to June 7, 1658, when the arms were more open, Ball's figure is exactly similar to Huyghens's, except a slight difference in the position of the obscure zone or belt.

Also, finally, the same remark applies to the figure of the planet from Jan. 3, 1659, to June 17 of the same year, when the ansæ were a little more widely opened.

Having made these comparisons between Ball's drawings of the planet and his own, Huyghens remarks that Ball was unacquainted with his hypothesis* (respecting the ring), and therefore could not be supposed to be

* Huyghens's *Systema Saturnium* only appeared in 1659.

biased by it, while he himself would not dare to represent the phenomena otherwise than they really were, since, if he did, he might at once be contradicted by the English observer.

This judgment of so competent an authority as Huyghens, made while he had before him all the materials for forming it, left no doubt on my mind as to the merit of Ball's observations.

In order to see whether any further light could be thrown on the subject, I have recently taken an opportunity of consulting the MSS. preserved in the archives of the Royal Society.

Among them I find there is a letter in William Ball's own hand, dated April 14, 1666, in which he makes reference to his observations of *Saturn*, although the greater part of the letter relates to other subjects. He mentions that the observations were made partly with a telescope thirty-eight feet in length, having a double eye-glass, and partly with another telescope twelve feet in length. In the postscript to this letter he gives a small sketch of *Saturn* as it appeared at that time (1666), and he mentions that the same appearance was presented by the planet in 1664. In this figure the external boundary of the ring has the form of a regular oval, without any notches or other irregularities.

No allusion is made to the very different appearance which, if the figure in the *Philosophical Transactions* is authentic, the planet must have presented in 1665.

It should be understood that the paper in the *Philosophical Transactions* which is now in question was not written by Ball himself. It contains, however, a quotation from a letter of Ball to a friend (probably Sir R. Moray), and in what appears to be the last clause of this quotation, the figure is said to be "a little hollow above and below." I cannot help thinking that this clause has been added or altered in some way to correspond with the given figure. The letter of Ball on which this paper was founded is not in the archives; but there is preserved, not a drawing, but a paper-cutting, representing the planet and its ring, which is no doubt the original of the figure engraved in the *Transactions*.

The defect in the paper-cutting probably originated in the following way. In order to make the cutting, the paper was first folded twice in directions at right angles to each other, so that only a quadrant of the ellipse had to be cut.

The cut started rightly in a direction perpendicular to the major axis, but through want of care, when the cut reached the minor axis, its direction formed a slightly obtuse angle with that axis instead of being perpendicular to it.

Consequently, when the paper was unfolded, shallow notches or depressions appeared at the extremities of the minor axis.

I imagine that the account in the *Philosophical Transactions* was written by some one inexperienced in astronomical observations, who took for granted that the figure was correct. The mistake being soon discovered, the plate which contained the erroneous figure of *Saturn*, together with two other figures relating to different subjects, was cancelled, and thus its appearance in only a few of the copies is accounted for. The other figures on the cancelled plate were repeated in a new plate which accompanied No. 24 in the same volume of the *Transactions*.

In Lowthorp's abridged edition of the *Transactions* the figure of *Saturn* has been corrected.

I find no evidence that Ball, any more than Huyghens, had noticed any indication of a division in the ring.

It may be interesting to give the original text of the passages of Huyghens's *Brevis Assertio Systematis Saturnii sui*, in which reference is made to Ball's observations.

The citations are taken from the third volume of Huyghens's *Opera Varia*, edited by 'S Gravesande, and published at Leyden in 1724.

"Credo et fasciam nigricantem in *Saturni* disco, liquido sibi conspici dixisset Eustacius, ni Fabrio visum fuisset eam nimium hypothesi meæ annulari favere. Cum autem ne optimis quidem suis perspicillis eam cerni affirmet, hinc quoque quanto illa meis deteriora sint perspicuum sit. Nam ne mihi phenomenon illud confictum credatur, idem et in Anglia pridem observari cœpisse sciendum est; et liquet ex literis viri clar. Joh. Wallisii, Oxonia ad me datis 22 Dec. 1658, quibus inter alia hæc scribit. *Monebam etiam iisdem literis* (nempe *datis* 29 *Maji* 1656) *de Saturni fascia quam jam ante observaverat D. Ball, et sciscitabar num tu eandem conspexeras*, &c. Eam porro fasciam à 5 Feb. 1656 ad 2 Jul., quo tempore rotundus *Saturnus* absque ansis apparuit, medium planetæ discum secare D. Ball adnotavit, ut in schemate ad me misso expressa est. Atque ita mihi quoque fuerat eo tempore observata, ut cernitur pag. 544 *Systematis Saturnii*, quam figuram hic repeto. Postmodum tamen renatis *Saturni* ansis cum difficillimè conspici eadem fascia cœpisset, minus rectè quoque a D. Ball, quantum ad situm attinet, depicta est. At in mearum observationum adversariis, die 26 Nov.

1656, et alias adscriptum invenio, lineam obscuram fuisse evidentissimam, eo nempe positu, qui pag. 545 *System. Saturnii.* memoratur."—Pp. 624, 625.

. . . .

"Non ægre nunc fidem habitum iri spero, tum mihi tum Anglis simul observatoribus, qui anno 1657 oblonga *Saturni* brachia disco utrinque conjuncta spectavimus, qualia exhibet figura *Systematis* mei pag. 545, quam hic repono; non autem binorum orbiculorum formâ a medio disco disjunctorum, ut Eustacius se illa eodem tempore vidisse dejerat. Adderem hic schema quod mihi à D. Ball, supra memorato, advenit, nisi planè simile esset huic nostro, hoc uno tantillum duntaxat abludens, quod brachia illa ubique paulo crassiora ille referat.

"Eam vero formam a 5 Nov. 1656 ad 9 Jul. 1657 sibi apparuisse scribit. Apertis autem brachiis, qualis pag. 547 *Systematis* mei et hic repræsentatur, talem à 9 Nov. 1657 ad 7 Jun. 1658, idem observator depingit, simillima prorsus figura, nisi quod ad positum zonæ obscuræ attinet, de quo dixi suprà. Ac denique à 3 Jan. 1659 ad 17 Jun. ejusdem anni, ansis paulo latius adhuc apertis. Et hæc quidem ille, ignarus adhuc meæ hypotheseos, ne ob præconceptam opinionem aliquid indulsisse sibi existimetur. Neque ego aliter quam se revera habent referre auderem, cum redarguere me, si fallam, autori observationum in promptu sit."—P. 626.

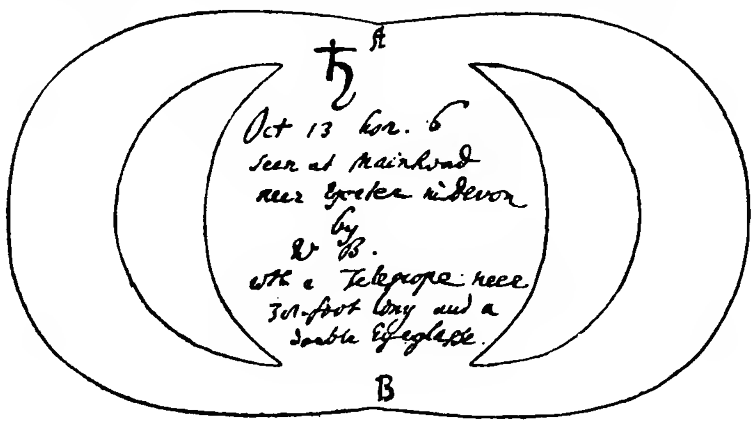
The following extract comprises all that is material in the Paper in the *Philosophical Transactions*:—

"This observation was made by Mr William Ball, accompanied by his brother, Dr Ball, October 13, 1665 at six of the Clock, at Mainhead [Mamhead] near Exeter in Devonshire, with a very good Telescope near 38 foot long, and a double Eye-glass as the observer himself takes notice, adding, that he never saw that planet more distinct. The observation is represented by Fig. 3 concerning which, the Author saith in his letter to a friend, as follows, This appear'd to me the present figure of Saturn, somewhat otherwise, than I expected, thinking it would have been decreasing, but I found it full as ever, and a little hollow above and below. Whereupon the Person, to whom notice was sent hereof, examining this shape, hath by letters desired the worthy Author of the *System of this Planet*, that he would now attentively consider the present Figure of his Anses, or Ring, to see whether the appearance be to him, as in this Figure, and consequently whether he there meets with nothing that may make him think, that it is not *one* body of a circular Figure, that embraces his Disk, but *two*."

From this it is clear that the suggestion of *two* rings was made, not by Ball himself, but by his anonymous correspondent.

By the kind permission of the President and Council of the Royal Society, I am enabled to make the following extracts from two letters in William Ball's own hand, and likewise to give exact representations of the form of the paper-cutting, and of Ball's small sketch of *Saturn*, referred to in the foregoing Paper, both of which have been kindly copied for me by our Assistant-Secretary, Mr Wesley.

The annexed figure shews the form of the paper-cutting.



The writing on the cutting appears to be in Oldenburg's hand.

The first letter is dated Mamhead, April 14, 1666, and is probably addressed to Oldenburg.

"I have seen ♄ two mornings this year (with a 12 foot glasse the longest I can use at this time with convenience) and find the figure the same as it was in -64. What his figure was last autumne (by mee observed with 38 foot glasse much better than that at Gresham Colledge) I suppose S^{r}. R. Moray hath communicated. I could not have a second sight, straining very much for that one, for the shadow of the body on his ring I doe not well understand the meaning but I suppose I saw the same thing; for I never had a clearer sight of him in any glasse I ever looked in, one thing I can boast of, sc. I am not prejudiced with any conceit of hypothesis which doth commonly send all observations to favour one side and soe there must bee a little added or diminished as the designe requires," &c. &c.

In a postscript is the following, with the little sketch:—

"I saw ♄ this morn. at 4 a clock with 12 foot glasse and judge him the same figure as in -64—that is just ovall with two black spotts and I thinke a faint shadow of a belt which I have alwaies seene, but will not be peremptory in itt."

The second letter is dated "Mamhead ♄ September 15, -66," and is addressed "For Sir Robert Moray K^{t} at Whitehall, These."

"I designe to send you all the figures of ♄. I promised them my L^{d} Brounker and hee was pleased most kindly to accept itt but I (like any thing you please to call mee bad enough) have hitherto shamfully failed, as alsoe of an account of husbandry to Mr Oldenburg. I am still gazing at the starrs though to very little purpose more then to keep my eyes in use," &c. &c.

It will be noticed that the passage in Ball's first letter in which he claims to be unbiased by any hypothesis, agrees with the statement of Huyghens respecting him.

The passage in the same letter, "for the shadow of the body on his ring I doe not well understand the meaning but I suppose I saw the same thing," I conjecture to refer to an attempted explanation by Huyghens, or some other astronomer, of the phenomenon observed by Ball, by attributing it to the shadow of the body of the planet cast on his ring.

It is plain that such an explanation would not be applicable, if similar depressions had been observed at the two extremities of the minor axis of the ring.

37.

ON THE CHANGE IN THE ADOPTED UNIT OF TIME.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. XLIV. (1884).]

THE December number of the *Monthly Notices* contains a paper by Major-General Tennant in which the author arrives at conclusions which appear to him to confirm Mr Stone's views respecting a change in the unit of mean solar time. In reality, however, those conclusions are quite consistent with my own as given in the same number of the *Monthly Notices*, (*see p.* 259 *above*) and not at all with Mr Stone's.

According to Major-General Tennant (*Monthly Notices*, p. 43), the factor by which the tabular mean motions should be multiplied in consequence of the change from Bessel's to Le Verrier's determination of the ratio of the mean solar to the sidereal day is what he calls

$$\frac{\text{Sidereal Seconds in Le Verrian Mean Day}}{\text{Sidereal Seconds in Besselian Mean Day}}.$$

Now, if n be the Sun's mean motion in a mean solar day as determined by Bessel, the sidereal seconds in a mean solar day will be

$$86400 \times \frac{360^\circ + n}{360^\circ}.$$

But if $n + \delta n$ be the Sun's mean motion in a mean solar day as determined by Le Verrier, the sidereal seconds in a mean solar day will be

$$86400 \times \frac{360^\circ + n + \delta n}{360^\circ},$$

and therefore the factor above referred to by Major-General Tennant will be

$$\frac{360^\circ + n + \delta n}{360^\circ + n} = 1 + \frac{\delta n}{360^\circ + n},$$

whereas, according to Mr Stone's views, this factor should be

$$\frac{n + \delta n}{n} = 1 + \frac{\delta n}{n},$$

where the difference from 1 is nearly 366 times greater than it should be.

The same thing may be otherwise shewn thus:—

If N denote the number of mean solar days in a mean tropical year, according to Bessel's determination, then $N+1$ will be the corresponding number of sidereal days in the same interval.

Consequently, the ratio of the length of a mean solar to that of a sidereal day will be

$$\frac{N+1}{N} = 1 + \frac{1}{N}.$$

But if $N + \delta N$ denote the number of mean solar days in a mean tropical year, according to Le Verrier's determination, then $N + \delta N + 1$ will be the corresponding number of sidereal days in the same interval.

And consequently the above-mentioned ratio will become

$$\frac{N + \delta N + 1}{N + \delta N} = 1 + \frac{1}{N + \delta N}.$$

Hence the ratio of the length of a mean solar to that of a sidereal day will be changed in the ratio of

$$\frac{1 + \dfrac{1}{N + \delta N}}{1 + \dfrac{1}{N}} = 1 - \frac{\delta N}{N(N+1)}, \text{ nearly,}$$

whereas, according to Mr Stone, the ratio which measures this change would be

$$\frac{N}{N + \delta N} = 1 - \frac{\delta N}{N}, \text{ nearly,}$$

where, as before, the difference from 1 is nearly 366 times too great.

Mr Stone's error appears to arise from his *equating* two things which are really different, and which are inconsistent with each other,—viz. Bessel's and Le Verrier's determinations of the Sun's mean motion in longitude in the same interval of time.

Major-General Tennant is wrong in supposing that solar observations are no longer employed in Observatories for the determination of mean solar time. If this were the case, it would only shew that the Observatories had taken a very retrograde step, since the final test whether the mean solar times have been correctly found can only be supplied by solar observations. Whenever the mean solar times are deduced from the observed sidereal times, it is tacitly assumed that the tabular mean longitudes of the Sun which have been employed are correct; and if this is not the case, the mean solar times deduced will require a corresponding correction, which can only be found by solar observations.

Thus mean solar time may be determined with reference to a natural phenomenon,—viz. the transit of the true Sun over the meridian of a given place; and the mean solar day is the average of all the apparent solar days defined as the intervals between two successive transits, and therefore has nothing arbitrary about it. To speak of Besselian mean time and Le Verrian mean time, or of the Besselian mean solar day and the Le Verrian mean solar day, can produce nothing but confusion in our ideas of the measure of time.

38.

ON NEWTON'S SOLUTION OF KEPLER'S PROBLEM.

[From the *Monthly Notices of the Royal Astronomical Society,* Vol. XLIII. (1882).]

OF all the methods which have been proposed for the solution of this problem, that which leads most rapidly to a result having any required degree of precision may be briefly explained as follows:—

The equation to be solved by successive approximations is

$$x - e\sin x = z,$$

where z is the known mean anomaly, e the eccentricity, and x the eccentric anomaly to be determined.

Suppose x_0 to be an approximate value of x, found whether by estimation, by graphical construction, or by a previous rough calculation, and let

$$x_0 - e\sin x_0 = z_0.$$

Then if

$$\delta x_0 = \frac{z - z_0}{1 - e\cos x_0},$$

and

$$x' = x_0 + \delta x_0,$$

x' will be a much more approximate value of x than x_0.

Similarly, if we put

$$x' - e\sin x' = z',$$

and if

$$\delta x' = \frac{z - z'}{1 - e\cos x'}$$

and $$x'' = x' + \delta x',$$

x'' will be a much more approximate value of x than x'; and so on, to any required degree of approximation.

If the error of the assumed value x_0 be supposed to be of the order i, when e is taken as a small quantity of the first order, then the error of the value x' will be of the order $2i + 1 = i'$ suppose, similarly the error of the value x'' will be of the order $2i' + 1 = 4i + 3$, and so on, so that the order of the error is more than doubled at each successive approximation.

The above explains the immense advantage of this process over the use of series proceeding according to powers of e, when great precision is required in the result; since, in this latter method, the addition of a new term only increases the order of the error by unity.

The degree of rapidity of the approximation may be still further increased by the following slight modification of the above process.

Starting, as before, with the value x_0, and calling $z - z_0 = \delta z_0$, we should obtain a much more accurate value than before of the correction δx_0 to be applied to x_0, by putting

$$\delta x_0 = \frac{z - z_0}{1 - e\cos(x_0 + \tfrac{1}{2}\delta x_0)} = \frac{\delta z_0}{1 - e\cos(x_0 + \tfrac{1}{2}\delta x_0)}.$$

Now, e being supposed to be small, δz_0 is an approximate value of δx_0, and may be written for it in the small term in the denominator.

Hence, if we put

$$\delta x_0 = \frac{\delta z_0}{1 - e\cos(x_0 + \tfrac{1}{2}\delta z_0)};$$

$$x' = x_0 + \delta x_0,$$

x' will be a nearer approximation to the true value of x than was obtained before by the corresponding operation.

Similarly, if $$x' - e\sin x' = z',$$

and $$z - z' = \delta z',$$

and if $$\delta x' = \frac{\delta z'}{1 - e\cos(x' + \tfrac{1}{2}\delta z')},$$

then $$x'' = x' + \delta x'$$

will be the next approximate value of x, and the process may be continued as far as we please.

If the error of x_0 be of the order i, that of x' will now be of the order $2i+2$, that of x'' will be of the order $2(2i+2)+2=4i+6$, and so on, so that the degree of rapidity of the approximation is still greater than before.

If we chose to take the mean anomaly itself as the first approximate value of the eccentric anomaly—that is, if we put

$$x_0 = z,$$

we should have

$$z_0 = z - e\sin z,$$

and the value of δx_0 given by the first method would be

$$\delta x_0 = \frac{e\sin z}{1 - e\cos z},$$

while that given by the second and more accurate method would be

$$\delta x_0 = \frac{e\sin z}{1 - e\cos\left(z + \tfrac{1}{2}e\sin z\right)};$$

and the error of $x' = x_0 + \delta x_0$ would be of the 3rd order in the former case, and of the 4th order in the latter.

In practice, however, a much nearer first approximate value of x may be always found by inspection, and of course the smaller the error of this value is, the more rapid will be the rate of the subsequent approximations.

The methods above explained have been long known. The first method is given at p. 41 of Thomas Simpson's *Essays on Several Subjects in Speculative and Mixed Mathematics*, published in 1740; and Gauss' method given at pp. 10—12 of the *Theoria Motus*, published in 1809, is essentially the same.

The second method, or rather the modification of the first, is given by Cagnoli in his *Trigonométrie*, at pp. 377, 378 of the first edition, published in 1786, and at pp. 418—420 of the second edition, published in 1808.

Now, my object in the present note is to point out that the first method explained above is exactly equivalent to that given by Newton in the *Principia*, at pp. 101, 102 of the second edition, and at pp. 109, 110 of the third edition, when Newton's expressions are put into the modern analytical form.

None of the subsequent authors, however, mentions this method as being Newton's, the unusual form in which Newton's solution is given having, no doubt, caused them to overlook it.

In the first edition of the *Principia* a modification of the method is given which was, I have no doubt, intended by Newton to be equivalent to the second method given above; but by some inadvertence, instead of the denominator of $\delta x'$ being

$$1 - c \cos\left(x' + \frac{1}{2}\delta z'\right),$$

when expressed in the above notation, he takes it to be what is equivalent to

$$1 - e \cos\left(x' + \frac{1}{2} e \sin x'\right),$$

which is only true for the first approximation when x_0 is taken $= z$.

In the second and third editions this error is corrected, but Newton contents himself with the more simple expression given by the first method.

We need not be surprised that Newton should have employed this method of solving the transcendental equation

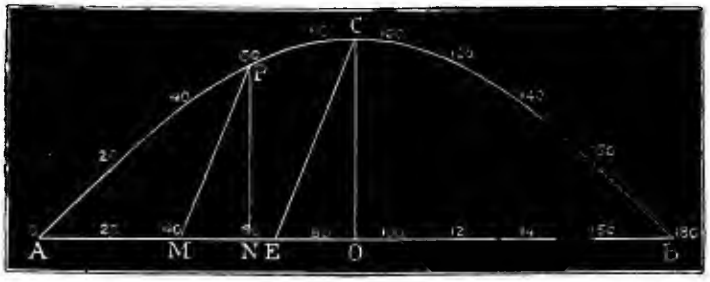
$$x - e \sin x = z,$$

since the method is identical in principle with his well-known method of approximation to the roots of algebraic equations.

For convenience of calculation, the approximate values x_0, x', x'', &c., should be so chosen that their sines may be taken directly from the tables without interpolation; and, since each approximation is independent of the preceding ones, this may always be done if x' be taken equal, not to $x_0 + \delta x_0$ itself, but to the angle nearest to $x_0 + \delta x_0$ which is contained in the tables, and if similarly x'' be taken equal to the tabular angle which is nearest to $x' + \delta x'$, and so on. In the first approximation it will be amply sufficient to use 5-figure logarithms, but in the subsequent ones tables with a larger number of decimal places should be employed.

A first approximate value of the eccentric anomaly corresponding to any given mean anomaly may be found by a very simple graphical construction, provided we have traced, once for all, a curve in which the ordinates are proportional to the sines of the angles represented on any given scale by the abscissæ.

This curve is commonly called "the curve of sines." It will be sufficient to trace the portion of the curve for which the ordinates are positive.



Let AOB be the line of abscissæ, and let AO be taken equal to OB, and let each of them be divided into 90 equal parts representing degrees of angle. Let AN be any abscissa representing the angle x, and let the corresponding ordinate $NP = c \sin x$; then the greatest ordinate will be $OC = c$, corresponding to the abscissa AO.

Suppose the curve line $APCB$ to be divided into 180 parts which correspond to equal divisions on the line of abscissæ $ANOB$.

Then if E be taken in AO so that $EO = e \times 57{\cdot}296$ divisions, or if $AE = 90 - e \times 57{\cdot}296$ divisions, and if CE be joined and PM be drawn parallel to it through P meeting the line of abscissæ in M, then AM will represent the mean anomaly corresponding to the eccentric anomaly represented by AN.

For, since the triangles PMN, CEO are similar,

$$\frac{MN}{EO} = \frac{PN}{CO} = \sin x,$$

and therefore $MN = EO \sin x = 57{\cdot}296\,(e \sin x)$.

Hence MN represents the number of degrees in $x - z$, and therefore AM represents the mean anomaly z.

Conversely, if AM represents any given mean anomaly, then if MP be drawn parallel to EC, it will cut the curve in the point P corresponding to the eccentric anomaly.

By the employment of a parallel ruler we may find the eccentric anomaly corresponding to any given mean anomaly, or conversely, without actually drawing a line. For if we lay an edge of the ruler across the points EC and then make a parallel edge to pass through the point M it will cut the curve in the point P required.

Thus we may always find a first approximate value of the eccentric anomaly, without making repeated trials, whether the eccentricity be large or small.

I described this graphical method of solving Kepler's problem at the Birmingham meeting of the British Association in 1849. It is referred to in a paper by Mr Proctor in Vol. XXXIII. of the *Monthly Notices*, p. 390.

The construction is so simple that it has probably been proposed before, though I have nowhere met with it.

Note on Professor Zenger's solution of the same problem given in Number 9 *of* Vol. XLII. *of the "Monthly Notices."*

The only peculiarity in this solution is in the mode of obtaining the first approximate value employed. The subsequent approximations are carried on by means of the first method given above. Professor Zenger's process may be represented in a slightly different form as follows:—

We have
$$x - z = e \sin x,$$
and therefore
$$\sin(x-z) = \sin(e \sin x) = e \sin x \left\{1 - \frac{1}{6} e^2 \sin^2 x + \frac{1}{120} e^4 \sin^4 x - \text{etc.}\right\},$$
or
$$\sin(x-z) = f \sin x;$$
where
$$f = e\left\{1 - \frac{1}{6} e^2 \sin^2 x + \frac{1}{120} e^4 \sin^4 x - \text{etc.}\right\}.$$
Hence
$$\tan(x-z) = \frac{f \sin z}{1 - f \cos z}.$$

Now, an approximate value of f is e, and the error in the determination of $\tan(x-z)$ if we were to put
$$\tan(x-z) = \frac{e \sin z}{1 - e \cos z},$$
would be of the 3rd order in e.

If we determine f so that the error in the determination of x shall vanish when

$$x=\frac{\pi}{2},$$

we shall have

$$f=e\left\{1-\frac{1}{6}e^2+\frac{1}{120}e^4-\text{etc.}\right\}=\sin e,$$

and the approximate equation for finding $x-z$ becomes

$$\tan(x-z)=\frac{\sin e \sin z}{1-\sin e \cos z}.$$

The error still remains in general of the 3rd order in e, but the maximum error will be smaller than when f is taken $=e$.

The value of x given by this equation is readily seen to be equivalent to that given by Professor Zenger's equation,

$$\cot x=\cot z-\frac{e \operatorname{cosec} z}{1+\frac{1}{6}\sin^2 e+\frac{3}{40}\sin^4 e+\text{etc.}},$$

where we may remark that the quantity

$$\frac{1}{1+\frac{1}{6}\sin^2 e+\frac{3}{40}\sin^4 e+\text{etc.}}$$

is equivalent to

$$\frac{\sin e}{e}, \text{ or to } 1-\frac{1}{6}e^2+\frac{1}{120}e^4-\text{etc.},$$

a series which converges much more rapidly than the series for its reciprocal, employed by Professor Zenger.

A still more advantageous result may, however, be obtained by determining f so that the error may vanish both when

$$x=\frac{\pi}{3},$$

and when

$$x=\frac{2\pi}{3},$$

that is when

$$\sin x=\frac{\sqrt{3}}{2},$$

so that
$$f = e\left\{1 - \frac{1}{8}e^2 + \frac{3}{640}e^4 -, \text{ etc.}\right\}.$$

The order of accuracy of the approximation will not be altered by confining ourselves to the first two terms of this value of f, so that we may take

$$\tan(x - z) = \frac{e\left(1 - \frac{1}{8}e^2\right)\sin z}{1 - e\left(1 - \frac{1}{8}e^2\right)\cos z}, \text{ nearly.}$$

The error is still of the 3rd order, but its maximum amount is less than before.

If f be taken
$$= e\left\{1 - \frac{1}{6}e^2 \sin^2 z\right\},$$

and
$$\tan(x - z) = \frac{f \sin z}{1 - f\cos z};$$

the error in the determination of $\tan(x - z)$, and therefore in the determination of x, will be only of the 4th order.

There are several misprints and some errors of calculation in Professor Zenger's paper, on which I need not dwell. *True* anomaly in line 8 of the paper should be *eccentric* anomaly, and the same error occurs on p. 448.

39.

NOTE ON DR MORRISON'S PAPER (ON KEPLER'S PROBLEM).

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. XLIII. (1883).]

THE reference to Hansen's paper should be made to *Abhandlungen der Süchsischen Gesellschaft der Wissenschaften*, Band IV. p. 249, instead of to Band II. as stated by Dr Morrison.

In this paper Hansen's object is not merely to express the coefficients of the series which gives the eccentric anomaly in powers of e, otherwise this might have been done much more simply in the following manner.

Calling g the mean, and x the eccentric anomaly, we have

$$g = x - e \sin x,$$

or

$$x = g + e \sin x,$$

which is in the proper form for the application of Lagrange's theorem for developing x or any function of x in terms of g and ascending powers of e.

Hence we have

$$x = g + e \sin g + \frac{e^2}{1 \,.\, 2} \frac{d}{dg} (\sin^2 g) + \frac{e^3}{1 \,.\, 2 \,.\, 3} \frac{d^2}{dg^2} (\sin^3 g) + \frac{e^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} \frac{d^3}{dg^3} (\sin^4 g) + \&c.,$$

whence by substituting for the powers of $\sin g$ their expressions in sines or cosines of multiples of g, and differentiating, we may readily obtain the function of g which multiplies any given power of e.

The numerical coefficient of the term in $(x-g)$ which involves

$$e^m \sin (m-2n)\, g$$

is

$$(-1)^n \left(\frac{m-2n}{2}\right)^{m-1} \frac{1}{(1\,.\,2\ldots n)\,(1\,.\,2\ldots \overline{m-n})}$$

where m is a positive integer, and n is either zero or a positive integer less than $\frac{m}{2}$, and $(1\,.\,2\ldots n)$ is to be put $=1$, when $n=0$.

The expressions for x and for the sines of multiples of x are developed to the 12th power of e by Schubert in the appendix to Bode's *Jahrbuch* for 1820. In the same appendix Schubert likewise gives the development of the true anomaly in terms of the mean to the 13th power of e.

Oriani had already given this last-mentioned development to the 11th power of e in the appendix to the Milan *Ephemeris* for 1805.

The numerical coefficients which he finds differ in four cases from those given by Schubert, but I have recomputed the coefficients in these cases, and find that Schubert's results are correct.

There is a misprint, however, in Schubert's expression for the true anomaly at the foot of p. 230, where the coefficient of $e^{12} \sin 12g$ should be

$$\frac{7218065}{2^{13}\,.\,3\,.\,7\,.\,11} \text{ instead of } \frac{7218065}{2^{13}\,.\,3^7\,.\,11}.$$

Delambre's formula is copied from Oriani's, and is therefore affected by the same errors, together with some additional typographical ones.

I have verified Schubert's result for (v), the true anomaly in terms of the mean, by the consideration that when $g=0$, the value of

$$\frac{dv}{dg} \text{ becomes } \frac{(1+e)^2}{(1-e^2)^{\frac{3}{2}}}$$

$$=1+2e+\frac{5}{2}e^2+3e^3+\frac{27}{8}e^4+\frac{15}{4}e^5+\frac{65}{16}e^6+\frac{35}{8}e^7+\frac{595}{128}e^8+\frac{315}{64}e^9+\frac{1323}{256}e^{10}$$

$$+\frac{693}{128}e^{11}+\frac{5775}{1024}e^{12}+\frac{3003}{512}e^{13}+\&c.$$

By comparing Schubert's result with that of Dr Morrison, we see that there are the following errata in the latter: viz. the coefficient of $e^{10}\sin 8M$ in the equation of the centre should be

$$-\frac{4745483}{2^9\,.\,3^4\,.\,5\,.\,7}\text{ instead of }-\frac{1182827}{2^7\,.\,3^4\,.\,5\,.\,7},$$

and the coefficient of $e^{12}\sin 10M$ should be

$$-\frac{76972457}{2^{11}\,.\,3^4\,.\,7\,.\,11}\text{ instead of }-\frac{769805651}{2^{12}\,.\,3^4\,.\,5\,.\,7\,.\,11}.$$

In Schubert's expression for $\frac{r}{a}$ in p. 231, which is also carried as far as e^{13}, there are the following errata, which are evidently merely typographical: viz. in the coefficient of $-\cos 3g$, instead of

$$-\frac{3^6\,.\,11}{2^{17}\,.\,5\,.\,7}e^{11}\text{ should be }+\frac{3^6\,.\,11}{2^{17}\,.\,5\,.\,7}e^{11},$$

and in the coefficient of $-\cos 12g$, instead of

$$\frac{2\,.\,3}{5^2\,.\,7\,.\,11}e^{12}\text{ should be }\frac{2\,.\,3^6}{5^2\,.\,7\,.\,11}e^{12}.$$

Oriani's formula for the radius vector has been examined and found correct.

A very good investigation of the general term of the expansion of the true anomaly in terms of the mean is likewise given in a paper by Mr Greatheed, in the first volume of the *Cambridge Mathematical Journal*, p. 208 (p. 228 in the second edition).

The approximate expression for the eccentric anomaly in terms of the mean given by Dr Morrison in the latter part of his paper coincides with the first two terms of the series found in Keill's *Astronomical Lectures*, p. 291 (5th edition, 1760), and the method of correcting an approximately known value which Dr Morrison quotes from Encke is identical with Newton's method for the same purpose, which is also explained in Keill's *Lectures*, p. 296 *et seq.*

On this subject reference may also be made to my paper in the *Monthly Notices* for December 1882, p. 43 (*see p.* 289 *above*).

In addition to the errata already specified, the following may be noticed:—

In Oriani's formula for the equation of the centre, in the Milan *Ephemeris* 1805, pp. 14 and 15,

In the coefficient of sin $4g$,

$$\text{instead of } -\frac{1367}{2^7 . 3^3 . 7}e^{10} \text{ read } -\frac{1619}{2^7 . 3^3 . 7}e^{10}.$$

In the coefficient of sin $5g$,

$$\text{instead of } -\frac{3649663}{2^{17} . 3^3 . 7}e^{11} \text{ read } -\frac{4305913}{2^{17} . 3^3 . 7}e^{11}.$$

In the coefficient of sin $6g$,

$$\text{instead of } +\frac{7751}{2^{10} . 6}e^{10} \text{ read } +\frac{7751}{2^{10} . 7}e^{10}.$$

In the coefficient of sin $11g$,

$$\text{instead of } \frac{63039512101}{2^{17} . 3^4 . 5^2 . 7 . 11}e^{11} \text{ read } \frac{62929017101}{2^{17} . 3^4 . 5^2 . 7 . 11}e^{11}.$$

As Delambre's formula is copied from Oriani, it is affected with the same errors, and in addition to these the following errata occur:—

In the Introduction to Delambre's Solar Tables, 1806,

In the coefficient of sin g,

$$\text{instead of } \frac{565879}{2^{16} . 3^2 . 5^2}e^{11} \text{ read } \frac{565879}{2^{16} . 3^3 . 5^2}e^{11}.$$

In the coefficient of sin $6g$,

$$\text{instead of } -\frac{7913}{2z . 5 . 7}e^{8} \text{ read } -\frac{7913}{2^7 . 5 . 7}e^{8}.$$

In the coefficient of sin $7g$,

$$\text{instead of } -\frac{1173271}{2^{14} . 3^2 . 5}e^{9} \text{ read } -\frac{1773271}{2^{14} . 3^2 . 5}e^{9}.$$

And in his *Astronomy*, 1814, vol. II. p. 52,

In the coefficient of sin $2g$,

$$\text{instead of } +\frac{677}{2^2 . 3^3 . 5}e^{10} \text{ read } +\frac{677}{2^9 . 3^3 . 5}e^{10}.$$

Also in Delambre's expression for $\frac{r}{a}$ the following errata occur:—

In the Introduction to his Solar Tables, 1806,

In the coefficient of $-\cos g$,

$$\text{instead of } -\frac{3}{2^2}e^3 \text{ read } -\frac{3}{2^3}e^3.$$

In the coefficient of $-\cos 5g$,

$$\text{instead of } +\frac{5^6}{2^{13}\,.\,9}e^9 \text{ read } +\frac{5^6}{2^{13}\,.\,7}e^9.$$

And in his *Astronomy*, 1814, vol. II. p. 51,

In the coefficient of $-\cos 5g$,

$$\text{instead of } \frac{53}{2^7\,.\,3}e^5 \text{ read } \frac{5^3}{2^7\,.\,3}e^5.$$

Also in Delambre's formula for the hyperbolic logarithm of the radius vector, the following errata occur:—

In the Introduction to his Solar Tables, 1806,

In the coefficient of $-\cos 2g$,

$$\text{instead of } -\frac{9}{240}e^6 \text{ read } -\frac{9}{640}e^8.$$

In the coefficient of $-\cos 8g$,

$$\text{instead of } \frac{47529}{2^{10}\,.\,5\,.\,7}e^8 \text{ read } \frac{47259}{2^{10}\,.\,5\,.\,7}e^8.$$

And in his *Astronomy*, 1814, vol. II. p. 50,

In the coefficient of $-\cos 7g$,

$$\text{instead of } \frac{355081}{2^{10}\,.\,3^2\,.\,5^7}e^7 \text{ read } \frac{355081}{2^{10}\,.\,3^2\,.\,5\,.\,7}e^7.$$

40.

ON NEWTON'S THEORY OF ASTRONOMICAL REFRACTION, AND ON HIS EXPLANATION OF THE MOTION OF THE MOON'S APOGEE.

[*British Association Report* (1884), p. 645.]

41.

ON THE GENERAL VALUES OF THE OBLIQUITY OF THE ECLIPTIC, AND OF THE PRECESSION AND INCLINATION OF THE EQUATOR TO THE INVARIABLE PLANE, TAKING INTO ACCOUNT TERMS OF THE SECOND ORDER*.

[From *The Observatory*, No. 109 (1886).]

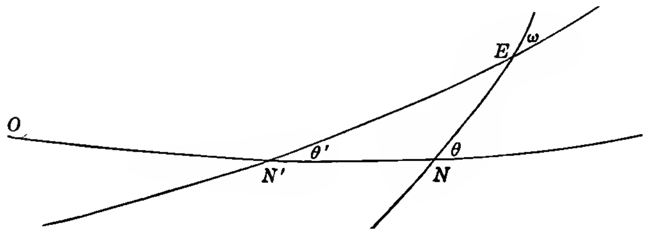
IF we adopt the values of the precession and nutation employed by Peters in his classical work *Numerus Constans Nutationis*, I find that the ratio of the sum of the masses of the Earth and Moon to the mass of the Moon is that of 82·834 to 1, a result which differs slightly from that found by Peters from the same data.

The amount of precession caused by the Sun's action depends in a slight degree on the eccentricity of the Earth's orbit. In order to find the precession for an indefinite period, it will be proper to employ the *mean* value of the square of this eccentricity instead of the value of this quantity at the present time.

Taking this circumstance into account, and also introducing the small correction of the coefficient of precession which depends on the square of the coefficient of nutation, I find that if ω be the obliquity of the

* Abstract of a paper read Sept. 11, 1884, at the Philadelphia meeting of the American Association for the Advancement of Science.

ecliptic at any time, the rate of the luni-solar precession at that time during a Julian year will be represented by $c \cos \omega$, where $c = 54''\cdot 94625$ nearly.



Now let $ON'N$ be the fixed plane of reference, which may be either the ecliptic at a given epoch, or, better still, the invariable plane of the system, or any other arbitrary fixed plane.

Also let $N'E$ be the position of the ecliptic and NE that of the equator $\Big\}$ at any time t,

so that the point E is the autumnal equinox at that time. $ON = \phi$, $ON' = \phi'$, O being a fixed point, θ and θ' the inclination of the equator and ecliptic respectively to the fixed plane, and ω the angle $N'EN$, or the obliquity of the ecliptic at time t. Also let $NE = \lambda$. Then the quantities $p = \tan\theta' \sin\phi'$ and $q = \tan\theta' \cos\phi'$ are known in terms of t from the theory of the secular variations of the plane of the Earth's orbit, and θ' may be considered as a small quantity of the first order, the square of which we propose to take into account.

In the triangle $N'EN$ we have

$$\cos\omega = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi'),$$

$$\sin\omega \cos\lambda = \sin\theta \cos\theta' - \cos\theta \sin\theta' \cos(\phi - \phi'),$$

$$\sin\omega \sin\lambda = \sin\theta' \sin(\phi - \phi'),$$

which give ω and λ when θ and ϕ are known.

From the instantaneous motion of the equator with reference to the ecliptic at time t, supposed for an instant to be fixed, it is easily seen that we have

$$\frac{d\phi}{dt} = -c\frac{1}{\sin\theta}\cos\omega \sin\omega \cos\lambda,$$

$$\frac{d\theta}{dt} = \quad c\cos\omega \sin\omega \sin\lambda,$$

or, substituting from above for $\cos\omega$, $\sin\omega\cos\lambda$ and $\sin\omega\sin\lambda$,

$$\frac{d\phi}{dt} = -c\,\frac{\cos^2\theta'}{\sin\theta}\{\cos\theta + \sin\theta\tan\theta'\cos(\phi-\phi')\}\{\sin\theta - \cos\theta\tan\theta'\cos(\phi-\phi')\},$$

$$\frac{d\theta}{dt} = c\cos^2\theta'\{\cos\theta + \sin\theta\tan\theta'\cos(\phi-\phi')\}\tan\theta'\sin(\phi-\phi'),$$

which are the differential equations for determining θ and ϕ, θ' and ϕ' being supposed to be already known in terms of t.

From the above we may deduce the following:—

$$\frac{d}{dt}\left(\frac{\cos\omega}{\cos\theta'}\right) = \frac{dq}{dt}(\sin\theta\cos\phi) + \frac{dp}{dt}(\sin\theta\sin\phi).$$

The integration of these equations may be readily effected by the method of indeterminate coefficients.

Suppose the values of p and q to be

$$p = \Sigma\gamma_i\sin(g_it+\beta_i),$$
$$q = \Sigma\gamma_i\cos(g_it+\beta_i),$$

where i takes the successive integral values 0, 1, 2, &c., equal in number to the number of planets considered, and the quantities γ_i, g_i, and β_i are known constants.

Then we may find that

$$\begin{aligned}\theta = h &+ \tfrac{1}{2}\tan h\Sigma a_i(a_i-1)\gamma_i^2 + \tfrac{1}{2}\cot h\Sigma\left(a_i-\tfrac{1}{2}\right)\gamma_i^2\\ &+ \Sigma a_i\gamma_i\cos\{(k-g_i)t+\alpha-\beta_i\}\\ &+ \Sigma a_{ii}(\gamma_i)^2\cos 2\{(k-g_i)t+\alpha-\beta_i\}\\ &+ \Sigma a_{ij}\gamma_i\gamma_j\cos\{(2k-g_i-g_j)t+(2\alpha-\beta_i-\beta_j)\}\\ &+ \Sigma a'_{ij}\gamma_i\gamma_j\cos\{(g_i-g_j)t+\beta_i-\beta_j\}.\end{aligned}$$

And

$$\begin{aligned}\phi = kt + \alpha &+ \Sigma b_i\gamma_i\sin\{(k-g_i)t+\alpha-\beta_i\}\\ &+ \Sigma b_{ii}(\gamma_i)^2\sin 2\{(k-g_i)t+\alpha-\beta_i\}\\ &+ \Sigma b_{ij}\gamma_i\gamma_j\sin\{(2k-g_i-g_j)t+(2\alpha-\beta_i-\beta_j)\}\\ &+ \Sigma b'_{ij}\gamma_i\gamma_j\sin\{(g_i-g_j)t+\beta_i-\beta_j\},\end{aligned}$$

in which i and j are supposed to be different integers.

Also

$$a_i = \frac{k}{k-g_i}, \text{ and therefore } a_i - 1 = \frac{g_i}{k-g_i};$$

$$a_{ii} = -\tfrac{1}{4}a_i(a_i^2-1)\tan h - \tfrac{1}{4}a_i^2 \cot h;$$

$$a_{ij} = -\tfrac{1}{2}\frac{k}{2k-g_i-g_j}\{(a_i^2+a_j^2-2)\tan h + (a_i+a_j)\cot h\};$$

$$a'_{ij} = \tfrac{1}{2}\frac{k}{g_i-g_j}\{a_i^2-2a_i-a_j^2+2a_j\}\tan h + \tfrac{1}{2}\frac{k}{g_i-g_j}(a_i-a_j)\cot h.$$

Also

$$b_i = -a_i(a_i-1)\tan h - a_i \cot h;$$

$$b_{ii} = \tfrac{1}{8}a_i^2(a_i-1)^2\tan^2 h + \tfrac{1}{4}a_i(a_i^2+a_i-1) + \tfrac{1}{4}a_i^2\cot^2 h;$$

$$\begin{aligned} b_{ij} = &-\frac{k}{2k-g_i-g_j}a_{ij}\tan h - \tfrac{1}{2}\frac{k}{2k-g_i-g_j}\{a_i(a_i-1)+a_j(a_j-1)\}\tan^2 h \\ &+\tfrac{1}{2}\frac{k}{2k-g_i-g_j}\{a_i^2+a_i-1+a_j^2+a_j-1-a_ia_j\} \\ &+\frac{k}{2k-g_i-g_j}(a_i+a_j)\cot^2 h; \end{aligned}$$

$$\begin{aligned} b'_{ij} = &-\frac{k}{g_i-g_j}a'_{ij}\tan h + \tfrac{1}{2}\frac{k}{g_i-g_j}\{a_i(a_i-1)+a_j(a_j-1)\}\tan^2 h \\ &-\tfrac{1}{2}\frac{k}{g_i-g_j}\{a_i^2+a_j^2+a_ia_j-5a_i-5a_j+6\}; \end{aligned}$$

or the value of this last coefficient may be otherwise expressed thus—

$$b'_{ij} = -\tfrac{1}{2}\frac{k}{g_i-g_j}\{(a_i-1)(a_j-1)(a_i+a_j)\tan^2 h + (a_i+a_j-2)(a_i+a_j-3)\}.$$

Also the value of ω, the obliquity of the ecliptic, is thus expressed in terms of the same quantities:

$$\begin{aligned} \omega = h &+ \Sigma(a_i-1)\gamma_i\cos\{(k-g_i)t+\alpha-\beta_i\} \\ &+\Sigma\left[-\tfrac{1}{4}a_i(a_i-1)^2\tan h - \tfrac{1}{4}(a_i-1)^2\cot h\right]\gamma_i^2\cos 2\{(k-g_i)t+\alpha-\beta_i\} \\ &+\Sigma\Big[-\tfrac{1}{2}\frac{k}{2k-g_i-g_j}(a_i^2+a_j^2-2)\tan h - \tfrac{1}{2}\frac{k}{2k-g_i-g_j}(a_i+a_j)\cot h \\ &\qquad +\tfrac{1}{2}(a_i^2+a_j^2-a_i-a_j)\tan h + \tfrac{1}{2}(a_i+a_j-1)\cot h\Big] \\ &\qquad\qquad \times\gamma_i\gamma_j\cos\{(2k-g_i-g_j)t+2\alpha-\beta_i-\beta_j\} \end{aligned}$$

$$+\Sigma\left[\tfrac{1}{2}\frac{k}{g_i-g_j}(a_i-a_j)(a_i+a_j-2)\tan h+\tfrac{1}{2}\frac{k}{g_i-g_j}(a_i-a_j)\cot h\right.$$
$$\left.-\tfrac{1}{2}(a_i^2+a_j^2-a_i-a_j)\tan h-\tfrac{1}{2}(a_i+a_j-1)\cot h\right]$$
$$\times\gamma_i\gamma_j\cos\{(g_i-g_j)\,t+\beta_i-\beta_j\}.$$

Also the value of k in terms of the constant c which, as stated before, is known from the theory of precession is

$$k=-c\cos h\{1-\Sigma\tfrac{1}{4}(a_i-1)(3a_i-5)\gamma_i^2\};$$

h and a are the arbitrary constants which enter into the complete integrals of our equations, and they are determined so as to make the initial values of θ and ϕ, or those of ω and ϕ, equal to the observed values.

It is to be remarked that one of the values of g is 0, and if the invariable plane of the system be taken as the fixed plane of reference, the corresponding value of γ will be also zero, so that the expressions for θ, ϕ, and ω will be considerably simplified by this choice of the fixed plane.

According to Stockwell's determination, in Vol. 18 of the *Smithsonian Contributions*, the longitude of the ascending node of the invariable plane on the ecliptic of 1850 is 106° 14′ 18″, and the inclination of this plane to the same ecliptic is 1° 35′ 20″.

Also, as already mentioned, if we make the invariable plane of the system our plane of reference, we have for $g_0=0$, $\gamma_0=0$; and the remaining values of g_i and those of β_i and $\log\gamma_i$ which correspond to them, according to Stockwell's determination, will be the following:—

	$i=1$	$i=2$	$i=3$	$i=4$
$g_i\ldots$	$-2''\cdot9161$	$-25''\cdot9350$	$-5''\cdot21365$	$-6''\cdot6693$
$\beta_i\ldots$	133° 57′	126° 20′	19° 7′	307° 17′
$\log\gamma_i\ldots$	7·20626	7·44481	8·01815	7·84525

	$i=5$	$i=6$	$i=7$
$g_i\ldots$	$-17''\cdot6266$	$-18''\cdot9365$	$-0''\cdot66166$
$\beta_i\ldots$	300° 1′	254° 43′	20° 31′
$\log\gamma_i\ldots$	7·59939	8·41184	7·12320

where the quantities g_i are expressed in seconds and have reference to a Julian year as the unit of time, and the quantities γ_i are expressed in circular measure.

Now in the figure before given the point N' is the descending node of the invariable plane on the ecliptic of 1850, so that the longitude of N' is $286^\circ\ 14'\ 18''$.

Also the longitude of the point E, which is the autumnal equinox, is 180°. Hence $N'E = 253^\circ\ 45'\ 42''$.

Whence we may find for 1850:

$$\begin{aligned} \theta &= \ \ 23^\circ\ \ 3'\ 43'' \\ \phi - \phi' &= 257\ \ 20\ \ 31 \\ \text{or} \qquad \phi &= 183\ \ 34\ \ 49 \end{aligned}$$

Also, according to Stockwell, the obliquity of the ecliptic in 1850 was

$$\omega = 23^\circ\ 27'\ 31''\cdot 0.$$

Hence by repeated approximation we may find:

$$\begin{aligned} h &= \ \ 23^\circ\ 18'\ 54'' \text{ nearly} \\ a &= 177\ \ 25\ \ 52 \quad\ ,, \\ \text{also} \qquad k &= -50''\cdot 4607 \end{aligned}$$

whence by substitution all the terms in θ, ϕ, and ω may be found numerically.

Addition.—If we wish to take into account the variability of the eccentricity of the Earth's orbit, the value of $-k$ should be taken

$$= 50''\cdot 4548 + 24''\cdot 034\,(e^2 - e_0^2),$$

and the quantity $-kt$ in the above formulæ should be replaced by

$$50''\cdot 4548\, t + \int 24''\cdot 034\,(e^2 - e_0^2)\, dt.$$

Where e is the eccentricity of the Earth's orbit at time t, and e_0^2 the *mean* value of the square of the eccentricity, which, according to Stockwell's determination, is

$$= \cdot 0009864.$$

42.

ADDRESS ON PRESENTING THE GOLD MEDAL OF THE ROYAL ASTRONOMICAL SOCIETY TO M. PETERS.

[From the *Memoirs of the Royal Astronomical Society*. Vol. XXI. (1852).]

It has already been announced to you that the medal of the Society has been awarded to M. Peters, for his two papers, entitled, "Numerus Constans Nutationis ex Ascensionibus Rectis Stellæ Polaris in Specula Dorpatensi Annis 1822 ad 1838 observatis deductus," and "Recherches sur la Parallaxe des Etoiles Fixes," which are published respectively in the third and fifth volumes of the sixth series of the *Mathematical and Physical Transactions of the Imperial Academy of Sciences of St Petersburg;* and it is now my duty to explain to you the grounds of this award, which (unless their effect be marred by my very imperfect statement of them) will, I doubt not, secure your approval.

These papers form part of a series emanating from the astronomers of the Pulkowa Observatory, and having for their object the advancement of sidereal astronomy; first, by a new and more accurate determination of the elements which affect the apparent places of all the stars, such as precession, nutation, and aberration; and, secondly, by an examination of the peculiarities affecting individual stars, such as annual parallax and proper motion, by which alone we can gain a knowledge of the scale on which the visible universe is constructed, and of the arrangement in space and of the relative motions of the bodies of which it is composed.

These important objects have been steadily pursued at the Pulkowa Observatory, under the guiding mind of its illustrious director, with an energy and success which have placed that establishment in a position with respect to sidereal astronomy, similar to that which our own observatory of Greenwich occupies with respect to the observation of the Moon.

The order of date, as well as the nature of the subjects treated of, leads me first to speak of M. Peters' paper on the constant of nutation. But before proceeding to give an account of the paper itself, it may not be out of place to advert rapidly to former researches respecting nutation.

When Newton traced the precession of the equinoxes to its cause in the attraction of the Sun and Moon on the protuberant equatoreal zone of the terrestrial spheroid, he perceived that the Sun's action would likewise cause a nutation of the Earth's axis, the period of which is half a year. He contents himself with remarking that this nutation can be scarcely sensible.

In the same way, of course, the Moon's action produces a small nutation, of which the period is half a month. Abstracting these nutations, the tendency of the Sun's action is to make the pole of the equator move in a circular arc about the pole of the ecliptic; and in a similar manner the Moon's action tends to make the pole of the equator describe a circular arc about the pole of the Moon's orbit for the time being. Now, as this latter pole moves in a circle about the pole of the ecliptic in a period of about nineteen years, it is easy to see that this will give rise to an inequality in the rate of precession, and to a change of the obliquity of the ecliptic, having the same period.

It is curious, however, that Newton does not allude at all to this, which constitutes by far the most important part of nutation; and this is the more remarkable, since the principles which he lays down in treating of precession are quite sufficient to obtain, by means of very simple geometrical reasoning, not only the law, but very approximately, the coefficients of the inequalities in the precession and obliquity due to this cause.

The state of practical astronomy, however, in Newton's time, was not sufficiently advanced to induce him to enter more fully into this subject; and it was, consequently, reserved for the immortal discoverer of aberration to detect these motions of the Earth's axis by means of his observations, and then to trace them to their true cause. While discussing the observations which led him to the discovery of aberration, Bradley noticed that the annual changes of declination of the stars did not exactly correspond with those which would be occasioned by precession, and he made allowance for this by employing in the reduction of his observations the changes deduced from the observations themselves.

No sooner, therefore, had Bradley determined the law and the cause of aberration, than a new subject of investigation presented itself, requiring a much longer course of observations for its complete examination. Comparing his observations of different stars, he found that their changes of declination were such as might be attributed to a real motion of the Earth's axis, and he was not slow in perceiving that the varying action of the Moon upon the equatoreal parts of the Earth, according to the different positions of the nodes of the lunar orbit, was the probable cause of this motion. During the course of the observations, Bradley communicated what he had observed to Machin, who was then "employed in considering the theory of gravity and its consequences with regard to the celestial motions," mentioning at the same time what he suspected to be the cause of these phenomena.

Machin confirmed this supposition, and shewed that the observed motions might be very nearly accounted for, by supposing that the pole of the equator described a small circle about its mean position as centre, during a period of the Moon's nodes.

Bradley remarked that his observations would be more completely represented by supposing the true pole to move about the mean pole in an ellipse instead of in a circle, the major axis being in the solstitial colure; and this conclusion is perfectly true, the minor axis being, however, a little smaller than he made it.

Bradley continued the observations during an entire revolution of the Moon's nodes, and then published an account of his discovery in the *Philosophical Transactions* for 1748, in a paper which is a perfect model of lucid statement and strict inductive reasoning.

In the following year, D'Alembert succeeded in determining the true motion of the Earth's axis by means of analysis, in his "Recherches sur la Précession des Equinoxes et sur la Nutation de l'Axe de la Terre," and since that time the subject has been repeatedly treated of by physical astronomers. The most complete and elegant theoretical investigation, however, of the motion of the Earth about its centre of gravity is that given by Poisson in the seventh volume of the *Mémoires de l'Institut.* The theoretical investigations with respect to nutation leave nothing to be determined by observation, except the value of one constant. This is generally chosen to be the coefficient of the principal inequality in the obliquity of the ecliptic. The accurate determination of this constant is important, not only from its being required for the reduction of star observations, but also from its affording one of the best means we have of determining the mass of the Moon.

In precession we see the effect of the joint action of the Sun and Moon, but by means of the observed quantity of nutation, we can ascertain what part of this is due to the Moon's action, and having thus obtained the ratio between the actions of the Sun and Moon, the Moon's mass easily follows.

The most trustworthy determinations of the constant of nutation, previous to this of M. Peters, are those of MM. Von Lindenau, Brinkley, Robinson, and Busch; and M. Peters begins his memoir with a critical examination of their labours.

The results of the three latter astronomers present an admirable agreement, while that of Von Lindenau differs from them by about a quarter of a second. Von Lindenau employed about 800 observations of right ascension of *Polaris*, made at different observatories, and therefore his result is liable to be vitiated by the different personal equations of the several observers. We shall find in the sequel that this remark is important.

Brinkley deduced his value of the constant from 1618 observations of ten stars, made about the times of two opposite maxima of nutation in declination with the Dublin meridian circle, the proper motions of the stars being determined by the comparison of his own declinations with those in the *Fundamenta*. As these observations embrace only half a period of the Moon's nodes, the result is liable to be affected by errors in the supposed proper motions.

Dr Robinson's investigation is contained in the eleventh volume of the *Memoirs* of the Royal Astronomical Society. He employs the declinations of the polar star, and of fourteen others observed at Greenwich between the years 1812 and 1835 with Troughton's mural circle. There can be no doubt of the high value of this investigation, but M. Peters thinks that, in consequence of the way in which the error of collimation is determined, errors of observation may exist with a yearly period, and that these may slightly affect the resulting value of nutation. Baily's coefficient of aberration is employed, the annual parallaxes of the stars are neglected, and the equations of condition are not treated by the method of least squares.

M. Busch has deduced the constant of nutation from Bradley's observations at Kew and Wansted. The reductions are made in the most strict manner, except that the annual parallaxes are neglected, and M. Peters regards the result as worthy of the highest confidence.

M. Peters then enters upon his own investigations, which are based on 603 right ascensions of *Polaris*, observed at Dorpat between 1822 and 1838, with Reichenbach and Ertel's meridian circle. Of these observations, the first 249 were made by M. Struve, and the remaining 354 by M. Preuss. These are compared with the right ascensions deduced from the *Tabulæ Regiomontanæ*, and the equations of condition thence arising are treated by the method of least squares, taking as the unknown quantities the correction of the constant of nutation, the correction of the constant of aberration, the annual parallax, the corrections for the position of the axis of the transit-circle (illuminated pivot east or west), the correction of the star's right ascension, and the personal equation of the two observers.

The equations are first solved, giving equal weight to all the observations. The observations are then divided into two groups (one for each observer), and the equations of each group are solved separately. There is a surprising agreement between the results found from the four years' observations of M. Struve, and the twelve years' observations of M. Preuss, the coefficients of nutation deduced differing by less than three-hundredths of a second. This investigation supplies a measure of the precision of the separate observations, and it is found that M. Struve's observations are entitled to greater weight than those of M. Preuss.

The whole of the observations are then combined, giving the proper relative weights just obtained, and the equations are re-solved. The values found for the unknown quantities differ extremely little from the results given by the supposition of equal weights.

One of the most striking results is the constant difference between the right ascension given by the two observers, or the personal equation, which amounts, for *Polaris*, to more than 0·8 of a second of time. The magnitude of this shews that the personal equation changes with the declination of the stars. Hence, also, we may easily understand that M. Lindenau's results may be vitiated by the omission of the consideration of personal equation, especially as the observations which he employed were made with different instruments, as well as by different observers.

While M. Peters was employed in these investigations, M. Lundahl was likewise engaged in discussing the observations of declination of the same star, made also at Dorpat within the same space of time. The value of the constant of nutation which he deduces agrees admirably with those found by MM. Peters and Busch.

Finally, M. Peters takes the mean of the three results, giving the proper relative weights to the several determinations, and he finds the most probable value of the constant to be $9''\cdot2231$, with the probable error $0''\cdot0154$. This value differs very little from Brinkley's, which has generally been employed by English astronomers, but M. Peters' determination undoubtedly possesses much greater weight.

M. Peters next enters upon a theoretical investigation of nutation, far more complete than any that had before appeared. Starting from the equations of Poisson's theory, he develops them, taking into account the ellipticities of the orbits of the Earth and Moon, and also the principal lunar inequalities. He thus obtains a great number of small terms which had previously been neglected. Most of these may be safely omitted; but there are two terms which should be taken into account in delicate investigations, as they have an annual period, and are therefore mixed up with the effect of aberration and parallax. M. Peters takes care to apply the requisite corrections to the coefficients of aberration, and to the parallax of *Polaris* given by his investigations. Although most of the new terms found by M. Peters are very small, yet these researches are not the less valuable, since it is always satisfactory to know what we really neglect.

M. Peters takes into account the effect of a possible difference between the ellipticities of the two hemispheres, which he determines by means of the pendulum experiments collected by Mr Baily in his "Report on the Experiments made by Foster," in the seventh volume of the *Memoirs* of the Royal Astronomical Society. It fortunately happens that this effect is insensible, as this difference of the two hemispheres is extremely doubtful.

The last part of M. Peters' paper contains researches on the obliquity of the ecliptic and the precession of the equinoxes, so that he treats of all the elements which relate to the apparent changes in the places of the stars, due to the motion of the pole of the Earth. He deduces the secular diminution of the obliquity of the ecliptic by comparing the obliquity for 1757, given by Bradley's observations, with that for 1825 given by the observations at Dorpat, both being reduced to the mean by the new value of nutation. The rate of the diminution so found agrees very well with that found by M. Le Verrier from theory, the difference not amounting to one second in a century. The true value of the obliquity of the ecliptic at a given epoch cannot, however, be considered as definitively settled, in consequence of the puzzling constant differences between the declinations determined at different observatories. For instance, the obliquity given by the mean of several years' observations at Greenwich exceeds by rather more than one second the obliquity for the same epoch given by M. Peters' investigations.

M. Peters' researches respecting precession are based on the results of M. Otto Struve's paper, which obtained our medal on a former occasion, combined with M. Le Verrier's determination of the secular change in the position of the ecliptic.

M. Otto Struve determines, independently, by observation, the values of two constants on which the precessions in right ascension and declination depend. Now, theory establishes a relation between these constants, and M. Peters is thereby enabled to find the most probable values which result from the combination of the observed values, and thence to derive complete formulæ for precession applicable to any given epoch.

I have no hesitation in regarding M. Peters' results, with respect both to precession and nutation, as definitive for the present state of astronomy.

I now come to M. Peters' second paper, which relates to the delicate subject of the parallax of the fixed stars.

The first part of this important paper contains an historical and critical review of the researches of astronomers respecting parallax from the time of Tycho to the year 1842. The second treats of the parallaxes of several stars as determined by M. Peters' own observations, made at Pulkowa by means of the great vertical circle of Ertel. In the third part, the results of the two former are applied to determine the mean parallax of stars of the second magnitude.

The historical part is drawn up with great care, and contains many curious and interesting discussions on particular points. For instance, M. Peters shews that the coefficient of aberration may be obtained with great accuracy from Flamsteed's observations of the zenith distance of the pole-star. The probable error of a single observation is found to be only 6″, which gives a far higher idea of the accuracy of Flamsteed's observations than has been generally entertained. Bradley himself remarked, that Flamsteed's observations of the pole-star agreed with his theory of aberration.

The celebrated controversy between Brinkley and Pond is discussed at considerable length, and the labours of the latter astronomer are criticised with great severity. M. Peters considers that Brinkley was far superior to his opponent in his knowledge of the theory of his instruments, and in the use of precautions to avoid error, though it is certain that Pond was the more correct in his conclusions respecting parallax.

The parallaxes determined by M. Struve at Dorpat, from 1818 to 1821, by means of observed differences of right ascension of circumpolar stars having nearly opposite right ascension, deservedly occupy a good deal of attention. The parallaxes thus found, though small, were almost all positive, and M. Peters confirms their reality by the following ingenious consideration. He shews that any diurnal variation of the instrument due to temperature will affect the coefficients of aberration and parallax in the same direction, and the former probably more than the latter. Now, the coefficient of aberration found from these observations is about $0''\cdot 08$ less than the definitive value given by the Pulkowa observations, and it is therefore probable that M. Struve's parallaxes should be increased by a few hundredths of a second.

It is unnecessary for me to follow M. Peters in his account of Struve's micrometrical measurements of the parallax of *α Lyræ*, of Bessel's well-known observations of 61 *Cygni* with the heliometer, and of the parallaxes of *α Centauri* and *Sirius*, as determined by MM. Henderson and Maclear at the Cape, as these have been fully discussed by Mr Main in an able paper in the twelfth volume of our *Memoirs*. The Council is also indebted to Mr Main for a careful report on M. Peters' paper, from which I have derived considerable assistance in drawing up my account of it.

The second and most important part of M. Peters' paper consists of an investigation of the parallaxes of eight stars, by means of observations of zenith distance made by M. Peters at Pulkowa, in 1842 and 1843, with Ertel's great vertical circle. The stars selected are *Polaris*, *Capella*, *ι Ursæ Majoris*, *Groombridge* 1830, *Arcturus*, *α Lyræ*, *α Cygni*, and 61 *Cygni*.

The utmost care is taken in the instrumental adjustments, in the equalisation of the interior and exterior temperatures, and in eliminating every imaginable source of error.

It would be impossible for me to convey an adequate idea to any one, unacquainted with M. Peters' paper, of the numerous precautions used by him for this purpose. For instance, the observations are made by placing the wire very near the star, and then waiting for the time when the star is exactly bisected by it. The large motions of the instrument are always made without touching either the telescope or the divided circle, or the pieces carrying the microscopes. In making the double observation (face East and face West) the micrometer-screw is always turned finally in the same direction, the reading of the levels is always commenced at the same end of the scale (though they are protected from heat by glasses). The effect of flexure of the telescope-tube is eliminated by an important arrangement, by which the eye-piece and object-glass are capable of being fixed at pleasure at either end of the tube. This transposition was made after every eight complete observations of the Sun.

At every observation the readings of the microscopes are taken for coincidence with both the preceding and succeeding divisions on the limb, and the utmost pains are employed to correct for any inequality in the micrometer-screw and for errors of division.

Again, in the reduction of the observations and the elimination of the unknown quantities, the same attention to minute accuracy is observable. Thus, small terms are introduced into the expressions for aberration and nutation which had hitherto been neglected, and an elaborate investigation is entered into respecting the proper motions of the stars observed. The unknown quantities to be determined are the correction to the assumed latitude, the flexure of the telescope-tube, the correction of the thermometrical coefficient of refraction, the correction of the assumed mean declination, the annual parallax, and the correction of the coefficient of aberration. Of these, the first three are found by means of the observations of the pole-star. All the equations are solved by the method of least squares, and the greatest care is used in estimating the probable errors of all the results, whether arising from probable errors of observation or uncertainty in the elements employed in the calculation.

There are also discussions on some curious points, such as the effect of clouds on refraction, the possible variability of latitude, &c. The resulting values for parallax are all positive, with the exception of that of α *Cygni*, which comes out a minute negative quantity; this, of course, only indicates that the real parallax of that star is probably extremely small.

The constant of aberration obtained by taking the mean of the several results for the different stars is $20''\cdot481$, which differs only $0''\cdot036$ from the definitive value found by M. Struve. The smallness of this difference gives great confidence as to the accuracy of the results for parallax, as there is no reason why the aberration should be found more accurately than the parallax.

Another strong confirmation is afforded by the fact, that the parallax of 61 *Cygni* determined by M. Peters is absolutely identical with that found by Bessel by means of the heliometer.

The last part of M. Peters' paper treats of the mean value of the parallax of stars of the second magnitude. M. Peters finds that there are thirty-five stars whose parallaxes are determined with sufficient accuracy to serve as a basis in this research. Of these, however, he excludes two stars which have very large proper motions, 61 *Cygni* and 1830 *Groombridge*, as exceptional, and therefore not properly to be included when an average is the quantity sought. Struve's scale of relative distances of stars of different magnitudes is employed in combining the observed parallaxes for different stars, although the final result is nearly independent of the assumed scale, inasmuch as the second magnitude is nearly the mean of all the magnitudes of the stars employed.

M. Peters shews his usual skill in estimating the probable errors which may arise from the defects of the hypotheses employed, such as that of the same absolute brightness of the stars, as well as from the errors of the observed parallaxes; and he finally arrives at the result, that the most probable value of the mean parallax of stars of the second magnitude is $0''\cdot116$, and that the probable error of this determination is only $0''\cdot014$.

M. Peters closes his paper with a most interesting result, deduced by combining his own researches with those of M. Otto Struve respecting the solar motion. M. Otto Struve finds that the annual apparent motion of the Sun, as seen at right angles from a point at the mean distance of stars of the first magnitude, is $0''\cdot339$. Now, according to M. Peters, the mean parallax of a star of the first magnitude is $0''\cdot209$; so that we are able to turn the former result into absolute measure. Thus the annual motion of the Sun with respect to the great body of the surrounding stars is equal to $1\cdot623$ times the radius of the Earth's orbit.

I cannot but regard this work of M. Peters as a perfect model of excellence, evincing consummate skill in the observer, as well as admirable power of turning the observations to the best account. It shews that it is possible by meridional observations to obtain absolute parallaxes almost as small as the relative parallaxes that can be measured by the heliometer, or by similar means; though to do so requires a most rare union of instrumental advantages, care and judgment in the observer, and analytical skill in combining in the best manner the results of observation.

No one can read the papers of M. Peters, or those of the Russian and German astronomers generally, without being struck with the constant employment of the method of least squares. It is to be wished that this method were more in use among English astronomers, as I believe not a little of the precision of modern determinations is due to it. We seem to entertain a distrust respecting the results of the calculus of probabilities, more particularly with regard to the estimation which it affords of the probable amount of error in any determination.

It should be borne in mind, that when we speak of the probable error being of a certain amount, it is not meant that it is improbable that the error should exceed that amount, but only that it is as probable *à priori* that the error falls short of, as that it exceeds it. If we know by independent means that the error of any determination is much greater than the probable error given by the observations, we may infer, with great probability, that some constant cause of error has occurred in the observations employed. In the estimation of probable error, only fortuitous causes of error are taken into account. The employment of the method of least squares does not render it less necessary to avoid all sources of constant error: it is not a substitute for, but an auxiliary to good observations, and enables us to obtain from them all that they are capable of yielding.

I cannot conclude without congratulating the Society on the improved prospects of that very delicate branch of astronomy which relates to the research of stellar parallax, especially as there is every reason to believe that this country will contribute its full share to the advancement of it. We may hope that the beautiful reflex zenith telescope of the Astronomer Royal, the magnificent heliometer which is in the able hands of Mr Johnson, and the improved method of recording star transits by means of galvanism, will enable us ere long to take many firm, though long-reaching, steps into regions of space hitherto untrodden.

(*The President then, delivering the Medal to Mr Hind, Foreign Secretary, addressed him in the following terms*):—

In transmitting this medal to M. Peters, you will assure him of our high appreciation of the importance of the results at which he has arrived, and of the admirable science and skill which he has shewn in obtaining them; and you will express our confident hope, that in his new sphere at Königsberg he will confirm and add to the reputation which he has so deservedly acquired at the Observatory of Pulkowa.

43.

ADDRESS ON PRESENTING THE GOLD MEDAL OF THE ROYAL ASTRONOMICAL SOCIETY TO MR HIND.

[From the *Memoirs of the Royal Astronomical Society.* Vol. XXII. (1853).]

GENTLEMEN,—You have heard from the Report which has just been read how much reason we have to congratulate ourselves on the present state and future prospects of our science. Never was there a time when greater vigour and activity were exhibited in the promotion of it. Nor is this activity confined to one country, or devoted merely to one department of astronomy. Whether we regard the introduction of improved instruments and methods of observation, or the more rigorous discussion to which the observations are submitted, the formation of extensive catalogues of stars, the discovery of new members of our planetary system, or the closer and more systematic scrutiny and examination of those which are already known, in every direction we find the most satisfactory evidences of progress.

One of the most prominent features of astronomical discovery for several years past, has been the continual addition of new members to the remarkable group of small planets between the orbits of *Mars* and *Jupiter*, and the year just ended has been distinguished beyond all precedent in this respect.

Since our last anniversary meeting no fewer than eight of these bodies have been brought to light, and the supply seems to be inexhaustible. New discoverers have made their appearance on the field, while those who have already distinguished themselves seem to have acquired a new aptitude in the search.

It is gratifying to find that one of our own body has been the very foremost in this noble career of discovery; and to him, in testimony of our appreciation of his well-directed and successful labours, the Council has awarded the medal, which it is my pleasing duty this day to present.

Skilfully using the excellent instrumental means placed at his disposal by the enlightened liberality and scientific zeal of Mr Bishop, and in spite of the interruptions occasioned by a climate, the disadvantages of which are peculiarly felt in researches of this nature, Mr Hind has added no fewer than eight planets to our system, four of which have been found in the course of the past year. After this, I feel that it is unnecessary to add another word in justification of the award of your medal. Mr Hind's discoveries are of a nature to be understood and appreciated by all; and I shall, therefore, confine myself to a very brief notice of some circumstances connected with them, and to a few remarks on the conclusions to which they seem to point, respecting the constitution of our planetary system.

The first five of Mr Hind's planets were found by comparing the heavens with the excellent and well-known star-maps of the Berlin Academy. These, however, are limited to 15° on each side of the equator, and therefore do not include the whole of the region about the ecliptic, which it is so desirable to examine; neither do they contain stars smaller than between the ninth and tenth magnitudes.

Mr Bishop, therefore, very soon determined to intrust to Mr Hind the formation of a series of ecliptic charts, which should contain all stars down to the eleventh magnitude, which were situate within 3° on each side of the ecliptic. Mr Hind has already begun to reap the fruits of these labours, the planet *Fortuna* having been detected in the course of preparing one of the charts, while *Calliope* and *Thalia* were found by the comparison of two of the completed charts with the heavens.

Eight of these valuable charts have now been published, and I understand that most of the remaining ones are considerably advanced. Other astronomers, particularly Mr Cooper of Markree, are engaged in the preparation of charts on a similar plan, and the path of future discoverers cannot fail to be singularly facilitated by their means.

The existence of such a numerous group of small planets in the same part of our system has naturally given rise to much speculation respecting their origin and mutual relations. When, instead of the single planet which was expected to fill up the gap between the orbits of *Mars* and *Jupiter*, *Ceres* and *Pallas* were found at very nearly the same mean distance from the sun, Olbers threw out the conjecture that they were fragments of a larger planet which had been rent asunder by some internal convulsion, and that many more such fragments probably existed. If this were the case, he reasoned, they would all, after longer or shorter periods, again pass through the point where the explosion took place, and though the perturbations which they would suffer, would, in the course of time, prevent them from continuing to pass exactly through the same point, yet it might be expected that they would not stray far from it, and that, therefore, the remaining fragments might be found by carefully watching the parts of the heavens corresponding to the two points in which the orbits of *Ceres* and *Pallas* approached towards intersecting.

Although the finding of *Juno* and *Vesta* appeared to give some countenance to this hypothesis, later discoveries have deprived it of much of its plausibility. Several of the orbits are everywhere far distant from each other, and where the contrary is the case, the points of nearest approach occur in various parts of the heavens. Probably one reason why Olbers did not discover more of these bodies, though he continued his examination for many years after detecting *Vesta*, was, that he was induced by his theory to confine the search within too narrow limits.

Several astronomers have endeavoured to find some general relations between the orbits of this group, similar to that imagined by Olbers; but it appears to me that they have only succeeded in shewing a kind of general resemblance, indicating rather that similar causes have operated in determining the orbits of these bodies than that they were originally identical.

If we allow ourselves to speculate on the formation of our planetary system, and adopt the nebular theory, it seems at least as easy to imagine that the nebulous matter, circulating in any particular region about the Sun, would, in cooling, collect into many small masses, as that it would all coalesce into one.

Although, as has been stated, there is no single point through which all the orbits nearly pass, yet many of them, taken two and two, approach very closely to each other. In the case of *Astræa* and *Hygeia*, in particular, the shortest distance between the two orbits is less than $\frac{1}{150}$th part of the Earth's mean distance from the Sun; so that, as M. D'Arrest remarks, the time of their actual intersection cannot be very distant from the present.

One of the most curious circumstances connected with this group is, that there are several cases in which the mean distances are nearly identical with each other. Thus the mean distances of *Ceres* and *Pallas* are so nearly equal that their order of magnitude is sometimes changed by perturbation. The same remark applies to *Iris* and *Metis,* and also to the three planets, *Astræa, Egeria,* and *Irene.*

It should be noticed that this identity of mean distance would not be at all explained by supposing the planets in which it occurs to have been originally one.

There are also some remarkable cases in which the mean motions are nearly commensurable. Thus the mean motions of *Juno* and *Vesta* are very nearly in the ratio of 5 to 6, while those of *Juno* and *Flora* are as 3 to 4, and consequently those of *Vesta* and *Flora* as 9 to 10*.

The extreme smallness of the apparent diameters of these bodies makes it very difficult to determine their real diameters by direct measurement. According to Sir W. Herschel's observations, the diameters of *Ceres* and *Pallas* would not be far from 140 English miles, while Schröter's observations would make them much larger. Stampfer has attempted to determine their diameter by means of their apparent brightness, supposing the reflective power of their surfaces to be the same as that which obtains in the case of *Jupiter, Saturn, Uranus,* and *Neptune.* This supposition is obviously rather precarious, especially as the reflective power of *Mars* is found to be much less than that of the other planets; but Stampfer's result agrees very closely with the above-mentioned determination of Sir W. Herschel. Several of the more recently-discovered planets appear to be much smaller than these; and it is not improbable that there are many more which, by their excessive minuteness, elude our telescopes altogether. In this point of view, these asteroids would seem to form a connecting link between the larger planets and the aerolites, the cosmical nature of which appears to be pretty well established.

* The mean daily sidereal motion of *Juno* is 814″·24; that of *Vesta,* 977″·20; and that of *Flora,* 1086″·08. Also $\frac{6}{5} \times 814{\cdot}24 = 977{\cdot}08$, and $\frac{4}{3} \times 814{\cdot}24 = 1085{\cdot}65$.

To the physical astronomer these bodies offer problems of great interest and difficulty. On account of the large eccentricities and inclinations of some of the orbits, methods of approximation which succeed in determining the perturbations of the older planets become quite inadequate to deal with these, and, consequently, astronomers have hitherto been compelled to have recourse to the method of mechanical quadratures in order to calculate their motions. But although this method may be employed in all cases, and the use of it becomes much simplified by applying it directly to the differential equations of motion, in the elegant manner which has been recently devised by Mr Bond and Professor Encke, yet it only enables us to follow the disturbed planet, as it were, step by step, and it is, therefore, very desirable to have a method by which the course of the planet might be traced through an indefinite number of revolutions, and the results of which might be embodied in tables.

Professor Hansen has attacked this very difficult problem with his characteristic originality and skill, and Sir J. Lubbock has also treated the same subject very ably in his tracts on the perturbations of the planets. Much, however, remains to be done before the application of the method of quadratures to these cases can be superseded. It will be quite indispensable to take into account the square and higher powers of the disturbing force.

It may be remarked, however, that the eccentricities and inclinations of the orbits of several of these new planets are so moderate, that there will be little difficulty in calculating their perturbations by the ordinary methods.

The disturbances which these bodies suffer from the action of *Jupiter* are so large as to afford an excellent means of determining the mass of that planet. It was thus that Nicolai found that the value of this mass which had been employed by Laplace and Bouvard was considerably too small,—a result which Mr Airy afterwards confirmed by direct measures of the elongations of the satellites. Considering the great degree of proximity to each other, to which these bodies sometimes attain, it does not seem improbable, notwithstanding their minuteness, that they may occasionally produce a sensible effect on each other's motions; in which case the astronomer would be able to weigh these minute atoms in the same balance which he has already applied to the larger bodies of our system.

In examining the heavens in search of small planets, Mr Hind has naturally been led to pay great attention to the variable stars, and he has consequently detected a considerable number of these objects among the smaller stars. Two of these I will mention, which are at opposite extremities of the scale, and which seem to imply the operation of totally different causes.

The first is that remarkable new star in *Ophiuchus* which Mr Hind noticed on the 27th of April, 1848, as being of the 6th magnitude, and occurring in a spot where he was certain no star even of the 9—10th magnitude had been visible three weeks before. After attaining to the 4—5th magnitude, so as to be conspicuous to the naked eye, it gradually faded away, and at present it is only of the 11th magnitude.

The other star to which I will refer appears to vary in a similar way to *Algol*. Its period, according to Argelander, is about $9^{d}\,11\frac{1}{2}^{h}$, but for 9 days of this time it shines as a star of the 8th magnitude, then suddenly descends to the 10—11th, and as quickly returns again to the 8th.

Variations of this latter kind appear to be most naturally accounted for by the periodical interposition of an opaque body in its revolution about the star, but those of the kind first mentioned seem to mock all our attempts at explanation.

In recording these discoveries, it is doubly gratifying to recollect that they emanate from an observatory founded and maintained by a private individual out of pure love of the science and zeal for its advancement. Of the judgment which Mr Bishop has shewn in the selection of his observers, and the choice of objects of observation, there can be no better proof than is afforded by the admirable double-star observations of Mr Dawes and the planetary discoveries of which we have just been speaking. Mr Bishop may well feel proud in the consciousness that his observatory has been the means of contributing so largely to science, and has thus become known wherever astronomy is cultivated.

Another subject of congratulation is the manner in which Mr Hind's services to science have been recognised by the Government of the country. It is sometimes asked, whether the progress of science is best promoted by private or by public means; but the truth is, that there is no such opposition between these modes of advancing it as is implied in the form of the question. In a country where the dignity of science, and the benefits which it confers, are properly estimated, both Government and people will harmoniously co-operate in its support, and each will easily find its appropriate sphere of action. Surely few objects can be mentioned more truly national in their character than the encouragement and reward of scientific discoveries, which at the same time reflect honour on the country, and give so powerful an impulse to the intellectual advancement of the people.

(*The President then, delivering the Medal to Mr Hind, addressed him in the following terms*):—

Mr Hind,—It is with peculiar pleasure that I present you with this Medal, in testimony of our appreciation of your eminent services to astronomy. The whole world will acknowledge how nobly it has been earned, and will join with us in the wish that your health may long be spared, and that thus you may be able to make many more additions to our knowledge in that field of science to which you have devoted yourself with so much energy and success.

44.

ADDRESS ON PRESENTING THE GOLD MEDAL OF THE ROYAL ASTRONOMICAL SOCIETY TO M. CHARLES DELAUNAY.

[From the *Monthly Notices of the Royal Astronomical Society.* Vol. xxx. (1870).]

GENTLEMEN,—It has been announced to you that the Society's Medal has been awarded to M. Ch. Delaunay for his great work on the Theory of the Moon.

The illness of our excellent President having made it impossible for him to be present on this occasion, the Council have done me the honour to request that I would occupy the chair, and in his stead lay before you the grounds of their award. I have acceded to their wishes with the more readiness because I have given some attention to special branches of the Lunar Theory, and my study of M. Delaunay's work has led me to form the highest opinion of its merits.

Of all the problems presented to us by physical astronomy none has so much engaged the attention of mathematicians as that of the determination of the motion of our satellite. The theoretical interest as well as the great practical importance of the results, has proved an irresistible attraction, and the mathematical difficulties have merely acted as a stimulus to the invention of various methods of surmounting them. It is fortunate that this has been the case, as the excessive labour involved in any theory of the Moon approaching to completeness, might otherwise have proved too great for human perseverance. The foundations of the theory were laid by

Newton in his *Principia*; and although his investigations are only fragmentary, being simply intended to shew how some of the leading lunar inequalities may be deduced from theory, yet they form one of the most admirable portions of that immortal work. Towards the middle of the eighteenth century the theory was more systematically entered upon by Clairaut, D'Alembert, and Euler, who severally shewed that the theory was competent to give very approximate values of all the inequalities which were then recognised by observation.

Still the theory was far from being sufficiently perfect to serve as a foundation for lunar tables accurate enough for the uses of navigation. This degree of accuracy was first attained by the tables of Mayer, who not only carried the approximations to the values of the coefficients of the various lunar inequalities further than his predecessors had done, but also corrected the theoretical coefficients thus obtained by comparison with his own observations. The theory was greatly advanced by Laplace, not only by his more accurate theoretical determination of the coefficients, but also by several important discoveries, especially that of the cause of the Moon's secular acceleration.

The improvements in the lunar tables, however, which were made successively by Bürg and Burckhardt, were founded, not on theory, but on comparison of the former tables with observations; and the empirical tables thus produced were far more accurate than any that could have been formed at that time by theory alone. Dissatisfied with this state of things, and wishing to see astronomy founded exclusively on the law of attraction, only borrowing from observation the necessary data, Laplace induced the Academy of Sciences to propose for the subject of the mathematical prize which it was to award in 1820 the formation, by theory alone, of lunar tables as exact as those which had been constructed by theory and observation combined. The prize was divided between two memoirs—one by M. Damoiseau, the other being the joint production of MM. Plana and Carlini. Damoiseau's memoir is printed in the third volume of the *Recueil des Savants Etrangers*. Plana's great work on the lunar theory, which appeared in 1832, is the development of the joint memoir by himself and Carlini. By these important works an immense advance was made in the theory, the approximations being carried to such an extent that the resulting coefficients were comparable in accuracy with those given by observation. In 1824 Damoiseau published tables founded entirely on his theory, which were found to be quite as exact as those of Burckhardt.

Both Damoiseau and Plana, following the example of Laplace, start from differential equations in which the Moon's longitude is taken as the independent variable; and after the equations have been integrated, they obtain the values of the Moon's coordinates in terms of the time by reversion of series. An important innovation, however, was introduced by Plana in the mode of conducting the investigation and exhibiting the results. The values of the Moon's coordinates being developed in series of sines and cosines of angles which vary uniformly with the time, the coefficients of the several terms of these series will depend on the eccentricities of the orbits of the Sun and Moon, the inclination of the Moon's orbit to the plane of the ecliptic, the ratio of the mean motions of the Sun and Moon, and the ratio of their mean distances from the Earth. Now Damoiseau, in common with all previous writers, having assumed certain values of the quantities just mentioned as given by observation, contented himself with determining the numerical values of the coefficients. Although this is all that is required for the construction of tables, yet, from a theoretical point of view, it leaves the mind unsatisfied, inasmuch as any coefficient in its numerical form shews no trace of its composition, that is of the manner in which its value depends on the value of the assumed elements. The several coefficients are far too complicated functions of the elements to be represented analytically, except in the form of infinite series, and Plana, accordingly, developes these coefficients in such series, proceeding by powers and products of the eccentricities, the tangent of the inclination, the ratio of the Sun's mean motion to that of the Moon, and the ratio of the Moon's mean distance to that of the Sun, all these quantities being assumed to be small, and the last mentioned ratio, which is much smaller than the others, being considered as a quantity of the second order.

In this mode of development, the numerical factor which enters into any term of the coefficient of any of the lunar inequalities is an ordinary fraction which admits of being determined not merely approximately, but with absolute accuracy. It is easy to see what great facilities are afforded by this circumstance for the verification of the work by a comparison of the results obtained by different methods. The greater or less degree of approximation will thus depend on the greater or less number of terms taken into account in the several series.

The numerical values of the several elements are not substituted in the formulæ until the work is completed, and this is attended with the important advantage that when a comparison of the theory with observation has supplied more accurate values of the elements, their corrected values can be at once substituted in the same formulæ, without requiring any additional work.

On the other hand, if the numerical values of the elements be introduced into the calculations from the first, then if it is desired to introduce corrected values of the elements, much additional investigation will be required for the purpose.

No doubt the labour required in order to obtain a given amount of numerical accuracy by this method is very much greater than is required when each coefficient, instead of consisting of a series of terms, is reduced to a simple numerical quantity, but the great theoretical advantage of knowing the composition of every coefficient in terms of the elements well repays the additional labour.

The degree of convergence of the series obtained for the several coefficients is in general sufficiently rapid, but in some few of the coefficients, on the contrary, the convergence is so slow, at least in the leading terms, that it is necessary to take into account terms which are analytically of a higher order than those to which the approximation is in general limited.

Thus Plana, who proposed to himself to determine the lunar inequalities completely to the fifth order, found it necessary in special cases to carry the approximation to the seventh and even to the eighth order, and in several cases he also added an estimated value of the remainder of the series founded on the observed law of diminution of the calculated terms.

Soon after the publication of Plana's great work, Sir John Lubbock formed the plan, which he partly carried out in his various tracts on the theory of the Moon, of verifying Plana's results by a totally different method, starting from differential equations in which the time is taken as the independent variable, and thus avoiding the necessity of reversion of series.

Later, M. de Pontécoulant undertook the same work on a similar plan, and carried it out more completely in the fourth volume of his *Théorie Analytique de Système du Monde.*

These works, while they corrected some errors which had crept into Plana's computations, confirmed their wonderful general accuracy, and with some few exceptions they do not extend the approximation beyond the order to which Plana restricts himself.

Meantime, M. Hansen had undertaken a completely new investigation of the lunar theory, by a remarkable method peculiar to himself and explained in his *Fundamenta nova investigationis orbitæ veræ quam Luna perlustrat*, which appeared in 1838.

In applying the method described in this work to the case of the Moon, M. Hansen throughout employs numerical values of the elements of the Moon's orbit, and consequently the coefficients of the lunar inequalities as obtained by him are also purely numerical. The process is one of successive approximations, which are repeated again and again until the values of the inequalities which are found from the last approximation sensibly coincide with those which were assumed in entering upon that approximation.

The numerical values of the coefficients thus finally obtained are undoubtedly very exact. The slight corrections which these coefficients still require are probably chiefly due to the small corrections required by the numerical elements on which the calculations are based, and in the method employed no provision is made for taking into account the effect of these corrections.

From his formulæ, M. Hansen constructed tables of the Moon, which were published in 1857, at the expense of the British Government; and these tables, having been found far superior in accuracy to all others, are now exclusively employed in the calculation of ephemerides.

A detailed account of the calculations leading to M. Hansen's last approximation, was given by him in the two parts of his *Darlegung der Theoretischen Berechnung der in den Mondtafeln angewandten Störungen*, which severally appeared in 1862 and 1864.

After the great works, to which we have thus briefly referred, had been either completed or were in progress, it might have been supposed that the matter was exhausted.

Our Associate M. Delaunay, however, was not of this opinion. Having devised, so long ago as 1846, a perfectly original and singularly beautiful method of integrating the differential equations of the Moon's motion, he determined to apply this method to the complete re-investigation of the theory, and to carry on the approximation to a much greater extent than had been done by his predecessors. The principal fruits of his labours, to which he has devoted himself with almost unexampled perseverance for so many years, are contained in the magnificent volumes which the Imperial

Academy of Sciences have done both M. Delaunay and themselves the honour of publishing among the volumes of their *Memoirs.* It is for this great work that your Council have awarded to M. Delaunay the Society's medal.

Strongly impressed with the advantages of determining the coefficients of the lunar inequalities in the analytical form, both as affording a solution more complete in itself and more satisfactory to the mind, as well as one offering facilities for the comparison of the results of different investigations, M. Delaunay did not hesitate to follow the example set in this respect by M. Plana, notwithstanding the immense length of the necessary calculations. M. Delaunay's results are thus obtained in a form which makes them directly comparable with those of M. Plana, while the methods employed in obtaining them are wholly different.

M. Delaunay chooses the *time* as the independent variable, and takes as his starting-point the differential equations furnished by the theory of the variation of the arbitrary constants. In an able Memoir which appeared in 1833, Poisson had advocated the employment of these equations in the theory of the Moon's motion, and he applied them to the discussion of some special points of that theory. These equations had been long used, almost exclusively, for the determination of the perturbations of the planets, and they offer peculiar advantages in the treatment of the secular inequalities and those of long period. In the case of the Moon, however, in consequence of the large perturbations caused by the disturbing force of the Sun, the ordinary mode of integrating these equations by successive approximations soon leads to calculations of inextricable complexity. In fact, these equations give the differential coefficients of the several elliptic elements taken with respect to the time, in terms of the elements themselves. In the case of the planets, where the disturbing forces are so small compared with the predominant central force of the Sun, very approximate values of the disturbed elements may be found by substituting in the values of the differential coefficients, the undisturbed instead of the disturbed values of the elements, and then integrating.

The perturbations of the elements thus found are said to be due to the first power of the disturbing force. If now the approximate values of the disturbed elements be substituted in the differential equations, and these be again integrated, we shall obtain a second approximation to the values of the disturbed elements, and the additional terms thus found are said to depend on the square of the disturbing force. In the theories of the planets it is only in special cases that terms depending on the square of the disturbing force need be taken into account, and it is scarcely ever necessary to consider terms of the next order of approximation.

In the case of the Moon, however, it would be necessary to repeat the process of approximation at least four or five times, in order to obtain results of the accuracy required in the present state of the theory. If we consider that the disturbing function consists of a great number of terms, and that each term gives rise to a corresponding term in the value of each of the disturbed elements, while powers and products of the corrections of all the elements in every possible combination, up to a certain order, have to be taken into account, it may be readily imagined how impracticable it would be by such a process to carry on the approximation to a greater extent than has been already done by Plana. Every process in which the approximations require to be repeated several times, is subject to the inconveniences that have been described, and these inconveniences are much greater when, as in the present case, we have to make successive approximations to the values of the *six* elements of the orbit, instead of to the values of the *three* coordinates of the Moon.

It was with the view of avoiding this excessive complication of the method of successive approximations that M. Delaunay devised his method of integrating the differential equations of the Moon's motion. The fundamental idea of this method consists in attacking the difficulty by small portions at a time, and in replacing these extremely complicated successive approximations by a much greater number of distinct operations, each of which is comparatively simple, so that it may be carried out to any degree of exactness that may be desirable, while the mind is relieved by being able readily to embrace the whole of each operation in one view.

It is difficult, without the use of algebraical symbols to give an idea of M. Delaunay's beautiful method, but I must endeavour, in some measure, to fulfil this task, and I must crave your indulgence should I fail in the attempt.

The theory of the variation of the arbitrary constants gives, as is well known, the differential coefficients of the elliptic elements with respect to the time, in terms of the elements themselves and the partial differential coefficients of a certain function, called the Disturbing Function, taken with respect to those elements. By a proper choice of elements, the differential equations may be reduced to their simplest, or to what is called their *canonical* form. In this form the six elements are divided into three pairs, the elements of each pair being conjugate to each other. Then the differential coefficient of any element with respect to the time is simply equal to the partial differential coefficient of the disturbing function taken with respect to the element which is conjugate to the former, the partial differential coefficients which occur in the two equations corresponding to a pair of conjugate elements being affected with opposite signs.

The disturbing function may be readily developed in a series of periodic terms involving cosines of angles, each of which is formed by the combination of multiples of the Moon's mean longitude, the distance of the Moon's perigee from its node, and the longitude of the node, together with angles which depend on the position of the disturbing bodies. The disturbing function likewise contains a non-periodic term, which, as well as the coefficients of the periodic terms, are all functions of the major semi-axis, the eccentricity and the inclination of the Moon's orbit.

Since the mean longitude of the Moon involves the time multiplied by the mean motion which is a function of one of the elements, it is obvious that the differentiation with respect to this element will give rise to terms in which the time occurs without its being included under a sine or a cosine. Such terms would render the equations very inconvenient for the determination of the lunar inequalities; and M. Delaunay accordingly avoids the introduction of them by taking the mean longitude itself instead of the *epoch* of mean longitude, as one of his elements, while by the simple yet novel expedient of adding to the disturbing function a non-periodic term which is a function of the major semi-axis alone and is independent of the disturbing forces, he preserves to the differential equations the same very simple form which they had at first. After this modification of the disturbing function, the time no longer enters into it explicitly except in so far as it is introduced by the values of the coordinates of the disturbing bodies, and consequently the difficulty which was before met with completely disappears.

The six elements employed by M. Delaunay are thus,—the Moon's mean longitude, the distance of the perigee of its orbit from the node, and the longitude of the node, which for distinction may be called the three *angular* elements, and three other elements which are respectively conjugate to the former, and which are determinate functions of the major semi-axis, the eccentricity and the inclination of the orbit.

The three coordinates of the Moon at any time are given in terms of the three angular elements and of the quantities last mentioned.

Now let us imagine, for a moment, that the disturbing function contained no periodic terms, but was reduced simply to its non-periodic part. Consequently the partial differential coefficients taken with respect to the angular elements would all vanish, and therefore the three conjugate elements would be all constant, as well as the major semi-axis, the eccentricity and inclination, of which those elements are functions. Hence, again, the partial differential coefficients taken with respect to the conjugate elements would be functions of those elements, and would therefore be constant. Hence each of the angular elements would consist of an arbitrary constant and a term proportional to the time, the multiplier of the time in each case being a known function of the three constant elements.

The object of M. Delaunay's method is, by means of a series of changes of the variables, to cause all the more important periodic terms to disappear from the disturbing function, one by one, while the differential equations continue to retain their canonical form, so that after each transformation we approach more nearly to the conditions of the ideal case which has just been considered.

In order to effect any one of these transformations, M. Delaunay supposes, for the moment, that the disturbing function is reduced to its non-periodic part, together with one of the periodic terms selected from among those which have the greatest influence in producing the lunar inequalities. With this simplified form of the disturbing function, the equations admit of being easily integrated. The elements with which we start may thus be expressed in terms of three new angular elements which vary uniformly with the time, and three new constant elements. M. Delaunay shews how the constant elements may be so chosen that they may be considered as respectively conjugate to the three new angular elements, so that, in fact, the quantities which are multiplied by the time in the expressions of these angular elements are respectively equal to the partial differential coefficients of a function of the new constant elements taken with respect to these elements.

Having thus found the relations between the old set of elements and the new ones by means of the simplified form of the disturbing function, M. Delaunay now restores the complete value of that function, and chooses new elements which are connected with the old ones by exactly the same relations as in the case just considered. Of course the three new angular elements will no longer vary uniformly with the time, and the three elements respectively conjugate to these will no longer be constant.

When, by means of the proper formulæ of transformation, the new variables have been substituted for the old ones in the disturbing function and in the expressions of the Moon's coordinates, M. Delaunay shews that—

1st. One of the important terms of the disturbing function disappears, viz., the periodic term which was selected in the preliminary investigation.

2nd. Various inequalities corresponding to this term are introduced into the values of the three coordinates of the Moon.

3rd. The values of the six new variables in terms of the time are determined by differential equations of exactly the same form as those which determined the values of the six variables for which they have been substituted.

One of the periodic terms having been in this manner caused to disappear from the disturbing function, a new operation of exactly the same kind causes another term of this function to disappear; similarly a third term may be taken away by means of a third operation, and so on to any number of terms.

In this way, after a suitable number of operations of this kind have been effected, the disturbing function will have been simplified by the removal from it of its most important periodic terms, after which the further process of integration becomes simple enough to be treated in the same manner as if we were concerned with the perturbations of a planet or of the Sun.

The whole difficulty in the determination of the lunar inequalities is caused by the great magnitude of the disturbing force of the Sun. M. Delaunay has therefore at first confined his attention to the investigation of the irregularities which are produced by this disturbing force, and the two magnificent volumes before us are entirely occupied with this investigation. Thus he has provisionally left out of consideration the very small inequalities due to some secondary causes, such as the attraction of the planets and the figure of the Earth; and, besides, he has omitted to consider the perturbations of the Sun's apparent motion about the Earth, intending in a supplementary volume to take into account the effects due to these several causes.

By means of repeated applications of the beautiful method of transformation which I have above attempted to describe, M. Delaunay proceeds to get rid of all the periodic terms of the disturbing function due to the Sun's disturbing force, which are capable of producing inequalities in the coordinates of the Moon of an order inferior to the fourth. For this purpose fifty-seven such operations are required to be performed. When these have been effected, the periodic terms which remain in the disturbing function are so small that their powers and products may be neglected, and consequently the differential equations which determine the six elements last introduced in terms of the time, may be integrated at once. Since the values of the Moon's coordinates are known in terms of the elements just mentioned and the time, we have only to substitute the values of the elements that have been found, in order to determine the Moon's coordinates in terms of the time.

The values of the elements, however, that would be found in this way are very complicated, and therefore the substitutions which would be required in order to find the Moon's coordinates would be excessively long. M. Delaunay, accordingly, prefers to get rid of the remaining periodic terms in the disturbing function, one by one, by means of transformations exactly similar to those which have been already effected. In order to carry on the approximation to the extent which he desires, M. Delaunay finds it necessary to perform no less than 448 of these secondary operations, but each such operation becomes very simple, since the squares of the coefficients of the periodic terms under consideration may be neglected.

Thus, at length, by means of 505 transformations, all the periodic terms of the disturbing function are removed, and the problem is reduced to the ideal case which was considered at the outset of our account of M. Delaunay's method.

After each transformation, by making the proper substitutions in the expressions for the Moon's coordinates, those coordinates are obtained in terms of the system of elements last introduced, so that finally the three coordinates are known in terms of the three final constants and angles which vary uniformly with the time.

It has been already mentioned that Plana, in his great work on the Lunar Theory, determined the analytical values of the coefficients of the lunar inequalities as far as terms of the fifth order inclusive, and that he only carried on the development to a greater extent in cases where the slowness of the convergence of the series appeared to him to render it necessary to take into account terms of higher orders than the fifth.

M. Delaunay has proposed to himself to carry on the approximation so as to include all terms of the seventh order, and in cases where the series converge slowly to take into account terms of the eighth, and even of the ninth order.

Those who have had any experience in calculations of this nature will readily understand how enormously the labour required has been increased by thus adding two orders more to those which Plana has considered. It is not merely that the terms of higher orders are far more numerous than those of the lower, but also that each of the terms of the former kind is much more difficult to calculâte, since it arises from a much greater number of combinations of terms of the inferior orders.

This enormous labour, which has occupied M. Delaunay for nearly twenty years, has been performed by him without assistance from any one. Indeed, from the nature of the calculations which are required, it would not have been easy to obtain any effective assistance. In order to insure accuracy, M. Delaunay has omitted no means of verification, and he has performed all the calculations, without exception, at two separate times, with a sufficient interval between them to prevent any special risk of committing the same error twice in succession.

The volumes before us are perfect models of orderly arrangement. Notwithstanding the great length and complication of the calculations, the whole work is so disposed that any part of it may be specially examined with the utmost readiness by any one who may wish to test its accuracy.

Finally, the analytical expressions which have been obtained for the Moon's coordinates are converted into numbers, by substituting for the elements the most accurate numerical values which the comparison of theory with observation has made known.

Such is an imperfect sketch of M. Delaunay's labours on the Theory of the Moon contained in these two magnificent volumes, the former of which appeared in 1860, and the latter in 1867. As I have already stated, they do not include a complete theory of the Moon, but only that which is by far the most difficult and complicated part of that theory, viz., the investigation of the perturbations due to the direct action of the Sun supposing its apparent motion about the Earth to be purely elliptic. Of the investigations which are required to take into account the remaining very small causes of disturbance, and which are intended by M. Delaunay to be included in a supplementary volume, some of the most important have been already completed by him, particularly the calculation of the

Secular Variation of the Moon's Mean Motion, and the investigation of the long inequalities due to the action of *Venus*.

I understand also that M. Delaunay is engaged in the construction of new Lunar Tables founded upon his theory.

Your Council, however, has decided that we ought not to await the appearance of M. Delaunay's supplementary researches before we mark emphatically our sense of the value of his labours.

The present work is complete in itself; in it the very difficult and complicated problem of determining the Moon's motion is attacked by a perfectly original method, and that one as powerful and beautiful as it is new. The work has been planned with admirable skill and has been carried out with matchless perseverance. The result is an enduring scientific monument of which our age may well be proud, and which we are happy to distinguish, on this occasion of our fiftieth anniversary, with the highest marks of our approval which it is in our power to bestow.

(*The Chairman, then delivering the Medal to M. Delaunay, addressed him in the following terms*):—

M. Delaunay, il ne me reste plus maintenant qu'à vous présenter cette médaille au nom de la Société Royale Astronomique, qui désire par ce tribut vous exprimer la haute appréciation qu'elle a de vos travaux. Notre Président regrette vivement que l'état de sa santé l'empêche de remplir cette tâche agréable. Il m'a prié de le remplacer dans cette circonstance, et je le fais avec d'autant plus de plaisir que depuis bien long-temps j'ai la plus grande estime pour vos hauts talents, et que j'ai étudié vos belles recherches avec la plus grande admiration, aussi je suis heureux de vous exprimer que notre Société vous a suivi dans votre immense travail avec le plus vif intérêt; et quoique ce travail ne soit pas entièrement terminé, elle sent qu'elle ne peut tarder plus long-temps à reconnaître la haute valeur de vos recherches. Nous sommes heureux de vous voir au milieu de nous à cette occasion, et nous faisons des vœux pour que votre santé et vos forces puissent durer de longues années afin d'enrichir la science de plus en plus du fruit de vos grands talents.

45.

ADDRESS ON PRESENTING THE GOLD MEDAL OF THE ROYAL ASTRONOMICAL SOCIETY TO PROFESSOR H. D'ARREST.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. xxxv. (1875).]

IT has been already announced to you that the Council have awarded the Society's Medal to Professor H. L. D'Arrest, Director of the Observatory of Copenhagen, for his Observations of Nebulæ contained in his *Resultate aus Beobachtungen der Nebelflecken und Sternhaufen* and in his later and much more extensive work, *Siderum Nebulosorum Observationes Havnienses*, as well as for his other recent astronomical labours. It now becomes my duty to lay before you the grounds of this award; and I feel confident that a plain statement of the nature and extent of the work accomplished by Professor D'Arrest will be sufficient to convince you that he richly deserves our medal.

Professor D'Arrest has been long well known for his contributions to our science. No reader of the *Astronomische Nachrichten* can fail to have been struck by the untiring activity shewn by his numerous communications to that periodical, so indispensable to the astronomers of all countries. Among his discoveries I may refer to that of the interesting periodical comet which bears his name, and likewise to that of the minor planet *Freia*, the 76th member of the group of small planets between *Mars* and *Jupiter*, the known number of which now amounts to 142, and is yearly increasing at a rate which shews no signs of slackening.

But of all the labours of Professor D'Arrest, unquestionably the most important are his observations of nebulæ contained in the two works mentioned at the commencement of this address.

These works would, in the opinion of your Council, even if they stood alone, amply justify the award of your medal.

Nearly forty years have elapsed since the Society's medal was awarded to Sir John Herschel for his Catalogue of Nebulæ and Clusters of Stars, printed in the *Philosophical Transactions* for 1833. In his address on that occasion, the Astronomer Royal gave an able sketch of the history of our knowledge of the nebulæ up to that time, which makes it quite unnecessary for me to go over the same ground, necessarily much more feebly. I may merely recall that the three catalogues of Sir William Herschel, published in the *Philosophical Transactions* for 1786, 1789, and 1802, contain the places and descriptions of 2500 nebulæ and star-clusters. Sir John Herschel's catalogue contains the results of his observations made at Slough, with his 20-foot reflector, between the years 1825 and 1833. These observations were undertaken for the purpose of reviewing the nebulæ and star-clusters discovered by his father. The catalogue comprises 2307 of these objects, about 500 of which are new.

Not content with having made this survey of the heavens visible in this latitude, Sir John Herschel resolved to undertake a similar survey of the southern heavens; and for this purpose he transported to the Cape of Good Hope the same instrument which he had employed in the northern hemisphere, "so as to give a unity to the results of both portions of the survey, and to render them comparable with each other."

The observations required in order to carry out this grand plan were made in the years 1834, 1835, 1836, 1837, and 1838, and the fruits of these prolonged labours appeared in 1847, in the magnificent work, *Results of Astronomical Observations made at the Cape of Good Hope.* The survey included the double-stars of the southern hemisphere, as well as the nebulæ and star-clusters. The work contains a catalogue of 1708 of these latter objects, entirely similar in its arrangement and construction to the Catalogue of Northern Nebulæ in the *Philosophical Transactions* for 1833, and reduced to the same epoch (1830), in order to facilitate the union of the two catalogues into one general one. Of these objects 89 are common to the two catalogues, so that the number of distinct nebulæ and clusters which they contain is 3926. Both of these works of Sir John Herschel contain engraved representations of some of the most remarkable nebulæ, whether of typical or of exceptional form, by means of which future observers may be able to ascertain whether any secular changes are perceptible in them.

The latter work also comprises valuable chapters on the apparent distribution of the nebulæ over the heavens, and on their classification, together with many general remarks on the phenomena presented by them, which have been suggested by the author's long experience.

By these labours of Sir William and Sir John Herschel, and by them almost exclusively, astronomers had now obtained a considerable amount of knowledge respecting the apparent distribution of the nebulæ over the heavens, and respecting their forms and physical structure as seen through powerful telescopes.

Their distances from us, however, and therefore their real distribution in space and their actual magnitudes remained matter of speculation only.

Sir William Herschel, having found that many nebulæ, which in inferior instruments shewed no traces of stellar composition, were, when viewed by his powerful telescopes, resolved entirely into stars, was at first inclined to believe that all nebulæ were so resolvable. Hence he was inclined to regard them as so many galaxies, similar in their nature to our Milky Way, and owing their nebulous appearance to the enormously greater distances from us at which they were situated. Longer experience, however, induced him completely to change his views.

Already in 1791, in a paper on Nebulous Stars, he had arrived at the conclusion that there exists a diffused self-luminous matter "in a state of modification very different from the construction of a sun or star," and that a nebulous star is one "which is involved in a shining fluid of a nature totally unknown to us," and "which seems more fit to produce a star by its condensation than to depend on the star for its existence."

Again, in his paper on the Construction of the Heavens, in the *Philosophical Transactions* for 1811, he shews that although the appearances presented by diffused nebulous matter and by a star are so totally dissimilar, yet that these extremes may be connected by a series of such nearly allied intermediate steps as to make it highly probable that every succeeding state of the nebulous matter is the result of the action of gravitation upon it while in a foregoing one, and that by such steps the successive condensation of it has been brought up to the condition of planetary nebulæ, and from this again to a stellar form.

From the appearances presented by the planetary nebulæ he infers that the nebulous matter is partially opaque, since the superficial lustre which these objects exhibit could not result "if the nebulous matter had no other quality than that of shining, or had so little solidity as to be perfectly transparent."

He also suggests that comets may be composed of nebulous matter in a highly condensed state, and that the faint nebulous branches which are often seen appended to a nucleus may be similar to the Zodiacal Light in relation to our Sun.

In the same paper he finds reason to conclude that the distance of the faintest part of the great nebula in *Orion* probably does not exceed that of stars of the 7th or 8th magnitude, but may be much less, perhaps even not exceeding the distance of stars of the 2nd or 3rd order, and consequently that "the most luminous appearance of this nebula must be supposed to be still nearer to us."

These views of Sir William Herschel respecting the gradual formation and growth of stars by the condensation of nebulous matter were still further confirmed and developed in his paper in the *Philosophical Transactions* for 1814.

Sir John Herschel's graphic description of the two Nubeculæ, or Magellanic clouds, likewise clearly shews that irresolvable nebulæ, resolvable nebulæ, and clusters of stars represent luminous matter in different conditions, but not necessarily at very different distances from us.

The direct measurement of the distance of a nebula by determining its annual parallax must be regarded as nearly hopeless. The nearest known fixed star has a parallax of scarcely one second. Now the error to which we are liable in the determination of the place of a nebula, although, as we shall see, it may under favourable circumstances be made much smaller than has been commonly supposed, still considerably exceeds one second. Hence, unless a nebula were much nearer to us than the nearest fixed star, there would be no chance of our being able to determine its parallax.

There is one method, however, by which we may expect ultimately to throw great light on the mutual relations of the nebular and sidereal systems, and on their relative distances from us: I mean by the study of their proper motions. Of course, no definite conclusion respecting the distance of an individual nebula could be drawn from the observation of its proper motion. For a nebula comparatively near to us might still have a very small proper motion, simply because its motion in space was nearly equal and parallel to our own. If a large number of instances, however, were taken, it might be asserted with a high degree of probability that those bodies which had a large proper motion were on an average nearer to us than those whose proper motion was small.

Now we know, at least approximately, the proper motions of many of the fixed stars, and materials are gradually accumulating which will give us a much more accurate and extensive knowledge respecting them; but of the proper motions of the nebulæ we know little or nothing.

Unfortunately for this object, the instruments of Sir William Herschel were not well adapted for the very accurate determination of the places of nebulæ. He himself estimates that after 1785 the uncertainty of his places might amount to $1\frac{1}{2}$ minute of space in R. A., and from $1\frac{1}{2}$ to 2 minutes in Declination, and that his earlier observations were liable to much greater errors. Hence these observations can scarcely be employed in such a delicate research as that of the determination of proper motions.

The degree of accuracy attained in Sir John Herschel's two catalogues is much greater. The author considers the probable error of a single observation in his northern catalogue not to exceed $1\frac{1}{2}$ seconds of time in R. A., and 30″ in Declination. In his Cape Observations he estimates that the error of a single observation will seldom exceed 30″ of space in the direction of the parallel, or 45″ in that of the meridian.

Both of these catalogues give the results of the separate determinations of the place of a nebula, and therefore afford the means of calculating the probable errors of the observed places.

Professor D'Arrest has thus found that the probable error of a single position is nearly 15″ in R. A. and 19″·5 in Declination.

Considering the comparatively recent date of these observations, however, it is plain that a considerable time must elapse before the comparison of Sir John Herschel's observations with later ones of a similar degree of accuracy can be expected to yield trustworthy results respecting the proper motions of the nebulæ.

M. Laugier was the first who attempted to determine the places of certain selected nebulæ with much greater precision than is attained in Sir John Herschel's catalogues, in order that they might furnish a secure foundation to future investigations respecting proper motion. In the *Comptes*

Rendus of December 12, 1853 (tome xxxvii. p. 874), he gives a catalogue of the places of 53 nebulæ for the beginning of 1850, selecting such as had well-defined centres or points of greatest brilliancy. It is to be regretted that no details are given respecting either the number of observations on which the places in the catalogue are founded, the mode of observation, or the telescope employed, so that the catalogue itself affords us no means of judging of the degree of accuracy of the places contained in it.

Professor D'Arrest's first series of observations on the nebulæ began in May 1855, and, like M. Laugier's, had for their object the accurate determination of positions for the express purpose of affording means in due time of studying the proper motions of the nebulæ, and thence arriving at more certain conclusions respecting the relations between the nebular and sidereal systems than could be attained by the mere contemplation and examination of the objects themselves, even with the aid of the most powerful telescopes. The results of these observations were published in the *Transactions* of the Royal Saxon Society of Sciences for 1856. The number of nebulæ observed amounts to 230. The observations were made at the Leipzig Observatory, of which Professor D'Arrest was then the Director, with the Fraunhofer refractor of $4\frac{1}{3}$ French inches in aperture and 6 feet focal length, by means of a Fraunhofer's double ring-micrometer. The magnifying power usually employed was 42 times. The nebulæ were thus directly compared with neighbouring stars out of Bessel's and Argelander's Zones. In one night usually three and sometimes four transits of a nebula and its comparison-star were observed, the transits being taken alternately in the northern and southern halves of the ring-micrometer. In order to guard against the uncertainty which may still remain in the places of the stars of comparison, Professor D'Arrest often gives, in his description, the observed differences of right ascension and declination. He also often gives the position of the nebula with respect to the nearest stars, frequently those of the 10th and 11th magnitude, which must ultimately prove most useful for the determination of the nebula's proper motion. In this last point he followed the excellent practice of Sir John Herschel; but he was able to make more repeated measures of this kind, since, on account of the comparatively small power of the instrument, the description of the objects was of secondary importance. It should be remarked that all these measures were taken with the ring-micrometer, no mere estimations being admitted except when they are expressly mentioned. The results derived from each night's observations are given separately. The places given in the catalogues of Sir William and

Sir John Herschel and in the small catalogue of Laugier are likewise reduced to the same epoch (1850) for the sake of comparison.

We are so much accustomed to think of the observations of nebulæ in connection with the most powerful instruments, that it will be no doubt a matter of surprise that a refractor of scarcely $4\frac{1}{2}$ inches aperture should have been found suitable for such work. Professor D'Arrest, however, from his experience with such an instrument, estimates that it is capable of shewing nearly a thousand nebulæ, that is about a third part of all that have been observed in our latitudes with the most powerful telescopes. He remarks also that the small nebulæ of Herschel, mostly round or elliptical in form, can have their places determined more accurately than the majority of telescopic comets. Besides, in observing nebulæ, there is the immense advantage of being able to repeat the observation of one and the same place on different nights. The prevailing central condensation in nebulæ, which sometimes attains a degree of concentration almost stellar, and which very frequently offers a well-defined nucleus, gives a great degree of definiteness to the observation. Those nebulæ which, for various reasons, cannot be observed accurately are, according to Professor D'Arrest, comparatively less numerous. Of the 53 nebulæ observed by Laugier, 31 have been re-observed by Professor D'Arrest. Excluding one of Laugier's right ascensions, which is evidently affected with a large error, and three of the declinations, which appear to be about $1'$ in error, perhaps through mistakes in copying, and assuming the probable error of one of Laugier's positions to be equal to that of the mean of three of his own single positions, Professor D'Arrest finds each of these probable errors to be about $6''$ both in right ascension and declination. By a provisional calculation of the probable error of his observations, founded on a comparison of the several determinations with their mean, Professor D'Arrest finds that the probable error of a definitive position, that is of the mean of the observations of three nights, generally depending on 9 transits, does not exceed 4 or 5 seconds of space in each coordinate.

Professor D'Arrest makes an interesting use of his comparisons of his own places with those of Sir John Herschel. The mean epoch of Sir John Herschel's observations is nearly 25 years earlier than that of his own. Hence the difference between the places of a nebula as given by the two authorities, and reduced to the same epoch, will include not merely the errors of the observations, but also the proper motion for 25 years and the difference of the star-places used in the reductions. Now, from the probable errors of Sir John Herschel's and Professor D'Arrest's places which have been already ascertained, we can at once obtain the value of the mean of the squares of the differences between those places, supposing the differences to be entirely due to casual errors of observation. The actual mean of the squares of the differences is found to be greater than the above-mentioned mean, and the excess is due partly to the proper motions of the nebulæ in the interval, partly to the differences in the star-places employed, and, very probably also partly to constant differences in the mode of observing the same nebula by the two observers. Hence Professor D'Arrest concludes that the probable amount of the annual relative motion of the nebulæ with respect to the sidereal system is less than $0''\cdot4$ measured in arc of a great circle.

I may appropriately conclude my remarks on Professor D'Arrest's *Resultate aus Beobachtungen der Nebelflecken und Sternhaufen* by a quotation from one who has himself done much in the same line of research. Speaking of Laugier's and D'Arrest's observations, Dr Schultz says: "These works have the high merit of having originated a new and important branch in the study of the nebulæ; and D'Arrest has done especial service to this study by shewing that, when what is required is simply good determinations of positions, a much greater number of nebulæ than has been usually supposed may be advantageously observed with instruments of but very moderate dimensions. But his series of observations is chiefly and especially important as proving beyond the possibility of a doubt that the positions of nebulæ in general are determinable with far greater accuracy than it had been previously usual to suppose; and D'Arrest's work thus made an epoch in the study of nebulæ, by freeing it from the deterring prestige which had before that period been attached to it."

Many other observers have since followed up the work thus begun by Professor D'Arrest. Very accurate positions of nebulæ have been observed by Auwers, Schmidt, Schönfeld, Vogel, Rümker, Stephan, Schultz, and others. I may particularly mention Schönfeld's Mannheim Observations of 235 Nebulæ, which appear to be extremely accurate and are published in a form that leaves nothing to be desired. This work also enjoys the immense advantage that the places of all the stars of comparison have been newly determined by the meridian observations of Professor Argelander. But a still more extensive work in the same field, and which promises to attain even a greater degree of accuracy, is that by Dr Schultz, from whom I have quoted above. This work consists of micrometrical observations of 500 nebulæ made at the University Observatory of Upsala, with the Steinheil 13-foot refractor, employing a parallel wire-micrometer with bright spider-lines on a dark field.

By means of the various series of observations to which I have referred, future astronomers will be provided with a rich store of materials for the study of the proper motions of the nebulæ, and we may hope that even in our own time some valuable results may be arrived at respecting them.

Professor D'Arrest's observations of nebulæ were interrupted for a time by his appointment as Director of the Observatory of Copenhagen. In no long time, however, his new position gave him the opportunity of resuming his observations with the aid of greatly increased optical power. In the year 1861, the Observatory acquired a magnificent refractor, by Merz, of 15 feet focal length and 10½ French inches in aperture, of which Professor D'Arrest has given an elaborate description in a separate publication, *De Instrumento magno æquatorio.* He considers this instrument to be intermediate, as regards optical power, between Sir John Herschel's 20-foot reflector in its best condition, and the excellent telescope with which Mr Lassell made his observations at Valletta. Finding that with this instrument he could not only perceive the very faintest of the nebulæ discovered by the two Herschels, but could make sufficiently precise observations of them, he resolved no longer to continue the work begun in Leipzig, where he confined his attention to selected nebulæ, but to enlarge his plan of operations and make a survey of the nebulæ of the whole of the northern heavens. At first, indeed, it was his intention to observe all the nebulæ he should meet with, whether previously known or not, with the utmost attainable precision, and that not once or twice only but repeatedly. He soon found, however, that to carry out such a plan, especially in such a climate, was beyond human powers, the number of the nebulæ far exceeding all expectation. After labouring assiduously and perseveringly at these observations for more than six years, Professor D'Arrest was at length compelled by failing health to bring his work to a close. He estimates that in those six years he had not been able to make more than about one-eighth of the total number of observations which would be required in order to form a catalogue of the approximate positions of those nebulæ which could be accurately observed with the Copenhagen refractor.

The results of these prolonged labours have been published in the great work, *Siderum Nebulosorum Observationes Havnienses*, 1867. This volume contains about 4800 single positions of 1942 different nebulæ. Of these about 390 have either not been previously observed, or have not had their places determined. Sir John Herschel's Northern Catalogue of Nebulæ and Clusters of Stars contains a larger number of objects, viz., about 2300. The difference between these numbers partly arises from the fact that D'Arrest has designedly omitted those objects in Herschel's catalogue which, in his judgment, should not be classed with the nebulæ, viz., clusters and collections of stars belonging to Sir William Herschel's sixth, seventh, and eighth classes. These clusters appear to have no necessary connection with true nebulæ, and they are distributed over the sphere in a totally different manner. The number of such clusters, especially near the Milky Way, might be easily greatly increased; and in making his sweeps, Professor D'Arrest has often been surprised to find certain clusters inserted in Herschel's catalogue, while several others in the same neighbourhood were omitted. The selection appears to him arbitrary and by no means natural. He thinks too that the introduction of these objects would tend to vitiate any inquiries into the law of distribution of the nebulæ.

By far the greater number of the nebulæ cannot be observed at all with bright wires, or at any rate can only be so observed by great expenditure of time and trouble. Hence Professor D'Arrest did not attempt to define their places with all the precision of which his instrument was capable, but brought each nebula into the centre of the ring-micrometer, the smallest radius of which was 3′ 40″. The power employed in determining all these approximate positions was 123. The hour circle was read off to integral seconds of time, and the declination circle to tenths of a minute of arc.

In fact, nearly the same method was followed which astronomers are accustomed to employ in finding the places of very faint comets. Thus everything was scrupulously avoided which would interfere with the keenness of vision, and the more precise definition of place was generally left to micrometrical observations and comparisons with minute stars situated in the immediate neighbourhood of the nebula.

The nebulæ were generally observed in zones of about 4° or 5° in breadth, and in each zone 4 or 5, or even sometimes 7 fixed stars of the 7th or 8th magnitude were included, whose places were taken from Bessel's or Argelander's zones, or sometimes from those of Lalande.

The work contains about 4000 micrometrical measures, chiefly made with the ring-micrometer. More rarely nebulæ were compared with the stars and with each other by means of the wire-micrometer. Bright and small nebulæ, having stellar nuclei, or at least an entirely regular form, were observed with all possible precision, and the differential determinations of their positions referred to neighbouring stars will, without doubt, be found of the greatest importance in the future study of their proper motions.

Excluding a few nebulæ, whose places do not admit of any accurate determination, Professor D'Arrest finds, from 1627 observations of declination of 525 nebulæ, that the probable error of a single observation of declination is $17''\cdot58$, while from 1552 right ascension observations of 497 nebulæ, he finds the probable error of a single observation of right ascension to be $0^s\cdot809 \sec\delta$.

These probable errors are slightly less than the corresponding probable errors of Sir John Herschel's catalogues.

Following the excellent example set by Sir John Herschel, Professor D'Arrest gives the results of each night's observations of a nebula separately, both as regards its place and its description.

The use of an equatorially-mounted telescope has no doubt rendered this catalogue comparatively free from incidental errors and mistakes in the identification of nebulæ, which will occasionally happen, in spite of the greatest care, when the observations are made with an instrument not so mounted.

Lord Rosse's valuable selection from the observations of nebulæ made with his gigantic reflector of 6-feet aperture appeared in the *Philosophical Transactions* for 1861, but, curiously enough, did not reach Professor D'Arrest's hands till 1864, when his own work was considerably advanced. This work contains sometimes brief and sometimes full descriptions of about 800 nebulæ, many of them being illustrated by figures. Professor D'Arrest found that not a few of the nebulæ which he had detected in the interval between 1861 and 1864 had been already observed by Lord Rosse and his assistants, and that his descriptions were generally confirmed by theirs. Very many "new" nebulæ, however, still remained which had not been observed by Lord Rosse; while, on the other hand, many which occur in Lord Rosse's work had escaped the notice of Professor D'Arrest. After this period he derived the greatest assistance from Lord Rosse's work. It is not surprising to find occasional differences and discrepancies in the descriptions of nebulæ given in these two works. Professor D'Arrest mentions that he has found and observed by far the greater part of those nebulæ which had been observed by Herschel, but had been inserted by Lord Rosse in a list of "nebulæ not found."

He also succeeded in verifying the existence and determining the places of many very faint nebulæ, which had been first discovered by means of Lord Rosse's telescope.

In the *Philosophical Transactions* for 1864, Sir John Herschel published his *General Catalogue of Nebulæ and Clusters of Stars*, and thereby laid astronomers under another very heavy obligation. This excellent catalogue contains all the nebulæ and clusters of stars, both northern and southern, actually known at that date, 5063 in number, arranged in order of right ascension, and reduced to the common epoch 1860. A short description of each nebula or cluster is given in abbreviated words, made out from an assemblage and comparison of all the descriptions of each object given in his father's and in his own observations.

It is not easy to over-estimate the boon which such a catalogue offers to an observer of nebulæ, by enabling him "at once to turn his instrument on any one of them, as well as to put it in his power immediately to ascertain whether any object of this nature which he may encounter in his observations is new, or should be set down as one previously observed." As Sir John Herschel remarks, "For want of such a general catalogue, a great many nebulæ have been from time to time, in the *Astronomische Nachrichten* and elsewhere, introduced to the world as new discoveries, which have since been identified with nebulæ already described and well known. Many a supposed comet, too, would have been recognised at once as a nebula, had such a general catalogue been at hand, and much valuable time been thus saved to their observers in looking out for them again."

While Sir John Herschel was engaged in the preparation of this catalogue, an important work by Dr Auwers appeared, entitled, *William Herschel's Verzeichnisse von Nebelflecken und Sternhaufen, bearbeitet von Arthur Auwers*, Königsberg, 1862. This contains a complete and most elaborate reduction to 1830, from the observed differences in right ascension and polar distance with known stars, recorded in the *Philosophical Transactions,* of all the nebulæ and clusters in Sir William Herschel's three catalogues; together with a separate catalogue of all those collected by Messier from his own observations or those of Méchain and others (101 in number), similarly reduced; another of Lacaille's southern nebulæ; and one of fifty "new nebulæ, comprising nearly all those observed by other astronomers (Lord Rosse excepted) in this hemisphere, all brought up to the same epoch."

Sir John Herschel states that a comparison with Dr Auwers' results led him to the detection of several grave errors in his own work which would otherwise have escaped notice, and whose rectification has added materially to its value.

Sir John Herschel's general catalogue contains the places and descriptions of 125 of the new nebulæ discovered by Professor D'Arrest, and reduced by him to the epoch of that catalogue.

At the end of his own work Professor D'Arrest gives a catalogue of the mean places of his 1942 nebulæ, reduced to the epoch 1860 for comparison with Herschel's general catalogue. He also gives a comparison of his own positions with the places of 223 nebulæ contained in the very accurate special catalogue by Schönfeld, which has been already mentioned.

In the above rapid sketch I have omitted to mention the many excellent descriptions and delineations of particular nebulæ which we owe to Mr Lassell, Professors W. C. Bond and G. P. Bond, Mr Mason, Otto von Struve, Padre Secchi, and others.

I must not terminate this very imperfect account of the principal additions to our knowledge of the Nebulæ which have been made in recent years, without referring to the entirely new mode of investigation to which they have been subjected by means of the spectroscope. By observations of this kind, Mr Huggins and others have thrown much additional light on the nature and constitution of these mysterious bodies. Already the spectra of about 140 nebulæ have been examined, and the light from many of them has been proved to emanate from glowing gas. This entirely confirms the mature view of Sir William Herschel, viz., that the condition of the luminous matter in many of the nebulæ is widely different from its condition in the fixed stars.

Professor D'Arrest has himself contributed to the spectroscopic observations of the nebulæ, and he has made the suggestive remark, that almost all the gaseous nebulæ are found either within or near the borders of the Milky Way, and that there is an entire absence of them in the regions near the poles of the galaxy, in which the other nebulæ so abound. I believe that a similar remark was made about the same time by Mr Proctor.

It is worth mentioning that one of the most remarkable of these gaseous nebulæ, viz. the planetary nebula numbered 4373 in Sir John Herschel's General Catalogue was observed as a fixed star by Lalande in 1790, and that by comparing its place so determined with the very accurate modern determinations of Schönfeld, D'Arrest, and others, it has been shewn that the proper motion of this nebula is quite insensible.

I trust that the statement, however bald and imperfect, which I have just laid before you respecting the labours of Professor D'Arrest, will have convinced you that your Council have been fully justified in awarding to him the Society's medal.

(*The President then, delivering the Medal to the Foreign Secretary, addressed him in the following terms*):—

Mr Huggins—In transmitting this medal to Professor D'Arrest, you will express to him the admiration we feel for the skill and perseverance which he has shewn in his observations of the nebulæ, and our high appreciation of the value of his labours. You may assure him of our ardent wishes that health and strength may long be spared to him, so that he may be able to make many further contributions to the progress of Astronomy.

46.

ADDRESS ON PRESENTING THE GOLD MEDAL OF THE ROYAL ASTRONOMICAL SOCIETY TO M. LE VERRIER.

[From the *Monthly Notices of the Royal Astronomical Society*, Vol. XXXVI. (1876).]

IT has been already announced to you that the Council have awarded the Society's medal to M. Le Verrier for his theories of the four great planets, *Jupiter*, *Saturn*, *Uranus*, and *Neptune*, and for his tables of *Jupiter* and *Saturn* founded thereupon. It now becomes my pleasing duty to explain to you the grounds of this award.

I need not, on the present occasion, enter into any detail respecting the previous achievements of our distinguished Associate, and the numerous and valuable researches with which he has enriched our science. These will be fresh in your recollection, and they have already been eloquently described to you from this chair.

It is not many years since our medal was awarded to M. Le Verrier for his theories and tables of the four planets nearest the Sun, viz. *Mercury*, *Venus*, the *Earth*, and *Mars*. Long before this he had been occupied with the larger planets, but before proceeding further with their theories he found it necessary to establish on solid foundations the theory of the motion of the Earth, on which all the rest depend, and this again naturally led him to investigate the theories of the three nearer planets which, with the Earth, constitute the inferior portion of the planetary system.

By the comparison of these theories with observation, M. Le Verrier was led to two interesting results. He found that in order to bring the theories of *Mercury* and *Mars* into accordance with observation, it was necessary and sufficient to increase the secular motion of the perihelion of *Mercury*, and also the secular motion of the perihelion of *Mars*.

Hence M. Le Verrier inferred that there existed, on the one hand, in the neighbourhood of *Mercury*, and on the other, in the neighbourhood of *Mars*, sensible quantities of matter, the action of which had not been taken into account.

This conclusion has been verified with respect to *Mars*. The matter which had not been considered turns out to belong to the Earth itself, the mass of which had been taken too small, having been derived from too small a value of the solar parallax. A similar increase of the mass of the Earth is indicated by the theory of *Venus*, and a corresponding increase of the solar parallax is likewise derived from the lunar equation in the motion of the Sun.

With respect to *Mercury*, a similar verification has not yet taken place, but the theory of the planet has been established with so much care, and the transits of the planet across the Sun furnish such accurate observations, as to leave no doubt of the reality of the phenomenon in question; and the only way of accounting for it appears to be to suppose, with M. Le Verrier, the existence of several minute planets, or of a certain quantity of diffused matter circulating about the Sun within the orbit of *Mercury*.

The results which M. Le Verrier had thus obtained from his researches on the motions of the interior planets added to the interest with which he now entered upon similar researches on the system of the four great planets which are the most distant from the Sun. Such researches might furnish information respecting matter, hitherto unknown, existing in the neighbourhood of these planets. Possibly they might afford indications of the existence of a planet beyond *Neptune*, and at any rate they would provide materials which would facilitate future discoveries.

As I shall have occasion to explain later on, the theories of the mutual disturbances of the larger planets are far longer and more complicated than those of the smaller, so that all that M. Le Verrier had yet done might be almost regarded as merely a prelude to what still remained to be done. Increased difficulties, however, far from deterring, seemed rather to stimulate him to greater exertions.

On the 20th of May, 1872, M. Le Verrier presented to the Academy an elaborate memoir, containing the first part of his researches on the theories of the four superior planets, *Jupiter*, *Saturn*, *Uranus*, and *Neptune*. This memoir contains an investigation of the disturbances which each of these planets suffers from the action of the remaining three. Throughout this investigation the development of the disturbing function, as well as that of the inequalities of the elements is given in an algebraical form, in which everything which varies with the time is represented by a general symbol, so that the expressions obtained hold good for any time whatever. Thus the eccentricities and inclinations, the longitudes of the perihelia and of the nodes are all left in the condition of variables. The mean parts of the major axes, which suffer no secular variations, are alone treated as given numbers.

At the end of the *résumé* of the contents of this memoir, given in the *Comptes Rendus*, M. Le Verrier lays down the following almost appalling programme of the work still remaining to be done.

It would be necessary, he says,

1. To calculate the formulæ, and to reduce them into provisional tables.
2. To collect all the exact observations of the four planets, and to discuss them afresh, in order to refer their positions to one and the same system of coordinates.
3. By means of the provisional tables, to calculate the apparent positions of the planets for the epochs of the observations.
4. To compare the observed with the calculated positions, to deduce the corrections of the elliptic elements of the four planets, and to examine whether the agreement is then perfect.
5. In the contrary case, to find the causes of the discrepancy between theory and observation.

Extensive as is this programme, it has already been completely carried out as regards the planets *Jupiter* and *Saturn*, and partly so as regards *Uranus* and *Neptune*.

Having received from the Academy the most effectual encouragement to pursue his researches, M. Le Verrier lost no time in bringing them gradually to completion, so that they might become available for practical use.

Accordingly, on the 26th of August, 1872, he presented to the Academy a memoir containing a complete determination of the mutual disturbances of *Jupiter* and *Saturn*, and thus serving as a base for the theories of both these planets, which are closely connected with each other.

Again, on the 11th of November, 1872, he presented his determination of the secular variations of the elements of the orbits of the four planets, *Jupiter*, *Saturn*, *Uranus*, and *Neptune*. These variations are mutually dependent on each other, and must be treated simultaneously. Their determination consequently involves the solution of sixteen differential equations, which are very complicated in form, and can only be integrated by repeated approximations.

This part of the work forms a necessary preliminary to the treatment of the theory of any one of these planets in particular.

On March 17, 1873, M. Le Verrier presented to the Academy the complete theory of *Jupiter*; and on July 14 in the same year he followed it up by the complete theory of *Saturn*.

On January 12, 1874, he presented his tables of *Jupiter*, founded on the theory which has just been mentioned, as compared with observations made at Greenwich from 1750 to 1830 and from 1836 to 1869, and with observations made at Paris from 1837 to 1867.

Again, on November 9, 1874, he presented to the Academy a complete theory of *Uranus*. Already in 1846, in his researches which led to the discovery of *Neptune*, M. Le Verrier had given a very full investigation of the perturbations of *Uranus* by the action of *Jupiter* and *Saturn*. In the memoir just mentioned he gives a fresh investigation, including a full treatment of the perturbations of *Uranus* by the action of *Neptune*.

On December 14, 1874, he presented a new theory of the planet *Neptune*, thus completing the theoretical part of the immense labours which he had undertaken with respect to the planetary system.

Finally, on August 23, 1875, he presented to the Academy the comparison of the theory of *Saturn* with observations.

Such is a bare enumeration of the various labours for which our science is already indebted to our illustrious Associate.

That any one man should have had the power and perseverance required thus to traverse the entire solar system with a firm step, and to determine with the utmost accuracy the mutual disturbances of all the primary planets which appear to have any sensible influence on each other's motions, might well have appeared incredible if we had not seen it actually accomplished.

I will now proceed to give a brief outline of the investigations relating to the motions of the four larger planets, with which we are now more particularly concerned. The most important parts of these investigations are printed in full detail in the volumes of *Memoirs* which form part of the *Annals of the Observatory of Paris.*

As in his former researches, M. Le Verrier here also exclusively employs the method of variation of elements, and the investigations are based on the development of the disturbing function given by him, in the first volume of the *Annals of the Paris Observatory*, with greater accuracy and to a far greater extent than had ever been done before.

The 18th Chapter of M. Le Verrier's researches, which forms nearly the whole of the 10th Volume of the *Memoirs*, is devoted to the determination of the mutual action of *Jupiter* and *Saturn*, which forms the foundation of the theories of these two planets.

These theories are extremely complicated, and I shall endeavour briefly to point out, and to explain as far as I can without the introduction of algebraical symbols, the nature of the peculiar difficulties which M. Le Verrier has had to encounter in their treatment, and which he has so successfully overcome. These difficulties either do not present themselves at all, or do so in a very minor degree in the theories of the smaller planets.

First, then, the masses of *Jupiter* and *Saturn* are far larger than those of the interior planets, the mass of *Jupiter* being more than 300 times and that of *Saturn* being nearly 100 times greater than the mass of the Earth. For this reason it is necessary to develop the infinite series in which the perturbations are expressed to a much greater extent when we are dealing with *Jupiter* and *Saturn*, than when we are concerned with the mutual disturbances of the interior planets. Also *Jupiter* and *Saturn* are so far removed from these latter planets that the disturbances which they produce in the motion of these planets are extremely small, in spite of the large masses of the disturbing bodies.

But the great magnitude of the disturbing masses is far from being the only reason why the theory of the mutual disturbances of *Jupiter* and *Saturn* is so complicated.

Another cause which aggravates the effect of the former is the near approach to commensurability in the mean motions.

Twice the mean motion of *Jupiter* differs very little from five times that of *Saturn*. In other words, five periods of *Jupiter* occupy nearly the same time as two of *Saturn*, so that if at a given time the planets were in conjunction at certain points in their orbits, then after three synodic periods they would be again in conjunction at points not far removed from their positions at starting. Hence, whatever uncompensated perturbations may have been produced in the motions of the two planets during these three synodic periods will be very nearly repeated in the next three synodic periods, and again in the next three, and so on.

Hence the disturbances will go on accumulating in the same direction during many revolutions of the two planets, and will become very important. The inequalities of long period thus arising will affect all the elements of the orbits of the two planets; but the most important are those which affect the mean longitudes of the bodies, since these are proportional to the square of the period of the inequalities, whereas the inequalities affecting the other elements are proportional to the period itself.

The principal terms of the inequalities of mean longitude are of the third order, if we consider the eccentricities of the orbits and their mutual inclination to be small quantities of the first order.

Terms of the same period, however, and those far more numerous and more complicated in expression, occur among those of the fifth and of the seventh order of small quantities, and M. Le Verrier has included these terms also in his approximations.

But the circumstance which contributes in the highest degree to cause the superior complexity of the theories of the larger planets is the necessity, in their case, of taking into account the terms which depend on the squares and higher powers of the disturbing forces.

I will endeavour to point out the nature of these terms and the manner in which they arise.

By the theory of the variation of elements we are able to express at any given time the rate of variation of any one of the elements in terms of the mean longitudes and the elements of the orbits of the disturbed and the several disturbing bodies. If this rate of variation were given in terms of the time and known quantities, we should at once find the value of the element for any given time by a simple integration. But this is not the case.

The method of variation of elements gives us, not a solution, but merely a transformation of our original differential equations of motion. The rates of variation are given in terms of the unknown elements themselves; and in order to find the elements from the equations so formed, we must employ repeated approximations.

Let us consider this matter a little more particularly.

The terms which express the rate of variation of any element may be divided into two classes:

1. Those which involve the mean longitudes of one or both of the planets concerned, as well as the elements of their orbits.

2. Those which involve the elements only.

The first are called periodic terms, since they pass from positive to negative, and *vice versâ*, in periods comparable with those of the planets themselves.

The second are called secular terms, and vary very slowly, since the elements on which they depend do so.

Each of the terms in the expression of the rate of variation of any element will involve the mass of one of the disturbing bodies as a factor.

Hence, if all these masses be very small, all the periodic inequalities of the elements will be likewise very small, and we shall obtain a value of the rate of variation which is very near the truth if we substitute for the complete value of any element its value when cleared of periodic inequalities.

Then the periodic inequalities in the element under consideration may be found by direct integration, supposing the elements to be constant in the terms to be integrated, and the mean longitudes only to vary.

Also the secular variation of the element considered, that is the rate of variation of the element when cleared of periodic inequalities, will be given by the secular terms taken alone.

If the disturbing masses, however, are not very small, this process is not sufficiently accurate, and the periodic inequalities thus found can only be regarded as a first approximation to the true values.

In order to find more correct values, we must substitute for the elements in the second member of the equation their secular parts augmented by the approximate periodic inequalities before found.

Now, if in any periodic term we increase any element by a periodic inequality depending on a different argument, that is involving different multiples of the mean longitudes, the result will evidently be to introduce new periodic terms which will involve the square of one of the masses or the product of two of them as a factor.

Similarly, if in any periodic term any element be increased by a periodic inequality depending on the same argument, the result will also introduce new terms of the second order which do not involve the mean longitudes, and which therefore constitute new secular terms.

These will be particularly important if the inequality in question be one of long period.

Also in the secular terms the result of increasing any element by a periodic inequality will be to introduce a new periodic term depending on the same argument.

Lastly, it should be remarked that in finding the periodic inequalities of any element by integration of the corresponding differential equation, we must take into account the secular variations of the elements which were neglected in the first approximation. The new terms thus introduced, like the others which we have just described, will evidently be of the second order with respect to the masses.

If the disturbing masses be large, as in the case of the mutual disturbances of *Jupiter* and *Saturn*, it may be necessary to proceed to a further approximation, and thus to obtain new terms, both periodic and secular, which involve the cubes and products of three dimensions of the masses.

The number of combinations of terms which give rise to these terms of the second and third orders is practically unlimited, and the art of the calculator consists in selecting those combinations only which lead to sensible results.

This is the chief cause of the great complexity of the theories of the larger planets, and more especially of those of *Jupiter* and *Saturn*.

M. Le Verrier lays it down as the indispensable condition of all progress that we should be able to compare the whole of the observations of a planet with one and the same theory, however great may be the length of time over which the observations extend. In order to satisfy this condition, he develops the whole of his formulæ algebraically, leaving in a general symbolical form all the elements which vary with the time, such as the eccentricities, the inclinations, and the longitudes of the perihelia and nodes. He treats in the same way the masses which are not yet sufficiently known.

All the work is given in full detail, and is divided as far as possible into parts independent of each other, so that any part may be readily verified.

All the terms which are taken into account are clearly defined, so that if it should ever be necessary to carry on the approximations still further, it will be easy to do so without having to begin the investigation afresh.

The whole work is presented with such clearness and method as to make it an admirable model for all similar researches.

After the development of the disturbing functions, and the formation of the differential equations on which the variations of the elements depend, the first step to be taken is to determine by integration of these equations the periodic inequalities of the elements of the orbits of *Jupiter* and *Saturn* which are of the first order with respect to the masses. As we have already said, the expressions of these periodic variations of the elements are given with such generality that, in order to obtain their numerical values at any epoch whatever, it is sufficient to substitute the secular values of the elements at that epoch. The calculation of the various terms under this general form is very laborious, and it requires great and sustained attention in order to avoid any error or omission of importance. On the other hand, by substituting from the beginning the numerical values of the elements at a given epoch, the calculation is rendered much shorter and admits much more readily of verification; but the result thus obtained only holds good for the given epoch, and is thus entirely wanting in generality.

In the determination of the long inequalities of *Jupiter* and *Saturn*, the approximation is carried to terms which are of the seventh degree with respect to the eccentricities and the mutual inclination of the orbits.

In the next place the terms of the first order in the secular variations of the elements of the orbits are determined.

After this the periodic inequalities of the second order with respect to the masses are considered. These are determined in the same form as the terms of the first order, in order that their expressions may hold good for any epoch whatever. The formulæ relating to these terms are necessarily very complicated. The coefficient belonging to a given argument depends, in general, on a great number of terms which are classed methodically.

Next are determined the terms of the second order in the secular variations of the elements of the orbits.

Afterwards, M. Le Verrier takes into account the influence of the secular inequalities on the values of the integrals on which the periodic inequalities depend.

The last part of this chapter is devoted to the completion of the differential expressions of the secular inequalities by the determination of certain secular terms in the rates of variation of the eccentricities and the longitudes of the perihelia, which are of the third and fourth orders with respect to the masses.

The 19th Chapter of M. Le Verrier's researches, which forms the first part of the 11th Volume of the *Annals of the Paris Observatory*, contains the determination of the secular variations of the elements of the orbits of the four planets, *Jupiter*, *Saturn*, *Uranus*, and *Neptune*.

In the first place are collected the differential formulæ which are established in the previous chapter, and which give the rates of secular change of the various elements at any epoch in terms of the elements themselves, which by the previous operations have been cleared of all periodic inequalities.

The terms of different orders which enter into these formulæ are carefully distinguished.

If we were to confine our attention to the terms of the first degree with respect to the eccentricities and inclinations of the orbits, and of the first order with respect to the masses, the differential equations which determine the secular variations would become linear, and their general integrals might be found, so as to give the values of the several elements for an indefinite period.

In the present case, however, the terms of higher orders are far too important to be neglected, and when these are taken into account the equations become so complicated as to render it hopeless to attempt to determine their general integrals.

Fortunately, however, these are not needed for the actual requirements of Astronomy, and for any definite period the simultaneous integrals may be determined with any degree of accuracy that may be desired by the method of quadratures.

In this way M. Le Verrier has determined the values of the elements for a period of 2000 years, starting from 1850, at successive intervals of 500 years. The first steps in this integration were attended with some difficulties, because the determination of the numerical values of the rates of change of the several elements at the various epochs depends on the elements themselves which are to be determined. Hence several approximations were necessary in order to obtain the requisite precision.

After this work of M. Le Verrier, however, the extension of the investigation to other epochs, past or future, is no longer attended with the same difficulties. In fact, from his results we may at once find, by the method of differences, very approximate values of the elements at an epoch 500 years earlier or later than those which he has considered. His general formulæ will then give the rates of change of the several elements at the epoch in question, and having these we can determine by a direct calculation the small corrections which should be applied to the approximate values of the elements first found.

This process may evidently be repeated as often as we choose.

It is important to remark that in the formulæ which give the rates of change of each of the elements at the five principal epochs considered, as well as in those which give the total variations of the elements at the same epochs, the masses of the several planets appear in an indeterminate form, so that it may be at once seen what part of the variation of any element is due to the action of each of the planets, and what changes would be produced in the value of any element at any epoch by any changes in the assumed values of the masses.

Consequently, when the astronomer of the future, say of 2000 years hence, has determined the values of the elements of the planetary orbits corresponding to that epoch, it will be easy for him, by comparing those values with the general expressions given by M. Le Verrier, to determine with the greatest precision the actual values of the masses, provided that all the disturbing bodies are known; and should there be any unknown disturbing causes, their existence would be indicated by the inconsistency of the values of the masses which would be found from the different equations of condition.

By means of the work which has just been described everything has been prepared which is required for the treatment of the theories of the several planets.

The remainder of the 11th Volume of the *Annals* is accordingly occupied by the complete theories of *Jupiter* and *Saturn*, the former theory being given in Chapter 20 and the latter in Chapter 21 of M. Le Verrier's researches.

The coefficients of the periodic inequalities of the mean longitudes and of the elements of the orbits are not only exhibited in a general form, but are also calculated numerically for the five principal epochs considered in Chapter 19 of these researches, viz. for 1850, 2350, 2850, 3350, and 3850.

The long inequalities of the second order with respect to the masses, depending on twice the mean motion of *Jupiter* plus three times the mean motion of *Uranus* minus six times the mean motion of *Saturn*, are also determined in a similar form.

Chapter 22 of M. Le Verrier's researches, forming the first part of the 12th Volume of the *Annals*, contains the comparison of the theory of *Jupiter* with the observations, the deduction of the definitive corrections of the elements therefrom, and finally the resulting tables of the motion of *Jupiter*.

The observations employed are the Greenwich observations from 1750 to 1830 and from 1836 to 1869, together with the Paris observations from 1837 to 1867.

To the results given in the Astronomer Royal's "Reduction of the Greenwich Observations of Planets from 1750 to 1830" M. Le Verrier has applied the corrections which he has found to be required by his own reduction of Bradley's observations of stars and his redetermination of the Right Ascensions of the fundamental stars, published in the 2nd Volume of the *Annals* (Chapter 10).

The equations of condition in longitude, for finding the corrections of the elements and of the assumed mass of *Saturn*, are divided into two series corresponding to the observations made from 1750 to 1830, and into two other series corresponding to the observations made from 1836 to 1869.

Moreover, in each of these series the equations are subdivided into eight groups, corresponding to the distances of the planet from its perihelion, 0° to 45°, 45° to 90°, and so on.

From these are formed four final equations, the solution of which gives the corrections of the epoch, of the mean motion, of the eccentricity, and of the longitude of the perihelion, in terms of the correction required by the mass of *Saturn*, which is left in an indeterminate form.

The substitution of these expressions in the thirty-two normal equations corresponding to the several groups above mentioned gives the residual differences between theory and observation in terms of the correction of the mass of *Saturn*.

No conclusion can be drawn from the ancient observations; but from the modern observations M. Le Verrier finds that the mass of *Saturn* assumed—which is that of Bouvard—should be diminished by about its $\frac{1}{200}$th part. This correction is very small, but M. Le Verrier regards it as well established.

On the other hand, Bessel's value of the mass of *Saturn*, founded on his observations of the Huyghenian satellite, exceeds Bouvard's by about its $\frac{1}{350}$th part.

The equations of condition in latitude are treated in a similar manner, being grouped according to the distances of the planet from its ascending node.

From these equations the corrections of the inclination of the orbit and longitude of the node are found separately from the ancient and from the modern observations. The results differ very little, but the second solution is employed in the construction of the tables.

After the application of these corrections to the elements, the agreement between theory and observation may be considered perfect; so that the action of the minor planets on *Jupiter* appears to be insensible, and there is no indication of any unknown disturbing causes.

There are some peculiarities in the mode of tabulating the perturbations caused by the action of *Saturn*. The perturbations of longitude and of radius vector are not, as usual, exhibited directly, but instead of them M. Le Verrier gives the perturbations, both secular and periodic, of the mean longitude, of the longitude of the perihelion, of the eccentricity, and of the semi-axis major of the orbit, and then from the elements corrected by these perturbations he derives the disturbed longitude and radius vector by the ordinary formulæ of elliptic motion.

Where the perturbations are large, M. Le Verrier considers this preferable to the ordinary method of proceeding.

The perturbations of latitude being small, he applies to the inclination and longitude of the node their secular variations alone, and then determines directly the periodic inequalities of latitude.

All these perturbations, whether of the elements or of the latitude, are developed in a series of sines and cosines of multiples of the mean longitude of *Saturn*, including a constant term, the coefficients multiplying these several terms being functions of the mean elongation of *Saturn* from *Jupiter*, which for a given elongation are developed in powers of the time reckoned from the epoch 1850.

These coefficients only are tabulated with the mean elongation as the argument, and the perturbations are thence calculated by means of the ordinary trigonometrical tables.

The intervals of the argument are so small, that the requisite interpolations are very simple, and the coefficients which relate to the four elements, and depend on the same argument, are given at the same opening of the tables.

The tables have been calculated specially for the 500 years included between the years 1850 and 2350. Nevertheless they may be applied to epochs anterior to 1850, by simply changing the sign of the time reckoned from 1850. For one or two centuries before 1850 this extension will have all the rigour of modern observations, while for still earlier times the accuracy of the tables will greatly surpass that of the observations which we have to compare with them.

M. Le Verrier's Tables of *Jupiter* are now employed in the computations of the *Nautical Almanac*, beginning with the year 1878.

The 13th Volume of the *Annals* is devoted to the theories of *Uranus* and *Neptune*. These theories are not unattended with difficulties.

In the first place, these planets are disturbed by the actions of the two great masses, *Jupiter* and *Saturn*, interior to their orbits, and these actions are modified by the great inequalities of *Jupiter* and *Saturn* depending on five times the mean motion of *Saturn* minus twice the mean motion of *Jupiter*.

In the next place, twice the mean motion of *Neptune* differs very little from the mean motion of *Uranus*, and thus arise inequalities of long period in the elements of their orbits which are large enough to produce very sensible terms of the second order.

Lastly, the mean elliptic elements of the two planets are not yet sufficiently well known.

In a preliminary chapter, the 24th, M. Le Verrier investigates formulæ which are specially applicable to the case of a planet disturbed by another which is considerably nearer to the Sun.

In this case it is easily seen that, by the direct action of the disturbing planet on the Sun, perturbations of large amount may be produced in the *elements* of the orbit of the disturbed planet, while the corresponding perturbations of the coordinates of the planet are comparatively small. Hence arises the advantage of considering this case apart.

We have seen how closely the theories of *Jupiter* and *Saturn* are related to each other. In a similar manner the theories of *Uranus* and *Neptune* are also closely related in consequence of the great perturbations introduced into the elements of their orbits by the near approach to commensurability in their mean motions.

Hence, before entering upon the separate theories, M. Le Verrier devotes Chapter 25 of his researches to the determination of the mutual actions of *Uranus* and *Neptune*, and this forms the base of the theories of both planets.

The method employed is similar to that adopted in the case of *Jupiter* and *Saturn*, and the results are exhibited in the same general form.

It is important to remark that the elements of *Uranus* and *Neptune* as determined from observations severally differ from their mean elliptic values by the amount of their perturbations of long period corresponding to the mean epoch of the observations.

The apparent elements of *Uranus* and *Neptune* for the epoch 1850 have been carefully determined by Professor Newcomb in his excellent work on the theory of those planets which obtained the Society's Medal in 1874.

By the application of his own general formulæ, M. Le Verrier deduces from these elements the values of the mean elliptic elements corresponding to the same epoch.

It may be remarked that the mean elements thus determined will depend on the assumed masses of the two planets, and will therefore require small corrections when more accurate values of the masses have been obtained.

When the secular variations of *Uranus* and *Neptune* given in Chapter 19 were found, the elements were less accurately known, and M. Le Verrier has therefore recalculated the values of the eccentricities and longitudes of the perihelia of the two planets for the same five epochs as before, starting from the mean elliptic values of the elements above referred to.

Chapter 26 contains the completion of the theory of *Uranus*. The last chapter, which contains the completion of the theory of *Neptune*, is not yet printed.

The 23rd Chapter also, which contains the comparison of the theory of *Saturn* with observations, together with the tables of the planet, and which will form the latter part of the 12th Volume of the *Annals*, is not yet printed. The results of this comparison of the theory with observations have, however, been fully published in the *Comptes Rendus*, and I understand that the tables will be used for computing the place of *Saturn* in the forthcoming volume of the *Nautical Almanac*.

Although the comparison of the theory of *Saturn* with observations shews in general a satisfactory accordance, there occur some discrepancies in individual years which are larger than might be desired.

During the thirty-two years over which the modern observations extend, viz. from 1837 to 1869, the discrepancy between theory and observation, however, remains constantly less than $2''\cdot5$ of arc, excepting in two instances, viz. in the years 1839 and 1844, when the differences amount to $4''\cdot5$ of arc.

In the ancient observations only, made in the time of Maskelyne, rather larger differences occur, amounting in two instances to nearly $9''$ of arc.

In order to test whether these discrepancies could be due to any imperfections in the theory, M. Le Verrier has not shrunk from the immense labour of forming a second theory of the planet independent of the former, employing methods of interpolation instead of the analytical developments.

I learn directly from M. Le Verrier that this second investigation entirely confirms the accuracy of the first as regards the periodic inequalities, but that the secular variations of the eccentricity and longitude of the perihelion are slightly changed.

The effect of these changes is to bring the theory into very satisfactory accordance with the observations of Bradley, but the discrepancies above mentioned in the time of Maskelyne and in the modern observations still remain unaffected.

The character of the discrepancies shewn by the modern observations makes it very improbable that they can be due to any errors in the theory.

In fact, the error appears to change almost suddenly from a positive one of $4''\cdot4$ in 1839 to a negative one of $5''\cdot0$ in 1844, a variation of nearly $9''\cdot5$ in five years. Now no terms or group of terms due to the action of the planets could thus suddenly disturb the motion in five years, at a given epoch, and then leave the motion unaffected during the following twenty-five years.

M. Le Verrier is therefore inclined to think that the discrepancies arise from errors in the observations, notwithstanding that the Greenwich and Paris observations are mutually confirmatory of each other.

He suggests that it is possible that the varying aspects presented at different times by the ring may affect the accuracy of the observations of the planet, and may cause changes in the personal equations of the observers, which, from being rather large in the case of the ancient observations, have gone on diminishing as the system of observation has become more perfect.

One unlooked-for result follows from M. Le Verrier's comparison of his theory of *Saturn* with the observations. Considering that the influence of *Jupiter* on the longitude of *Saturn* may amount to $3800''$, it might have been expected that from observations of the planet extending over 120 years the mass of *Jupiter* could have been determined with great precision. M. Le Verrier has found, however, that this is not the case.

The equations of condition furnished by the comparison of the heliocentric longitudes of *Saturn* as deduced from theory and observation contain five unknown quantities, viz. the corrections of the assumed values of four elements and the correction of the assumed mass of *Jupiter*.

On solving the equations with respect to the first four unknown quantities, the corrections to be applied to the elements are found to be greatly influenced by the indeterminate correction of the mass of *Jupiter*, and after they have been substituted in the equations of condition, the coefficients of the correction of the mass of *Jupiter* in great part destroy each other, nowhere amounting in the resulting equations to one-tenth part of their values in the primitive equations. Hence these equations are insufficient to determine the mass of *Jupiter* with any precision.

Consequently, in the formation of the Tables of *Saturn*, M. Le Verrier has employed the value of the mass of *Jupiter* determined by the Astronomer Royal from his observations of the 4th satellite.

The result which has just been noticed will appear to be less paradoxical if we consider that by far the larger part of the disturbances which *Jupiter* produces in the motion of *Saturn* is represented by the inequalities of long period which affect the mean longitude and the elements of the orbit. Now in the course of 120 years these inequalities have run through only a small part of their whole period, and therefore, during this interval, the greater part of their effects may be represented by applying changes to the several mean elements equal to the mean value of the corresponding long inequalities during the interval. It is only from the residual disturbances, which are comparatively small in amount, that any data can be obtained for the correction of the mass of *Jupiter*.

In the course of a few centuries, when these long inequalities, as well as the secular variations of the elements of *Saturn*, shall have had time to develop themselves, it will be possible to determine the mass of *Jupiter* from them with all desirable precision.

I trust that the review which I have just given, however hasty and imperfect, of the work of our distinguished Associate has been sufficient to convince you that your Council have done well in according him your Medal.

In conclusion, I may be allowed to express the great satisfaction I have felt in becoming the mouthpiece of the Council on this occasion, and in thus joining in doing honour to the eminent Astronomer whose untiring labours have added so greatly to our knowledge of the motions of the principal members of our Solar System.

(*The President then, delivering the Medal to the Foreign Secretary, addressed him in the following terms*):—

Dr Huggins—In transmitting this Medal to M. Le Verrier, you will express to him the interest with which we have followed his unwearied researches, and the admiration which we feel for the skill and perseverance by which he has succeeded in binding all the principal planets of our system, from *Mercury* to *Neptune*, in the chains of his Analysis. You can tell him how sorry we are not to see him among us on the present occasion, and how glad we shall be to welcome him if he is able to visit us later in the session. We hope that he will then have finished the printing of his "Tables of *Saturn*" and his "Theory of *Neptune*," and thus be able to rest awhile and re-establish his health—shaken, we fear, by his too arduous labours—until he goes forth again, with fresh vigour, to win new triumphs in the fields of Physical Astronomy.

47.

ASTRONOMICAL OBSERVATIONS MADE AT THE OBSERVATORY OF CAMBRIDGE, UNDER THE SUPERINTENDENCE OF PROFESSOR ADAMS.

[Extracts from the Introduction to Vol. XXI. (1861—1865).]

Corrections for Collimation, Level, and Azimuth.

Up to the end of 1863 the corrections for Collimation, Level, and Azimuth were applied in the usual way, by the aid of Professor Challis's calculating machine: thence forward, they were thrown into the form

$$m + n \text{ cotan N.P.D.} + c \text{ cosec N.P.D.}$$

where c denotes the collimation error, considered positive when the angle between the line of sight and the eastern half of the axis is less than a right angle;

n, the elevation of the west end of the axis above the plane of the equator;

and m, the deviation of the west end of the axis southward in the plane of the equator.

m, n, and c are expressed in seconds of time.

It is easy to see that, if a and b denote the deviations of the axis horizontally and vertically, or the azimuthal and level errors, expressed in seconds of time, and ϕ the latitude,

$$m = \quad a \sin \phi + b \cos \phi = b \sec \phi - n \tan \phi,$$
$$n = -a \cos \phi + b \sin \phi,$$

consequently

$$a = \quad m \sin \phi - n \cos \phi = b \tan \phi - n \sec \phi.$$

The collimation and level errors were found by observing the reflection of the wires in a trough of mercury, with a Bohnenberger's eye-piece, before and after reversing the Instrument. The deviation of the line of sight from the vertical, in one position of the Instrument, which was assumed to be illumination West, being $b+c$, in the other position, illumination East, it will be $b-c$. The value of c thus obtained at any reversal of the Instrument was, up to the end of 1863, in most cases supposed constant till the next reversal and used for finding b by means of intermediate observations of the reflection of the wires. Subsequently mean values of c were generally taken.

This method assumes that the position of the Y's is unaltered during the process of reversal, a supposition which was by no means borne out by the examination of the pivots in May, 1864, and it was thought better to adopt some mode of determining the errors independently for each position of the Instrument.

In default of Collimating Telescopes, a star near the pole, usually Polaris, was observed both directly and by reflection at the same culmination; from the times of transit reduced to the centre wire and corrected for irregularity of Pivots, the level error was easily found thus,

if α be the star's Right Ascension, δ its Declination,

T the time of the direct observation, reduced to the centre wire and corrected for irregularity of Pivots,

T' the time of the reflected observation,

E the Clock correction,

a, b, c the Azimuth and Level errors, and the Collimation error of the centre wire,

$$\alpha = T + E + a\frac{\sin(\phi-\delta)}{\cos\delta} + b\frac{\cos(\phi-\delta)}{\cos\delta} + \frac{c}{\cos\delta}$$

$$= T' + E + a\frac{\sin(\phi-\delta)}{\cos\delta} - b\frac{\cos(\phi-\delta)}{\cos\delta} + \frac{c}{\cos\delta},$$

whence
$$T - T' + 2b\frac{\cos(\phi-\delta)}{\cos\delta} = 0,$$

and
$$b = \frac{1}{2}(T'-T)\frac{\cos\delta}{\cos(\phi-\delta)}.$$

The observation of the reflection of the wires gave $b+c$ or $b-c$; thence c was obtained. This mode was adopted almost exclusively from September 24, 1864, till the Instrument was finally dismounted.

The coefficient for diurnal aberration, $-0'',19 = -0^s,013$, is, in every case, incorporated with the Collimation error.

Correction for Curvature of Star's path.

When the object is not bisected precisely on the meridian a small correction is necessary for curvature of path.

For stars near the pole the correction (C) may be calculated from the formula

$$C = \frac{1}{\sin 1''} \sin 2\Delta \sin^2 \frac{\theta}{2},$$

where Δ is the North Polar Distance, and θ the hour angle.

Differentiating, and expressing $d\Delta$ in seconds of arc, we have

$$dC = 2 \cos 2\Delta \sin^2 \frac{\theta}{2} d\Delta.$$

So that, for the Polar Distance

$$\Delta + n'', \quad C = \frac{1}{\sin 1''} \sin 2\Delta \sin^2 \frac{\theta}{2} + 2 \cos 2\Delta \sin^2 \frac{\theta}{2} . n''.$$

For Polaris,

$$\Delta = 1^\circ\ 25' + n'', \quad C = [4{\cdot}00842] \sin^2 \frac{\theta}{2} + [0{\cdot}30050] \sin^2 \frac{\theta}{2} . n''.$$

For 51 Cephei,

$$\Delta = 2^\circ\ 46' + n'', \quad C = [4{\cdot}29861] \sin^2 \frac{\theta}{2} + [0{\cdot}29900] \sin^2 \frac{\theta}{2} . n''.$$

For δ Urs. Min.,

$$\Delta = 3^\circ\ 25' + n'', \quad C = [4{\cdot}38991] \sin^2 \frac{\theta}{2} + [0{\cdot}29793] \sin^2 \frac{\theta}{2} . n''.$$

For λ Urs. Min.,

$$\Delta = 1^\circ\ 6' + n'', \quad C = [3{\cdot}89862] \sin^2 \frac{\theta}{2} + [0{\cdot}30071] \sin^2 \frac{\theta}{2} . n''.$$

For convenience of calculation these quantities are given in Tables I, II, III, at the end of this Introduction, for values of the hour angle taken at intervals of 10^s and extending to a sufficient distance from the meridian.

When the star is not very near the pole, since θ is very small, we may write

$$\frac{1}{4}\sin^2\theta \text{ for } \sin^2\frac{\theta}{2}$$

which gives

$$\text{correction} = \frac{1}{2\sin 1''}\sin\Delta\cos\Delta\sin^2\theta.$$

But if E be the equatorial interval corresponding to the apparent distance from the meridian of the point at which the bisection was made, then

$$\sin\Delta\sin\theta = \sin E;$$

therefore

$$\sin^2\theta = \frac{\sin^2 E}{\sin^2\Delta},$$

and

$$\text{correction} = \frac{1}{2\sin 1''}\cot\Delta\sin^2 E;$$

or, if E be expressed in seconds of time,

$$\text{correction} = \frac{\sin^2 15''}{2\sin 1''}E^2\cot\Delta$$

$$= \frac{225}{2}\sin 1''\,.\,E^2\cot\Delta.$$

In the Mural Circle, one equatorial interval of the wires $= 16^s{\cdot}6$.

Hence, if I be the number of intervals in the distance of the point of bisection from the meridian,

$$\text{correction} = \frac{225}{2}\sin 1''\,(16{\cdot}6)^2 I^2\cot\Delta$$

$$= [9''{\cdot}17694]\,I^2\cot\Delta$$

$$= 0''{\cdot}1503\,I^2\cot\Delta.$$

In practice, the middle wire is always so nearly in the meridian that I may be taken to be the number of intervals in the distance of the point of bisection from the middle wire.

The values of the correction for different values of I and Δ are given in Table IV. at the end of this Introduction.

Correction for Change of Declination.

In the case of the Sun and Planets a small correction is required for the motion in Declination in the interval between the time of crossing the meridian and the time of observation.

This interval is $16^s{\cdot}6\, I \operatorname{cosec} \Delta,$

where I has the same signification as before, and therefore the correction will be

$$\frac{16^s{\cdot}6}{3600}\, I \operatorname{cosec} \Delta \times \text{Var. of Decl}^{\text{n}}. \text{ in 1 hour of longitude.}$$

The last factor is obtained from an Ephemeris.

The multiplier of I in this expression, or the value of the correction for one interval, is given by means of Table V. at the end of this Introduction, so that the correction may be deduced by multiplying the number taken from the Table by I, the number of intervals stated in the eleventh column. The sign to be given to the correction is stated in the precept at the foot of the Table.

The Micrometer-wire was always so nearly adjusted equatorially that no correction for error of its position has been thought necessary.

The Pointer, which is used for setting the Telescope to observe an object either directly or by reflection, the setting angle to the nearest minute having been previously computed, is placed below Microscope A at an interval of $10^\circ\ 45'$ nearly from the zero of its reading. The graduation proceeding in the direction from the microscope downwards, the Pointer reading is the number of degrees and minutes of that division which in the order of graduation comes next before the position of the Pointer.

It is unnecessary to place the Pointer reading in a separate column, as it may be at once inferred from the concluded Circle reading, the minutes being always an integral number of $5'$.

The concluded Circle reading in the *twelfth column* is the Pointer reading added to the mean of the Microscope readings with all the above-mentioned corrections applied. It is therefore the reading which would have been given by the Circle, if the microscopes had been in accurate adjustment for runs, and the object had been bisected by the fixed wire at the middle vertical wire. For the Polar stars the concluded reading applies to the time of meridian passage.

The Circle reading corresponding to the position of the Telescope when directed exactly to the zenith is called the *Zenith Point.*

The adopted Zenith point is obtained by means of the collimating eye-piece, and is therefore more strictly the Circle reading corresponding to the Nadir point increased by 180°.

The Collimating eye-piece employed is of the same form as that used by Professor Challis, and consists of a common inverting microscope of three lenses, to which is attached, beyond the third lens, a piece of plate-glass, inclined at an angle of 45° to the axis of the microscope. The eye-piece of the Telescope being removed, this apparatus is put in its place, so that the plate-glass is between the wires and the microscope; and when the Telescope is directed vertically to a trough of mercury, the wires and their images by reflection become visible as dark lines on a bright ground, by throwing the light of a lamp on the plate-glass.

The Micrometer reading for coincidence of the micrometer-wire with its image is deduced from at least six readings for coincidence, or for alternate contact.

The Microscope readings for the determination of the Zenith point are inserted among those for the observations of the celestial objects named in the second column. The concluded Circle reading obtained by reducing an observation of Nadir point in the same manner as the other observations are reduced, and then increasing the result by 180°, is in general the adopted Zenith point. The limits within which any value is used are indicated by bars across the column of "concluded circle readings." If two observations of Zenith point occur within the same limits, the value used is the mean between the two results.

The temperature of the Circle room at the times of taking the Zenith point is given in the Table of observations of Runs.

The apparent Zenith distance in the direct observation of any object is the algebraic excess of the concluded Circle reading above the adopted Zenith point, and for a reflection observation it is the algebraic excess of the Nadir point above the concluded Circle reading. The object is South or North of the zenith according as the excess is in either case positive or negative. The apparent Zenith distance thus obtained is used with the data in the three next columns for the calculation of *refraction.*

The thirteenth column contains the height of the barometer, as shewn by a cistern-barometer constructed by Dollond and attached to the Circle pier. The lower surface of the mercury is raised by a screw pressing the bag till the light seen below a brass edge is excluded; and a brass slider is brought to the upper surface to shut out the light in the same way.

Before calculating the refraction, a correction of +0·01 in. was applied to these Barometer-readings [see Introduction to Vol. xx., p. cxvi.] for Index-error; but a comparison with a very fine Standard Barometer by Adie, which was mounted in the Transit Room in July, 1872, seems to shew that this correction is too small. A large number of comparisons made between August, 1872, and the end of the year, shew that the reading of Adie's Barometer exceeds that of Dollond's by 0·055 in., and the correction of Adie's Barometer, by comparisons with the Standard Barometer at Kew, is only −0·001 in. Probably the error of the old Barometer had been gradually increasing.

The fourteenth column contains the reading of the thermometer whose bulb is plunged in the cistern of the barometer.

The fifteenth column contains the reading of an external thermometer, which is fixed to a stage near the north shutter-opening at a distance of four feet from the wall of the building and nine feet from the ground. It is protected from radiation and from the weather, and contiguous parts of the building prevent the direct rays of the Sun from falling upon it.

The refraction is calculated by Bessel's Tables, using the convenient form in which they are given in the Appendix to the Greenwich Observations for 1836. In this mode of calculation the reading of the attached is supposed to be the same as that of the external thermometer. The former reading, though not made use of, is inserted in the printed columns, to allow of correcting for the error of this supposition, if it is thought necessary.

By adding the refraction to the apparent Zenith distance North or South, the true Zenith distance is found, and by adding algebraically the true Zenith distance, considered negative when north of the Zenith, to the assumed co-latitude of the Observatory, viz. 37° 47′ 8″·00, the

Apparent N.P.D. from the observation, given in the seventeenth column, is obtained. Accordingly, when a circumpolar star is observed below the pole, in which case S.P. is appended to the name of the star in the second column, this apparent N.P.D. is affected with the negative sign.

Occultations of Fixed Stars by the Moon.

The following are the formulæ employed in obtaining the Equations of Condition given in this volume.

Let T = mean local time of observation.

l = assumed longitude of place of observation, + when West.

$T+l=t$ = approximate time on first meridian.

α, δ, the Moon's Right Ascension and Declination.

π, σ, the horizontal equatorial parallax and semi-diameter, all calculated from the Ephemeris for the time t.

Up to the end of 1861 the quantities given in the *Nautical Almanac* are

$$\frac{\sin \pi}{\sin 1''}, \text{ and } \frac{\sin \sigma}{\sin 1''};$$

subsequently the quantities given are π and $\dfrac{\sin \sigma}{\sin 1''}$.

Hansen gives $\sigma = [4{\cdot}750519] \sin \pi = [9{\cdot}436094] \dfrac{\sin \pi}{\sin 1''}$.

ρ = radius vector of place of observation, taking the Earth's equatorial radius to be unity.

ϕ' = geocentric latitude.

θ = sidereal time corresponding to time T.

α', δ', the Right Ascension and Declination of the star occulted.

Find
$$x = \frac{\sin (\alpha - \alpha')}{\sin 1''} \cos \delta,$$

$$y = \frac{\sin (\delta - \delta')}{\sin 1''} + x \sin \delta' \tan \tfrac{1}{2} (\alpha - \alpha'),$$

$$\xi = \frac{\sin \pi}{\sin 1''} \cdot \rho \cos \phi' \sin (\theta - \alpha'),$$

$$\eta = \frac{\sin \pi}{\sin 1''} \rho \{\sin \phi' \cos \delta' - \cos \phi' \sin \delta' \cos (\theta - \alpha')\},$$

$$\tan \chi = \frac{y - \eta}{x - \xi},$$

$$S = \frac{x - \xi}{\cos \chi} = \frac{y - \eta}{\sin \chi}.$$

Also let Δl be the correction of the assumed longitude in seconds of time;

ΔT the correction of T in seconds of time;

$\Delta\alpha$, $\Delta\delta$, &c. the corrections of α, δ, &c. in seconds of arc;

$\frac{d\alpha}{dt}$ and $\frac{d\delta}{dt}$, the changes of α and δ in a second of time, estimated in seconds of arc;

$\frac{\sin \pi}{\sin 1''}(1+p)$ and $\frac{\sin \sigma}{\sin 1''}(1+s)$, the sines of the true horizontal equatorial parallax and semi-diameter, each divided by sin 1″.

Calculate the following quantities:

$$(\alpha) = \cos \delta\, [\cos \chi + \sin \chi \sin \delta' \sin (\alpha - \alpha')],$$

$$(\delta) = \sin \chi - \cos \chi \sin \delta \sin (\alpha - \alpha'),$$

$$(l) = (\alpha) \frac{d\alpha}{dt} + (\delta) \frac{d\delta}{dt},$$

$$m = \rho \sin \pi \cos \phi' [\cos \chi \cos (\theta - \alpha') + \sin \chi \sin \delta' \sin (\theta - \alpha')],$$

$$(\alpha') = m - (\alpha),$$

$$(\delta') = \rho \sin \pi \sin \chi [\sin \phi' \sin \delta' + \cos \phi' \cos \delta' \cos (\theta - \alpha')] - \sin \chi,$$

$$(T) = (l) - (1{\cdot}00274)\, 15m,$$

$$(\phi') = -\rho \sin \pi [\sin \chi \cos \phi' \cos \delta' + \sin \chi \sin \phi' \sin \delta' \cos (\theta - \alpha') - \cos \chi \sin \phi' \sin (\theta - \alpha')],$$

$$(p) = -\xi \cos \chi - \eta \sin \chi,$$

$$(s) = -\frac{\sin \sigma}{\sin 1''}.$$

Then the final equation of condition will be

$$\frac{\sin \sigma}{\sin 1''} - S = (\alpha)\, \Delta\alpha + (\delta)\, \Delta\delta + (\alpha')\, \Delta\alpha' + (\delta')\, \Delta\delta' + (T)\, \Delta T + (l)\, \Delta l + (\phi')\, \Delta\phi' + (p)\, p + (s)\, s.$$

Correction for Refraction.

The seventh and eighth columns contain the excess of the Comet's refraction above that of the Star, in Right Ascension and North Polar Distance respectively.

If the Transits of the two objects be observed across a wire placed accurately in the apparent circle of declination, which is usually the case in these observations, we shall have

Excess of Comet's refraction in R.A. in seconds of time

$$= \Delta\delta \times k \sec^2(\delta' - PQ) \frac{\tan ZQ}{15} \cos(2\delta' - PQ) \operatorname{cosec}^2 \delta',$$

Excess of Comet's refraction in N.P.D. $= \Delta\delta \times k \sec^2(\delta' - PQ)$.

Where the symbols have the following significations :

$\Delta\delta$ is the excess of the Comet's N.P.D. in seconds of arc,

PZM being the spherical triangle formed by the pole, the zenith and the middle point between the true places of the Comet and the Star, ZQ is the perpendicular from Z upon PM.

δ' is the N.P.D. of the point M, or the mean of the N.P.D. of the two bodies.

k is a quantity depending on the zenith distance of M, and on the state of the barometer and thermometer.

PQ and ZQ are found from the hour angle (h) by means of the equations

$$\tan PQ = \cot\phi \cos h$$

$$\cos ZQ = \frac{\cos\phi \cos h}{\sin PQ} = \frac{\sin\phi}{\cos PQ},$$

where ϕ is the latitude of the Observatory.

Also ζ, the zenith distance of M, is given by the equation

$$\cos\zeta = \cos ZQ \cos(\delta' - PQ).$$

These formulæ are equivalent to those of Bessel in his *Untersuchungen*, Band I. p. 168, PQ being the quantity there denoted by N, and ZQ being the complement of n.

Professor Challis has constructed Tables similar to Bessel's, and specially adapted to facilitate the calculation of refraction for this Observatory. These tables, together with the precepts for their use, are printed at the end of this Introduction. By their means the total refractions in R.A. and N.P.D. may be found if required, as well as the differential refractions spoken of above.

When the Comet is compared with a Star in N.P.D. only, with the Clock going, it is usual to bisect the two objects alternately, beginning and ending with the Star.

The micrometer readings for the Star will vary in consequence of the variation of the refraction in N.P.D. From two consecutive readings, the reading corresponding to the intermediate time of bisection of the Comet may be deduced on the supposition that the readings vary proportionally to the time, and the result may be treated as if the bisections of the Comet and the Star had been simultaneous.

In this case, if $\Delta\alpha$ and $\Delta\delta$ denote the approximate excesses of the Comet's R.A. and N.P.D. respectively, we have

Excess of the Comet's refraction in N.P.D.

$$= -\frac{15k}{\cos^2\zeta}\sin\phi\cos\phi\sin h\times\Delta\alpha + \frac{k}{\cos^2\zeta}\left[1-\cos^2\phi\sin^2 h\right]\times\Delta\delta,$$

where the other symbols have the same signification as before.

For the observations of Mars made in 1862, for the purpose of determining the Sun's Parallax, the micrometer-wire was adjusted so as to be at right angles to the apparent diurnal path of a star across the field of view.

In this case, we have

True excess of the planet's R.A. above that of the star

$$= \text{apparent excess of planet's R.A.} - \frac{2k}{\cos^2(\delta'-PQ)}\cdot\frac{\tan ZQ}{15}\cdot\frac{\sin(\delta'-PQ)}{\sin\delta'}\times\Delta\delta,$$

employing the same notation as before.

The ninth and tenth columns respectively contain the excesses of the Comet's R.A. and N.P.D. above the R.A. and N.P.D. of the Star, as given by the observations when cleared from the effects of refraction.

In the same columns are placed the coefficients for finding the Comet's Parallax in R.A. and N.P.D. respectively. From the nature of the case, no confusion can arise from placing two such different quantities in the same column, half of the space in which would otherwise be wasted.

In cases in which each comparison with a Star is complete in itself, the differences of R.A. and N.P.D. are placed opposite to the name of the Star, and the coefficients of Parallax opposite to that of the Comet; but in the cases in which the observations are made with the clock going, and each bisection of the Comet is compared with the result obtained from combining the two bisections of the Star which immediately precede and follow it, the differences of R.A. and N.P.D. are placed opposite to the Comet and the coefficients of Parallax opposite to the Star, and usually in the line above the former quantities.

These coefficients represent respectively

$$\text{Comet's Parallax in R.A.} \times \Delta$$

$$\text{and Comet's Parallax in N.P.D.} \times \Delta,$$

where Δ is the distance of the Comet from the Earth, considering the Earth's mean distance from the Sun to be unity.

Hence, to find the Parallax in R.A. and in N.P.D. respectively, these coefficients must be divided by Δ.

If $PZ'C$ be the spherical triangle formed by the pole, the geocentric zenith and the apparent place of the Comet, and if $Z'Q'$ be a perpendicular from Z' upon PC, then the values of these coefficients will be as follows:

$$\text{For R.A.}\quad \text{Coefficient} = \frac{\rho\pi \cos\phi' \sin h}{15 \sin\delta} = \frac{\rho\pi \sin Z'Q'}{15 \sin\delta},$$

$$\text{For N.P.D.}\quad \text{Coefficient} = -\frac{\rho\pi \sin\phi' \sin(\delta - PQ')}{\cos PQ'} = -\rho\pi \cos Z'Q' \sin(\delta - PQ'),$$

where π denotes the Sun's mean equatorial horizontal parallax,

ρ the distance of the point of observation from the Earth's centre, considering the equatorial radius to be unity,

ϕ' the reduced or geocentric latitude,

h the hour angle,

and δ the N.P.D. of the Comet or Planet.

The quantities PQ' and $Z'Q'$ are given by the equations

$$\tan PQ' = \cot\phi' \cos h,$$

$$\sin Z'Q' = \cos\phi' \sin h, \quad \text{or} \quad \cos Z'Q' = \frac{\sin\phi'}{\cos PQ'}.$$

48.

ON THE MEAN PLACES OF 84 FUNDAMENTAL STARS, AS DERIVED FROM THE PLACES GIVEN IN THE GREENWICH CATALOGUES FOR 1840 AND 1845, WHEN COMPARED WITH THOSE RESULTING FROM BRADLEY'S OBSERVATIONS.

[From *Appendix II. to Astronomical Observations made at the Cambridge Observatory.* Vol. XXII. (1866—1869.)]

INTRODUCTION.

THE present Appendix contains the formulæ and instructions which I drew up, many years ago, for the formation of a proposed New Fundamental Catalogue, to be used in the computation of the Star places given in the *Nautical Almanac.* The proposed plan was eagerly accepted by my friend, the late Lieutenant Stratford, who was then the superintendent, and my instructions were ably carried out by Mr R. Farley, then the principal assistant in the *Nautical Almanac* Office. The mean places were thus calculated for the beginning of each of Bessel's so called fictitious years from 1830 to 1870. The results for the years from 1857 to 1870 inclusive have already appeared in the several volumes of the *Nautical Almanac.* It has been thought desirable to collect together these results as well as those for the previous years, so as to exhibit at one view a set of mean places of each star, for the beginning of each year from 1830 to 1870, founded on consistent elements. It should be remarked that in all these calculations the actual proper motion of each star is supposed to be uniform and to take place in a fixed great circle. Hence no attempt is made to take into account the variability in the observed proper motions of

Sirius and Procyon. Indeed one of the principal objects which I had in view in the formation of this Catalogue was to test how far the observed proper motions of those stars which had been long and carefully observed, could be reconciled with the hypothesis that the proper motion, when referred to the equator or ecliptic of a given date, was really uniform.

The rule laid down in my instructions to Mr Farley embodies a very simple mode of representing the apparent variability of proper motion arising from the change of position of the great circles to which the star is referred, whenever the star is not very near to the pole.

When the star is very near the pole, the Right Ascension and Declination for the time $1800+t$ when referred to the Equator and Equinox of 1800 is first found by adding the proper motions in R.A. and Decl. for t years to the Right Ascension and Declination for 1800, and then this Right Ascension and Declination is converted into the corresponding Right Ascension and Declination referred to the Equator and Equinox of $1800+t$ by the proper Trigonometrical formulæ given below. These formulæ are founded upon the elements of precession given by Dr Peters in his classical work *Numerus Constans Nutationis*. It should be noticed that the corresponding formulæ given by Mr Carrington at p. xxx of the Introduction to his valuable Catalogue of Circumpolar Stars are not sufficiently accurate. The quantities which he denotes by $z+\nu$, $z'-\nu'$ and θ, and which he employs in reducing the place of a star from one epoch $1800+t$ to another $1800+t'$, ought to vanish identically when $t=t'$, whereas, according to Mr Carrington's Table of Precession Constants, when $t=t'=55$, the value of $z+\nu$ is $-0''\cdot73$ and that of $z'-\nu'$ is $+0''\cdot73$.

In the rule which I gave to Mr Farley for forming the value of the secular variation of the Precession to be employed in reducing the observed Right Ascension and Declination from 1840 to 1845, it is not taken into account that different Elements of Precession are employed by Argelander and Bessel from those which are employed in the *Nautical Almanac*. The slight inaccuracy thence arising will, however, scarcely be appreciable.

It should be remarked that the Polar Star 51 Cephei was not observed by Bradley, and consequently that this star, although included among the 84 Stars to which Mr Farley's calculations refer, does not, properly speaking, fall within the scope of my plan. The coordinates of this star for 1800, which I gave to Mr Farley as part of his fundamental data, were the means of two discordant determinations of those elements by Piazzi. Hence it is not surprising that the predicted places of this star when tested by comparison with more recent observations, should prove to be sensibly in error.

The following Table gives the places and the proper motions for 1800 of the remaining 83 stars embraced in the calculations.

MEAN PLACES AND ANNUAL PROPER MOTIONS FOR 1800, DEDUCED FROM PLACES FOR 1755 AND 1845 AND PRECESSIONS FOR 1755, 1800 AND 1845.

Name of Star	Mean R.A. 1800.0	Annual Proper Motion	Mean Decl. 1800.0	Annual Proper Motion
	h. m. s.	s.	° ′ ″	″
γ Pegasi	0. 2.57,112	−0,00087	14. 4.16,02	−0,0193
α Cassiop.	0.29.14,688	+0,00610	55.26.18,02	−0,0393
β Ceti	0.33.32,660	+0,01291	−19. 5.11,77	+0,0207
Polaris	0.52.25,375	+0,08822	88.14.24,49	+0,0055
θ¹ Ceti	1.14. 1,762	−0,00665	− 9.13.10,81	−0,2204
α Arietis	1.55.55,763	+0,01290	22.30.34,96	−0,1487
γ Ceti	2.32.57,049	−0,01047	2.23. 6,73	−0,1823
α Ceti	2.51.50,367	−0,00277	3.17.47,92	−0,1114
α Persei	3.10. 7,011	+0,00288	49. 8.11,48	−0,0487
η Tauri	3.35.37,319	−0,00031	23.28.30,58	−0,0600
γ¹ Eridani	3.48.42,288	+0,00259	−14. 5.13,39	−0,1162
α Tauri	4.24.27,571	+0,00423	16. 5.39,11	−0,1747
α Aurigæ	5. 1.56,233	+0,00863	45.46.38,07	−0,4294
β Orionis	5. 4.55,918	−0,00090	− 8.26.37,83	−0,0202
β Tauri	5.13.39,578	+0,00157	28.25.25,58	−0,1980
δ Orionis	5.21.47,582	+0,00113	− 0.27.32,14	−0,0380
α Leporis	5.23.54,707	+0,00167	−17.58.33,48	+0,0042
ε Orionis	5.26. 4,201	−0,00091	− 1.20.29,95	−0,0148
α Orionis	5.44.20,863	+0,00108	7.21.24,58	−0,0026
μ Geminorum	6.10.51,481	+0,00540	22.36. 7,10	−0,1269
α Can. Maj.	6.36.20,106	−0,03520	−16.27. 7,75	−1,2273
ε Can. Maj.	6.50.46,005	+0,00075	−28.42.32,65	−0,0109
δ Geminorum	7. 8. 9,908	+0,00007	22.20.13,49	−0,0160
α² Geminorum	7.21.48,902	−0,01238	32.18.43,73	−0,0758
α Can. Min.	7.28.49,438	−0,04674	5.43.35,76	−1,0351
β Geminorum	7.33. 3,462	−0,04772	28.29.46,37	−0,0619
15 Argus	7.59. 1,667	−0,00615	−23.44.11,91	+0,0668
ε Hydræ	8.36.10,393	−0,01223	7. 8.34,54	−0,0384
ι Ursæ Maj.	8.45.26,714	−0,04659	48.48.57,75	−0,2769
α Hydræ	9.17.45,456	−0,00214	− 7.47.56,31	+0,0322
θ Ursæ Maj.	9.19.23,808	−0,10677	52.34.46,70	−0,5656
ε Leonis	9.34.28,320	−0,00402	24.41.15,97	−0,0182
α Leonis	9.57.42,369	−0,01770	12.56.19,30	+0,0086
α Ursæ Maj.	10.51.15,542	−0,01647	62.49.38,58	−0,0888
δ Leonis	11. 3.27,011	+0,01167	21.37. 1,82	−0,1441
δ Hyd. & Crateris	11. 9.21,011	−0,00876	−13.41.52,38	+0,1777
β Leonis	11.38.50,858	−0,03532	15.41.21,56	−0,1022
γ Ursæ Maj.	11.43.14,559	+0,01142	54.48.24,17	−0,0042
β Corvi	12.23.54,679	−0,00737	−22.17.19,90	−0,0673
12 Can. Ven.	12.46.38,984	−0,02185	39.24. 4,58	+0,0573
α Virginis	13.14.40,472	−0,00445	−10. 6.46,22	−0,0386
η Ursæ Maj.	13.39.38,578	−0,01176	50.18.57,68	−0,0231
η Bootis	13.45. 9,590	−0,00362	19.24.20,17	−0,3543
α Bootis	14. 6.32,585	−0,08003	20.13.46,04	−1,9747
ε Bootis	14.36.15,145	−0,00467	27.55.28,28	+0,0046
α² Libræ	14.39.50,311	−0,00927	−15.12. 6,88	−0,0592
β Ursæ Min.	14.51.26,890	−0,00565	74.58.23,66	−0,0361

MEAN PLACES AND ANNUAL PROPER MOTIONS FOR 1800, DEDUCED FROM PLACES FOR 1755 AND 1845 AND PRECESSIONS FOR 1755, 1800 AND 1845.

Name of Star	Mean R.A. 1800.0	Annual Proper Motion	Mean Decl. 1800.0	Annual Proper Motion
	h. m. s.	s.	° ′ ″	″
β Libræ	15 . 6 . 15,726	−0,00768	− 8 . 38 . 7,19	−0,0146
α Cor. Bor.	15 . 26 . 13,406	+0,00813	27 . 23 . 44,55	−0,0730
α Serpentis	15 . 34 . 25,570	+0,00744	7 . 3 . 51,91	+0,0553
β[1] Scorpii	15 . 53 . 49,729	−0,00131	−19 . 14 . 44,95	−0,0202
δ Ophiuchi	16 . 3 . 52,659	−0,00524	− 3 . 10 . 6,94	−0,1222
α Scorpii	16 . 17 . 10,043	−0,00195	−25 . 58 . 28,37	−0,0287
ε Ursæ Min.	17 . 6 . 57,962	+0,01472	82 . 20 . 33,63	−0,0012
α Herculis	17 . 5 . 31,976	−0,00193	14 . 37 . 44,02	+0,0441
β Draconis	17 . 25 . 55,273	−0,00284	52 . 27 . 17,71	+0,0027
α Ophiuchi	17 . 25 . 39,375	+0,00604	12 . 42 . 58,90	−0,2101
γ Draconis	17 . 51 . 57,881	+0,00077	51 . 31 . 5,12	−0,0396
μ[1] Sagittarii	18 . 1 . 48,341	−0,00313	−21 . 5 . 48,16	−0,0063
α Lyræ	18 . 30 . 10,051	+0,01747	38 . 36 . 19,75	+0,2854
δ Ursæ Min.	18 . 36 . 38,748	+0,03237	86 . 33 . 43,42	+0,0231
β Lyræ	18 . 42 . 41,906	−0,00181	33 . 8 . 21,26	−0,0282
ζ Aquilæ	18 . 56 . 13,274	−0,00571	13 . 34 . 35,13	−0,0732
δ Aquilæ	19 . 15 . 24,798	+0,01465	2 . 43 . 37,11	+0,0983
γ Aquilæ	19 . 36 . 45,029	−0,00054	10 . 8 . 9,47	+0,0028
α Aquilæ	19 . 41 . 1,390	+0,03526	8 . 21 . 1,96	+0,3785
β Aquilæ	19 . 45 . 29,267	+0,00076	5 . 55 . 2,55	−0,4769
α[2] Capricorni	20 . 6 . 56,817	+0,00170	−13 . 9 . 13,73	−0,0003
α Cygni	20 . 34 . 37,004	−0,00043	44 . 34 . 18,28	+0,0005
λ Ursæ Min.	20 . 51 . 33,984	−0,05293	88 . 41 . 16,41	+0,0123
61[1] Cygni	20 . 57 . 56,873	+0,33999	37 . 46 . 24,22	+3,2233
ζ Cygni	21 . 4 . 25,881	−0,00264	29 . 24 . 47,98	−0,0695
α Cephei	21 . 13 . 47,721	+0,02174	61 . 44 . 31,83	+0,0052
β Aquarii	21 . 21 . 1,193	+0,00014	− 6 . 26 . 36,99	+0,0053
β Cephei	21 . 26 . 1,574	+0,00084	69 . 41 . 6,41	−0,0412
ε Pegasi	21 . 34 . 21,675	+0,00282	8 . 57 . 52,37	+0,0020
α Aquarii	21 . 55 . 30,413	−0,00098	− 1 . 17 . 8,49	−0,0130
ζ Pegasi	22 . 31 . 29,549	+0,00177	9 . 47 . 28,06	+0,0025
α Pisc. Aust.	22 . 46 . 34,099	+0,02319	−30 . 40 . 42,46	−0,1745
α Pegasi	22 . 54 . 48,447	+0,00307	14 . 7 . 54,00	−0,0218
ι Piscium	23 . 29 . 40,032	+0,02554	4 . 32 . 36,90	−0,4512
γ Cephei	23 . 31 . 15,471	−0,01994	76 . 31 . 0,21	+0,1516
α Andromedæ	23 . 58 . 4,639	+0,00886	27 . 59 . 8,39	−0,1542

Mr Farley has remarked that one of these stars, viz. ε Ursæ Minoris, is too near the pole to allow the treatment of it as an ordinary Non-polar Star to be quite satisfactory. In this case it would be preferable to use the formulæ for the reduction of star places which are specially appropriate to the Polar Stars. In two other cases, viz. β Ursæ Minoris and γ Cephei, the polar distances, though larger, are sufficiently small to make it expedient to use the same formulæ when the greatest degree of accuracy is required.

ON A PROPOSED NEW FUNDAMENTAL CATALOGUE.

I have frequently felt great inconvenience from the changes which have been made from time to time, in the Fundamental places of the Standard Stars in the *Nautical Almanac*. At present, also, different astronomers use different Fundamental places, so that it is impossible accurately to compare the observations made at different observatories, or at the same observatory in different years, without a troublesome preliminary investigation of the mean differences of the several catalogues employed to determine the Clock error.

The appearance of the Greenwich Twelve-year Catalogue seems to me to afford an excellent opportunity for the formation of such a catalogue as astronomers in general would be likely to employ in the reduction of their observations. By comparing the places in the Greenwich Catalogue with those of Bradley given in Bessel's *Fundamenta*, places would be obtained, which for many years to come, might be more depended on, than those given by a year or two's observations, however near these might be to the time for which the places were wanted. In order, however, to ensure this general assent of astronomers and to do justice to the excellence of the materials, the most scrupulous accuracy should be attended to in the reduction of the places to the proposed epoch, and in the calculation of the coefficients of the 1st and 2nd powers of the time which are required and wanted in order to find the places for any other epoch.

A short Appendix should be added to the *Nautical Almanac* in which the proposed Catalogue is given, fully explaining the method employed in its formation, in order that astronomers might use it with confidence.

I proceed to point out the method which it appears to me most desirable to adopt for this purpose.

The R.A. for 1840 and 1845 given in the Greenwich Catalogue are not referred to the same Fundamental position of the Equinox.

The mean corrections of the R.A. of the Fundamental Catalogue in the *Nautical Almanac* for 1834, given by the observations of the first 6 years and of the last 6 years, differ by $0^{s}{\cdot}067$. Part of this difference, however, arises from the proper motions having been omitted, except in a few cases, in the *Nautical Almanac* Catalogue, so that the mean corrections would vary with the time. By the comparison of the R.A. for 1840 and 1845, of the 30 stars common to the Greenwich Clock List and the *Tabulæ Regiomontanæ*, using as a basis Bradley's places for 1755, I find that in order to refer the R.A. to the most probable position of the Equinox as determined from the observations of the whole 12 years, the R.A. for 1840 must be increased by $0^s{\cdot}028$ and those for 1845 diminished by the same quantity.

The mean epoch of the observations on which the Catalogue for 1840 depends is the beginning of 1839, and the observations may be looked upon as giving the places for that time, independently of any assumed proper motion. The proper motions for 1 year should therefore be added to the places for 1840 of those stars whose proper motions have not been taken into account, and to the places of the other stars should be added, for the sake of uniformity,

Adopted proper motion for 1 year − Proper motion employed in the reductions.

The proper motions employed may be those given in the Fundamental Catalogue in the *Nautical Almanac* for 1848, which are those of Argelander as far as he gives them, the rest being taken from the B.A. Catalogue.

The proper motions used by the Astronomer Royal in his reductions are those given in the *Nautical Almanac* for 1834. For two stars, proper motions are mentioned in the notes to the Catalogue of 1439 stars, which are not given in the *Nautical Almanac*, viz. for α *Aquilæ*, a proper motion of $-0''{\cdot}32$ in N.P.D., and for ι *Piscium*, a proper motion of $+0^s{\cdot}025$ in R.A., both being taken from Baily. These however are not included in the Annual Precessions of that Catalogue, and I am not quite certain that they have been used in obtaining the places for 1840. The Astronomer Royal should be consulted on this point.

The R.A. for 1755 given in the *Fundamenta* should be diminished by $0^s{\cdot}020$ in consequence of Bessel having employed too large a value of the coefficient of nutation in his reductions.

The next step is to reduce the places for 1840 to the epoch 1845.

If α denote the R.A. for 1755, $\alpha_{,}$ that for 1840, and half the secular variation of the precession in R.A. be denoted by p, as in the *Nautical Almanac* Catalogue, then the R.A. for 1845 will be

$$\alpha_{,} + \frac{\alpha_{,} - \alpha}{17} + \frac{9}{2}p,$$

and similarly for the Declination.

The value of p may be taken at once from the *Nautical Almanac* for 1848. The value there given, however, does not include the small terms due to proper motion, and they are only partially included in the secular variations of precession given by Argelander and Bessel.

To be rigorously exact, we should take for the value of p

Secular Variation of Precession from Argelander or Bessel – Value of p given in *Nautical Almanac.*

Argelander gives the secular variation in his Catalogue; and for stars not in that Catalogue, it may be deduced from the change of precession for 45 years, given in the *Fundamenta,* bearing in mind that Bessel's precessions in R.A. are expressed in *arc.*

From the places thus reduced to 1845 and those given for the same epoch in the Greenwich Catalogue, the final places are to be deduced, giving to each determination a weight proportionate to the number of observations on which it depends.

The precessions should be calculated for 3 epochs, viz., 1755, 1800 and 1845. M. Peters' elements of precession should be employed; these are given by M. Struve in the *Astron. Nachr.* No. 486, and are founded on Otto Struve's investigations respecting precession combined with Le Verrier's determination of the changes of the plane of the Ecliptic.

The constants to be employed are:

For 1755.

$$m = 46''{\cdot}0495 \qquad \log n = 1{\cdot}302430,$$

$$\frac{m}{15} = 3{\cdot}06997 \qquad \log \frac{n}{15} = 0{\cdot}126339.$$

For 1800.

$$m = 46''{\cdot}0623 \qquad \log n = 1{\cdot}302346$$

$$\frac{m}{15} = 3{\cdot}07082 \qquad \log \frac{n}{15} = 0{\cdot}126255$$

For 1845.

$$m = 46''{\cdot}0751 \qquad \log n = 1{\cdot}302262,$$

$$\frac{m}{15} = 3{\cdot}07167 \qquad \log \frac{n}{15} = 0{\cdot}126171.$$

If a denote the R.A. in 1755 and a' the R.A. finally adopted for 1845, the R.A. for 1800 will be

$$\frac{1}{2}(a + a') - 20{\cdot}25\,p,$$

p having the same signification as before.

Similarly, the Declination for 1800 may be found.

Hence the precession in R.A. for 1800 may be calculated. Let this $= c$. Then the proper motion in R.A. for the same epoch will be

$$\frac{a' - a}{90} - c,$$

and similar formulæ hold for the Declination.

In consequence of the change of the plane to which the stars are referred, the proper motions in R.A. and Declination will not be strictly uniform, even if the actual proper motions be so. This variability of the proper motion may be very conveniently taken into account in the following manner.

To the R.A. and Declination for 1845 *add* the proper motions for 45 years just found, and with the places thus obtained calculate the precessions. These combined with the proper motions found for 1800 will give very approximately the annual variations for 1845.

Similarly, from the R.A. and Declination for 1755 *subtract* the proper motions for 45 years, and with the places thus obtained calculate the precessions. These combined with the proper motions for 1800 will give very approximately the annual variations for 1755.

Now let $c_{\prime}$ be the annual precession calculated in this way for 1755, c that for 1800, and c' that for 1845, and let the differences of these quantities be taken according to the following scheme,—

$$\begin{array}{ccc} c_{\prime} & & \\ & \Delta c_{\prime} & \\ c & & \Delta^2 c \\ & \Delta c & \\ c'. & & \end{array}$$

Then one-half the secular variation of precession for 1850,

or
$$p=\frac{10}{9}\left\{\Delta c+\frac{11}{18}\Delta^2 c\right\}.$$

Annual rate of variation for 1850,

or
$$k=\frac{\alpha'-\alpha}{90}+p-\frac{127}{162}\Delta^2 c,$$

α' and α being as before the R.A. for 1845 and 1755 respectively.

Also, R.A. for 1850,
$$=\alpha'+5k-\frac{1}{4}p+\frac{5}{486}\Delta^2 c.$$

Similar formulæ, of course, hold for the Declination.

If the difference between the determinations for 1845 exceed $0^s{\cdot}05$ for R.A. or 1″ for Declination, it should be ascertained whether the places have been rightly derived from those given in the several volumes of the Greenwich Observations. I found, for instance, a discrepancy in the R.A. of α Ceti, and on examination it appeared that the R.A. for 1840 should be $2^h\,53^m\,55^s{\cdot}23$ instead of $2^h\,53^m\,55^s{\cdot}32$; the correction $-0^s{\cdot}09$ mentioned in the Introduction to the Catalogue having apparently been omitted.

The calculation of the Fundamental places should be carried to 3 places of decimals in R.A., and 2 in Declination, and the calculation of the Precessions and Secular Variations should be carried to 5 places in R.A. and 4 in Declination.

I may mention here that the Secular Variations of Precession given in the British Association Catalogue do not include the terms which depend on the variation of m and n. Also that for Bradley's Stars the proper motions are calculated by using Bessel's old values of the precession given in the *Fundamenta*, and therefore ought not to be combined with the annual precessions given in the same Catalogue, which are founded on his later elements. Consequently, with the Precessions, Secular Variations, and proper motions of the Catalogue, we cannot reproduce the places for 1755, which were taken as the basis of calculation.

Example of the Application of the Method just explained to find the Place &c. of α Canis Majoris for 1850.

		R. A.		Decl.	
		s.		″	
	Prop. motion (Arg.)	−0·035		−1·23	
	Do. employed by Airy	−0·034		−1·14	
	Difference	−0·001		−0·09	
	const.	+0·028			
	Gr. Catalogue 1840	6 38 5·89		−16 30 6·98	
$a_{,}$	Adopted place 1840	6 38 5·917		−16 30 7·07	
a	Do. 1755	6 34 20·953	=Bessel's R.A. − 0s·020	−16 23 53·80	
	17)	3 44·964	17)	−6 13·27	
		13·233		−21·96	
	Sec. Variation from Argelander	+0·0004		−0·379	
	(p) Naut. Almanac	+0·00061		−0·1919	
	Difference = p adopted	−0·00021	p'	−0·1871	
		h. m. s.		° ′ ″	
		6 38 5·917		−16 30 7·07	
		13·233		−21·96	
	$\frac{9}{2}p$	−0·001		− 0·84	
		6 38 19·149	129 obs.	−16 30 29·87	234 obs.
	Place in Cat. for 1845 R.A. diminished by 0s·028	6 38 19·172	127 obs.	−16 30 27·02	58 obs.
	Adopted place for 1845	6 38 19·160		−16 30 29·30	
	Do. 1755	6 34 20·953		−16 23 53·80	
	Mean	6 36 20·057		−16 27 11·55	
	$-(20\frac{1}{4})p$.	+·004		+ 3·79	
	Place 1800	6 36 20·061		−16 27 7·76	
		or 99° 5′ 0″·91			

Calculation of Precession for 1800.

$\frac{n}{15}$	0·126255	n	1·302346
$\sin\alpha$	9·994519	$\cos\alpha$	−9·198314
$\tan\delta$	−9·470270		−0·500660
	−9·591044	Precession in Decl.	−3″·1671
	s. −0·38998		
	3·07082		
Precession in R. A.	2·68084		
$(\alpha'-\alpha)$	m. s. 3 58·207	$\delta'-\delta$	−6′ 35″·50
$\frac{1}{90}(\alpha'-\alpha)$	2·64674	$\frac{1}{90}(\delta'-\delta)$	−4·3944
Proper motion 1800	−0·03410		−1·2273
Do. in 45 years	−1·534		−55·23

Calculation of Precession for 1755.

	h. m. s. 6 34 20·953	−16° 23′ 53″·80
Correction	+1·534	+55·23
Place to be used in calculating Precession	6 34 22·487	−16 22 58·57
	or 98° 35′ 37″·30	

$\frac{n}{15}$	0·126339	n	1·302430
$\sin\alpha$	9·995097	$\cos\alpha$	−9·174427
$\tan\delta$	−9·468336		−0·476857
	−9·589772	Precession in Decl.	−2·9982
	−0·38884		
	3·06997		
Precession in R. A.	2·68113		

Calculation of Precession for 1845.

	R.A.		Decl.
	h. m. s. 6 38 19·160		° ′ ″ −16 30 29·30
Correction	−1·534		−55·23
Place to be used in calculating Precession	6 38 17·626		−16 31 24·53
	or 99° 34′ 24″·39		
$\frac{n}{15}$	0·126171	n	1·302262
sin α	9·993909	cos α	−9·220923
tan δ	−9·472258		−0·523185
	−9·592338	Precession in Decl.	−3·3357
	−0·39115		
	3·07167		
Precession in R.A.	2·68052		

Collecting and Differencing the Results.

	R. A. (s.)			Decl. (″)		
1755	2·68113			−2·9982		
		−29			−1689	
1800	2·68084		−3	−3·1671		+3
		−32			−1686	
1845	2·68052			−3·3357		

CALCULATION OF PLACE FOR 1850 AND ANNUAL VARIATIONS &c. FOR SAME TIME.

$\Delta c + \frac{11}{18}\Delta^2 c$	$-\ 0{\cdot}00034$		$-\ 0{\cdot}1684$	
$p = \frac{10}{9}(\ldots)$	$-\ 0{\cdot}00038$	p'	$-\ 0{\cdot}1871$	Half Sec. variation.
$\frac{\alpha' - \alpha}{90}$	$+\ 2{\cdot}64674$	$\frac{\delta' - \delta}{90}$	$-\ 4{\cdot}3944$	
$-\frac{127}{162}\Delta^2 c$	$+\ 0{\cdot}00002$		$-\ {\cdot}0002$	
k	$+\ 2{\cdot}64638$	k'	$-\ 4{\cdot}5817$	Annual variation.
$5k$	$+13{\cdot}232$	$5k'$	$-22{\cdot}91$	
$-\frac{1}{4}p$	${\cdot}000$	$-\frac{1}{4}p'$	$+\ 0{\cdot}05$	
	$+13{\cdot}232$		$-22{\cdot}86$	
α'	6 38 19·160	δ'	−16 30 29·30	
	6 38 32·392		−16 30 52·16	Place for 1850.

[Here follows Table of Elements for calculating the Mean Places of the Standard Stars, extracted from Mr Farley's Calculations of Fundamental Stars for 1850.]

The Right Ascension for the time $1850+t$ is

$$(\text{R. A. } 1850) + kt + \frac{p}{100}t^2 + \frac{\Delta^2 c}{12150}t^3,$$

and the Declination for the time $1850+t$ is

$$(\text{Decl. } 1850) + k't + \frac{p'}{100}t^2 + \frac{\Delta^2 c'}{12150}t^3,$$

where $\Delta^2 c$ and $\Delta^2 c'$ are the 2nd differences of the respective precessions given in the Table.

By these formulæ the places were calculated for every 5th year from 1830 to 1870, the results differenced, and then interpolated for every year.

Bessel's Fictitious Year.

The value of the precession given by Dr Peters refers to the tropical year as the unit of time, and the places of the Stars given by him and all the other German Astronomers correspond to the beginning of Bessel's fictitious year, viz. to the instant when the Mean Longitude of the Sun $=280°$. It seems desirable for the sake of uniformity to adopt the same usage, and therefore the places of the Stars found from Airy will require a small correction.

Greenwich Times at the commencement of the Fictitious Years							
	d.		*d.*		*d.*		*d.*
1830 Jan.	0+,361	1840 Jan.	0+,783	1850 Jan.	0+,205	1860 Jan.	0+,628
1	0+,603	1	0+,026	1	0+,448	1	0−,130
2	0+,846	2	0+,268	2	0+,690	2	0+,112
3	0+,088	3	0+,510	3	0−,068	3	0+,354
4	0+,330	4	0+,752	4	0+,174	4	0+,597
5	0+,572	5	0−,006	5	0+,417	5	0−,161
6	0+,814	6	0+,237	6	0+,659	6	0+,081
7	0+,057	7	0+,479	7	0−,099	7	0+,323
8	0+,299	8	0+,721	8	0+,143	8	0+,565
9	0+,541	9	0−,037	9	0+,385	9	0−,192
						1870	0+,050

The Epochs to which the Greenwich Catalogues of 1840 and 1845 most nearly correspond follow the beginnings of the several fictitious years by $0^{d}\cdot580$ and $0^{d}\cdot627$, that is by $0^{y}\cdot001588$ and $0^{y}\cdot001716$, respectively. Hence we have

Correction to the Greenwich Place for $1840 = -\overset{y.}{0}\cdot001588 \times$ (Ann. Var. for 1840)

" " " $1845 = -0\cdot001716 \times$ (Ann. Var. for 1845).

Le Verrier's Corrections of the Right Ascensions of Maskelyne's 35 Fundamental Stars for 1755.

Mr Farley's preliminary calculations were completed when Le Verrier published in the *Comptes Rendus* the corrections which a new and more complete reduction of Bradley's observations of these Stars shewed to be required to be applied to the Right Ascensions for 1755 as given in the *Tabulæ Regiomontanæ*.

The same corrections were subsequently published in the *Monthly Notices* for January, 1853, and Mr Farley made the modifications which were required in order that the results might coincide with those which would have been found if the above mentioned small corrections to the places for 1755, 1840, and 1845 had been first applied, and the calculations before described had been made with the places so corrected.

These modifications are as follows:

As explained before, in the preliminary calculations Mr Farley applied the constant correction $-0^s{\cdot}02$ to the Right Ascensions for 1755 given in the *Tabulæ Regiomontanæ*. Hence the correction to be further applied to the Right Ascension for 1755 will be = Le Verrier's correction $+ 0^s{\cdot}02$.

The corrections of Declination for 1755 will be 0, as well as the corrections of Right Ascension for the same date of Stars not included in Le Verrier's list.

Again the correction of the place for 1845 as deduced from that for 1840

$$= \text{correction for } 1840 + \frac{\text{correction for } 1840 - \text{correction for } 1755}{17},$$

and the mean of this value and of the correction for 1845 derived independently, as before mentioned, is to be taken according to the number of observations on which they respectively depend, and we shall have the adopted correction for 1845.

Also,

Adopted correction for 1845

$$+\tfrac{4}{90}\,(\text{adopted correction for } 1845 - \text{correction for } 1755)$$
$$= \text{correction for } 1857 \text{ to be applied to former results.}$$

The correction of the Proper Motion before found will be

$$= \tfrac{1}{90}\,(\text{adopted correction for } 1845 - \text{correction for } 1755).$$

[Here follows a table shewing the results of calculations made in conformity with the above.]

Polar Stars.

Adopted places and proper motions of the 4 Polar Stars for the beginning of 1800, *to be employed in obtaining the places for every* 5th *year from* 1830 *to* 1870.

	R. A. 1800	Annual Proper Motion in R. A.	Decl. 1800	Annual Proper Motion in Decl.
Polaris	13° 6′ 20″·631	+1″·32332	88° 14′ 24″·493	+0″·00549
51 Cephei	90 41 51·950	−1·90675	87 16 34·340	−0·09101
δ Ursæ Min.	279 9 41·220	+0·48557	86 33 43·415	+0·02306
λ Ursæ Min.	312 53 29·762	−0·79394	88 41 16·413	+0·01234

Constants and formulæ to be employed in reducing the above places to other epochs.

If θ denote the inclination of the Equator of $1800+t$ to the fixed Equator of 1800, and if $90°-z$ denote the Right Ascension of the intersection of the Equator of $1800+t$ with that of 1800, reckoned upon the latter, and $90°+z'$ denote the Right Ascension of the same intersection reckoned on the Equator of $1800+t$, then

$$\theta = 33'\,26''{\cdot}077\left(\frac{t}{100}\right) - 0''{\cdot}430758\left(\frac{t}{100}\right)^2 - 0''{\cdot}04184025\left(\frac{t}{100}\right)^3,$$

$$z = 38'\,23''{\cdot}1165\left(\frac{t}{100}\right) + 0''{\cdot}3105775\left(\frac{t}{100}\right)^2,$$

$$z' = 38'\,23''{\cdot}1165\left(\frac{t}{100}\right) + 1''{\cdot}1156955\left(\frac{t}{100}\right)^2,$$

and the values of θ, z and z' for the several Epochs mentioned will be as follows:

	θ	z	z'
1755	−15′ 2″·8181	−17′ 16″·33953	−17′ 16″·17650
1800	0	0	0
1830	+10 1·7832	+11 30·9629	+11 31·0354
1835	11 42·0724	13 26·1288	13 26·2274
1840	13 22·3592	15 21·2963	15 21·4251
1845	15 2·6436	17 16·4653	17 16·6284
1850	16 42·9256	19 11·6359	19 11·8372
1855	18 23·2051	21 6·8080	21 7·0516
1860	20 3·4821	23 1·9817	23 2·2716
1865	21 43·7566	24 57·1569	24 57·4971
1870	23 24·0285	26 52·3337	26 52·7282

The following is the process to be employed in reducing the above star places from 1800 to $1800+t$.

First to the above places for 1800 apply the proper motion for t years.

Let the resulting Right Ascension and Declination be called α and δ respectively. Take out from the above table the values of θ, z and z' for the year $1800+t$.

Then if α' and δ' be the Right Ascension and Declination for the year $1800+t$, these quantities will be obtained from the following Equations.

Assume

$$\tan\phi = \frac{\cos(\alpha+z)}{\tan\delta}.$$

Then

$$\tan(\alpha'-z') = \frac{\sin\phi}{\sin(\phi-\theta)}\tan(\alpha+z),$$

and

$$\tan\delta' = \frac{\cos(\alpha'-z')}{\tan(\phi-\theta)}.$$

As a check the following formula may be employed,

$$\sin(\alpha+z)\cos\delta = \sin(\alpha'-z')\cos\delta'.$$

But as a more severe check, and in order to find still more accurately the places for $1800+t$, we may employ the following.

Let $$\alpha+z=A, \quad \alpha'-z'=A'.$$

Then

$$\sin\tfrac{1}{2}(A'-A) = \sin\tfrac{1}{2}(A'+A)\tan\tfrac{1}{2}(\delta'+\delta)\tan\tfrac{1}{2}\theta,$$

$$\tan\tfrac{1}{2}(\delta'-\delta) = \frac{\cos\frac{1}{2}(A'+A)}{\cos\frac{1}{2}(A'-A)}\tan\tfrac{1}{2}\theta.$$

The differences $A'-A$ and $\delta'-\delta$ may be more accurately found from the logarithmic tables by these formulæ than A' and δ' themselves can be by the formulæ given before.

The above was the process followed by Mr Farley, except that he calculated the values of θ, z and z' for each 4th year, differenced the results and interpolated the places for every year.

[Here follow the star places thus found for every year from 1830 to 1870.]

PURE MATHEMATICS.

49.

ACCOUNT OF SOME TRIGONOMETRICAL OPERATIONS TO ASCERTAIN THE DIFFERENCE OF GEOGRAPHICAL POSITION BETWEEN THE OBSERVATORY OF ST JOHN'S COLLEGE AND THE CAMBRIDGE OBSERVATORY.

[From the *Cambridge Philosophical Society's Proceedings.* Vol. I. (1852).]

THE observations, especially those of eclipses and occultations, which were made during many years by the late Mr Catton at the Observatory of St John's College, and which have recently been reduced under the superintendence of the Astronomer Royal, render it a matter of some importance to determine the exact geographical position of that Observatory. The simplest and most accurate means of doing this appeared to be, to connect it trigonometrically with the Cambridge Observatory. For this purpose, a base was measured along the ridge of the roof of King's College Chapel, by means of two deal rods terminated by brass studs, the exact lengths of which were determined by comparison with a standard belonging to Professor Miller. The extremities of the base were then connected by a triangle, with a station on the roof of the Observatory at St John's, from which, as well as from the two former points, a signal post on the roof of the Cambridge Observatory could be seen. The angles at the extremities of the base, combined with the corresponding ones at the station at St John's, furnished two determinations of the distance of the Cambridge Observatory, which served to check one another. The meridian line of the transit instrument at St John's passes through King's College Chapel, so that by observing the point at which it intersected the base, the azimuths of the sides of the triangles could be immediately found.

The result thus obtained is, that the transit instrument of the Cambridge Observatory is 2313 feet to the north, and 4770 feet to the west of that at St John's College. Hence it follows that the difference of latitude is $22''\cdot8$, and the difference of longitude $5''\cdot10$; and the latitude of the Cambridge Observatory being $52° \; 12' \; 51''\cdot8$, and its longitude $23''\cdot54$ east of Greenwich, we have finally for the geographical coordinates of the Observatory of St John's College,

Latitude ...... $52° \; 12' \; 29''\cdot0$

Longitude ...... $0° \; 0' \; 28''\cdot64$ E. of Greenwich.

These operations, of course, furnish incidentally a very exact determination of the orientation of King's College Chapel. The line of the ridge of the roof points $6° \; 20'\cdot3$ to the north of east.

50.

PROOF OF THE PRINCIPLE OF AMSLER'S PLANIMETER.

[From the *Cambridge Philosophical Society's Proceedings.* Vol. I. (1857).]

Let O be the fixed point,

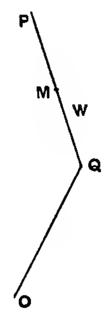
P the tracer,

Q the hinge,

W the centre of wheel,

M the middle point of PQ,

$OQ = a, \quad PQ = b, \quad MW = c.$



The area of any closed figure whose boundary is traced out by P, is the algebraical sum of the elementary areas swept out by the broken line OQP in its successive positions.

Let ϕ and ψ be the angles which OQ, QP at any time make respectively with their initial positions.

s the arc which the wheel has turned through at the same time.

If now OQP take up a consecutive position, and ϕ, ψ, s receive the small increments $\delta\phi$, $\delta\psi$, δs, we see that $\delta s =$ motion of W in direction perpendicular to PQ.

Hence motion of M in the same direction $=\delta s + c\,\delta\psi$, and therefore the elementary area traced out by $QP = b\,(\delta s + c\,\delta\psi)$. Also elementary area traced out by $OQ = \frac{1}{2}a^2\delta\phi$.

Hence the whole area swept out by OQP in moving from its initial to any other position is

$$\tfrac{1}{2}a^2\phi + bc\psi + bs.$$

If OQP returns to its initial position without performing a complete revolution about O, the limits of ϕ and ψ are 0, and the area of the figure traced out by P is bs.

If OQP has performed a complete revolution, the limits of ϕ and ψ are 2π, and the area traced out is

$$\pi\,(a^2 + 2bc) + bs.$$

51.

NOTE ON THE RESOLUTION OF $x^n + \frac{1}{x^n} - 2\cos n\alpha$ INTO FACTORS.

[From the *Cambridge Philosophical Society's Transactions.* Vol. XI., Part 2 (1868).]

THE relation between successive values of $x^m + \frac{1}{x^m}$ corresponding to successive integral values of m is

$$x^{m+1} + \frac{1}{x^{m+1}} = \left(x + \frac{1}{x}\right)\left(x^m + \frac{1}{x^m}\right) - \left(x^{m-1} + \frac{1}{x^{m-1}}\right),$$

when $m = 1$ this becomes

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x + \frac{1}{x}\right) - 2.$$

An exactly similar relation holds good between the successive values of $2\cos m\theta$, thus

$$2\cos(m+1)\theta = (2\cos\theta)(2\cos m\theta) - 2\cos(m-1)\theta,$$

when $m = 1$ this becomes

$$2\cos 2\theta = (2\cos\theta)(2\cos\theta) - 2.$$

Now let v_0, v_1, v_2 &c. v_n be a series of quantities, the successive terms of which are connected by the same relation as that which we have seen to exist between the successive values of $x^m + \frac{1}{x^m}$ and of $2\cos m\theta$, viz.

$$v_{m+1} = v_1 v_m - v_{m-1}.$$

Also as in those cases let $v_0 = 2$, but let v_1 be any quantity whatever, thus we have

$$v_2 = v_1 v_1 - v_0 = v_1^2 - 2,$$
$$v_3 = v_1 v_2 - v_1 = v_1^3 - 3v_1,$$
$$\&c. \qquad \&c.$$

Then it is evident

(1) that v_n is a definite integral function of v_1 of n dimensions, and that the coefficient of v_1^n in it is unity.

(2) that if $v_1 = x + \frac{1}{x}$, then $v_n = x^n + \frac{1}{x^n}$.

(3) that if $v_1 = 2\cos\theta$, then $v_n = 2\cos n\theta$.

Hence $v_n - 2\cos n\alpha$ will vanish when v_1 is equal to any one of the n quantities,

$$2\cos\alpha, \quad 2\cos\left(\alpha + \frac{2\pi}{n}\right), \quad 2\cos\left(\alpha + 2\frac{2\pi}{n}\right), \ \ldots\ldots\ 2\cos\left(\alpha + \overline{n-1}\,\frac{2\pi}{n}\right),$$

and therefore

$$v_n - 2\cos n\alpha = [v_1 - 2\cos\alpha]\left[v_1 - 2\cos\left(\alpha + \frac{2\pi}{n}\right)\right]\left[v_1 - 2\cos\left(\alpha + 2\frac{2\pi}{n}\right)\right]\ldots\ldots$$
$$\times\left[v_1 - 2\cos\left(\alpha + \overline{n-1}\,\frac{2\pi}{n}\right)\right],$$

for all values whatever of v_1.

Now, put $v_1 = x + \frac{1}{x}$;

$$\therefore\ x^n + \frac{1}{x^n} - 2\cos n\alpha$$
$$= \left[x + \frac{1}{x} - 2\cos\alpha\right]\left[x + \frac{1}{x} - 2\cos\left(\alpha + \frac{2\pi}{n}\right)\right]\left[x + \frac{1}{x} - 2\cos\left(\alpha + 2\frac{2\pi}{n}\right)\right]\ldots\ldots$$
$$\times\left[x + \frac{1}{x} - 2\cos\left(\alpha + \overline{n-1}\,\frac{2\pi}{n}\right)\right],$$

which is the required resolution.

Similarly, if we put $v_1 = 2\cos\theta$, we have

$$2\cos n\theta - 2\cos n\alpha$$
$$= [2\cos\theta - 2\cos\alpha]\left[2\cos\theta - 2\cos\left(\alpha + \frac{2\pi}{n}\right)\right]\left[2\cos\theta - 2\cos\left(\alpha + 2\frac{2\pi}{n}\right)\right]\ldots\ldots$$
$$\times\left[2\cos\theta - 2\cos\left(\alpha + \overline{n-1}\,\frac{2\pi}{n}\right)\right].$$

Hence we see that the two equations just found are particular cases of the general equation from which they have been derived, v_1 being in one case numerically not less than 2, and in the other not greater than 2.

If either $x=1$ or $\theta=0$, v_1 becomes $=2$, and either of the equations gives

$$2-2\cos n\alpha=[2-2\cos\alpha]\left[2-2\cos\left(\alpha+\frac{2\pi}{n}\right)\right]\left[2-2\cos\left(\alpha+2\frac{2\pi}{n}\right)\right]\ldots\ldots$$
$$\times\left[2-2\cos\left(\alpha+\overline{n-1}\,\frac{2\pi}{n}\right)\right].$$

Similarly, if either $x=-1$ or $\theta=\pi$, $v_1=-2$, and either of the equations gives

$$2(-1)^n-2\cos n\alpha=[-2-2\cos\alpha]\left[-2-2\cos\left(\alpha+\frac{2\pi}{n}\right)\right]$$
$$\left[-2-2\cos\left(\alpha+2\frac{2\pi}{n}\right)\right]\ldots\ldots\times\left[-2-2\cos\left(\alpha+\overline{n-1}\,\frac{2\pi}{n}\right)\right].$$

52.

ON A SIMPLE PROOF OF LAMBERT'S THEOREM.

[From the *British Association Report* (1877).]

THE following proof of Lambert's Theorem, which I find among my old papers, appears to be as simple and direct as can be desired.

Let a denote the semiaxis major and e the eccentricity of an elliptic orbit, n the mean motion, and μ the absolute force.

Also let r, r' denote the radii vectores, and u, u' the eccentric anomalies at the extremities of any arc, k the chord, and t the time of describing the arc.

Then
$$r = a(1 - e\cos u), \quad r' = a(1 - e\cos u'),$$
$$k^2 = a^2(\cos u - \cos u')^2 + a^2(1 - e^2)(\sin u - \sin u')^2,$$
and
$$nt = \left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} t = u - u' - e(\sin u - \sin u').$$

Or
$$\frac{r + r'}{2a} = 1 - \left(e\cos\frac{u + u'}{2}\right)\cos\frac{u - u'}{2},$$
$$\frac{k^2}{4a^2} = \sin^2\frac{u + u'}{2}\sin^2\frac{u - u'}{2} + (1 - e^2)\cos^2\frac{u + u'}{2}\sin^2\frac{u - u'}{2}$$
$$= \sin^2\frac{u - u'}{2}\left\{1 - e^2\cos^2\frac{u + u'}{2}\right\},$$
and
$$nt = u - u' - 2\left(e\cos\frac{u + u'}{2}\right)\sin\frac{u - u}{2}.$$

Hence we see that if a, and therefore also n, be given, then $r+r'$, k, and t are functions of the *two* quantities

$$u-u' \text{ and } e\cos\frac{u+u'}{2}.$$

Let $$u-u'=2\alpha \text{ and } e\cos\frac{u+u'}{2}=\cos\beta.$$

Then $$\frac{r+r'}{2a}=1-\cos\alpha\cos\beta,$$

$$\frac{k}{2a}=\sin\alpha\sin\beta;$$

therefore $$\frac{r+r'+k}{2a}=1-\cos(\beta+\alpha),$$

and $$\frac{r+r'-k}{2a}=1-\cos(\beta-\alpha);$$

also $$nt=2\alpha-2\sin\alpha\cos\beta,$$
$$=[\beta+\alpha-\sin(\beta+\alpha)]-[\beta-\alpha-\sin(\beta-\alpha)].$$

The first two of these equations give $\beta+\alpha$ and $\beta-\alpha$ in terms of $r+r'+k$ and $r+r'-k$, and the third equation is the expression of Lambert's Theorem.

An exactly similar proof may be given in the case of an hyperbolic orbit.

Let $$\frac{1}{2}(\epsilon^u+\epsilon^{-u}) \text{ be denoted by } \operatorname{csh}(u),$$

and $$\frac{1}{2}(\epsilon^u-\epsilon^{-u}) \text{ by } \operatorname{snh}(u),$$

which quantities may be called the hyperbolic cosine and hyperbolic sine of u

Then we have

$$\operatorname{csh}^2(u)-\operatorname{snh}^2(u)=1,$$

$$\operatorname{csh}(u)+\operatorname{csh}(u')=2\operatorname{csh}\frac{u+u'}{2}\operatorname{csh}\frac{u-u'}{2},$$

$$\operatorname{csh}(u)-\operatorname{csh}(u')=2\operatorname{snh}\frac{u+u'}{2}\operatorname{snh}\frac{u-u'}{2},$$

$$\operatorname{snh}(u)-\operatorname{snh}(u')=2\operatorname{csh}\frac{u+u'}{2}\operatorname{snh}\frac{u-u'}{2}.$$

The coordinates of any point in the hyperbola referred to its axes may be represented by

$$x = a \operatorname{csh}(u),$$

$$y = a\sqrt{e^2-1}\operatorname{snh}(u).$$

If u, u' denote the values of u corresponding to the two extremities of the arc, we have

$$r = a(e\operatorname{csh}(u)-1), \qquad r' = a(e\operatorname{csh}(u')-1),$$

$$k^2 = a^2[\operatorname{csh}(u)-\operatorname{csh}(u')]^2 + a^2(e^2-1)[\operatorname{snh}(u)-\operatorname{snh}(u')]^2;$$

or

$$\frac{r+r'}{2a} = \left(e\operatorname{csh}\frac{u+u'}{2}\right)\operatorname{csh}\frac{u-u'}{2} - 1,$$

$$\frac{k^2}{4a^2} = \operatorname{snh}^2\frac{u-u'}{2}\left[e^2\operatorname{csh}^2\frac{u+u'}{2} - 1\right].$$

Also twice the area of the sector limited by r and r'

$$= a^2\sqrt{e^2-1}\,[(e\operatorname{snh}u - u) - (e\operatorname{snh}u' - u')]$$

$$= a^2\sqrt{e^2-1}\left[2\left(e\operatorname{csh}\frac{u+u'}{2}\right)\operatorname{snh}\frac{u-u'}{2} - (u-u')\right],$$

and twice the area described in a unit of time is

$$\sqrt{\mu a(e^2-1)}.$$

Hence

$$t = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}\left[2\left(e\operatorname{csh}\frac{u+u'}{2}\right)\operatorname{snh}\frac{u-u'}{2} - (u-u')\right];$$

and therefore if a be given, then $r+r'$, k, and t are functions of the *two* quantities $e\operatorname{csh}\dfrac{u+u'}{2}$ and $u-u'$.

Let $u-u' = 2\alpha$, and $e\operatorname{csh}\dfrac{u+u'}{2} = \operatorname{csh}(\beta)$, which is always possible since e is greater than 1.

Then

$$\frac{r+r'}{2a} = \operatorname{csh}(\beta)\operatorname{csh}(\alpha) - 1,$$

$$\frac{k}{2a} = \operatorname{snh}(\beta)\operatorname{snh}(\alpha);$$

therefore

$$\frac{r+r'+k}{2a} = \operatorname{csh}(\beta+\alpha) - 1,$$

and

$$\frac{r+r'-k}{2a} = \operatorname{csh}(\beta-\alpha) - 1.$$

Also
$$t = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} [2 \operatorname{csh}(\beta) \operatorname{snh}(\alpha) - 2\alpha],$$
$$= \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} [\operatorname{snh}(\beta+\alpha) - (\beta+\alpha) - \operatorname{snh}(\beta-\alpha) + (\beta-\alpha)].$$

As before, the first two of these equations give $\beta+\alpha$ and $\beta-\alpha$ in terms of $r+r'+k$ and $r+r'-k$, and the last equation is the expression of Lambert's theorem in the case of the hyperbola.

When the orbit is parabolic, a becomes infinite; and since $r+r'$ and k are finite, the quantities α and β become indefinitely small.

Hence
$$\frac{r+r'+k}{2a} = 1 - \cos(\beta+\alpha) = \frac{1}{2}(\beta+\alpha)^2 \text{ ultimately},$$
$$\frac{r+r'-k}{2a} = 1 - \cos(\beta-\alpha) = \frac{1}{2}(\beta-\alpha)^2 \text{ ultimately};$$
also
$$t = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} \{\beta+\alpha - \sin(\beta+\alpha) - (\beta-\alpha) + \sin(\beta-\alpha)\}$$
$$= \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} \left\{\frac{1}{6}(\beta+\alpha)^3 - \frac{1}{6}(\beta-\alpha)^3\right\} \text{ ultimately}$$
$$= \frac{1}{6}\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} \left\{\left(\frac{r+r'+k}{a}\right)^{\frac{3}{2}} - \left(\frac{r+r'-k}{a}\right)^{\frac{3}{2}}\right\} \text{ ultimately}$$
$$= \frac{1}{6\sqrt{\mu}} \{(r+r'+k)^{\frac{3}{2}} - (r+r'-k)^{\frac{3}{2}}\},$$
which is Lambert's theorem in the case of the parabola.

53.

ON THE ATTRACTION OF AN INDEFINITELY THIN SHELL BOUNDED BY TWO SIMILAR AND SIMILARLY SITUATED CONCENTRIC ELLIPSOIDS ON AN EXTERNAL POINT.

[*Abstract.*]

[From the *Cambridge Philosophical Society's Proceedings.* Vol. II. (1871).]

No problem has more engaged the attention of mathematicians, or has received a greater variety of elegant solutions, than that of the determination of the attraction of a homogeneous ellipsoid on an external point.

Poisson's solution, which was presented to the Academy of Sciences in 1833, is founded on the decomposition of the ellipsoid into infinitely thin shells bounded by similar surfaces. By a theorem of Newton's, it is known that such a shell exerts no attraction on an internal point, and Poisson proves that its attraction on an external point is in the direction of the axis of the cone which envelopes the shell and has the attracted point for vertex, and that the intensity of the force can be expressed in a finite form, as a function of the coordinates of the attracted point.

In 1834, Steiner gave, in the 12th volume of Crelle's *Journal*, a very elegant geometrical proof of Poisson's theorem respecting the *direction* of the attraction of a shell on an external point. He shews that if the shell be supposed to be divided into pairs of opposite elements with respect to the point in which the axis of the enveloping cone meets the plane of contact, then the resultant of the attraction of each pair of such elements acts in the direction of the axis of the cone, and consequently the attraction of the whole shell acts in the same direction.

About three years later, M. Chasles shewed that Poisson's solution might be greatly simplified by the consideration that the axis of the enveloping cone is identical with the normal to the ellipsoid which passes through the attracted point and is confocal with the exterior surface of the shell.

This mode of enunciating the direction of the attraction has the advantage of making known the level surfaces with respect to the attraction of the shell on external points.

In 1838, M. Chasles presented to the Academy of Sciences a very simple and elegant investigation, in which he arrives at Poisson's results respecting the attraction of a shell on an external point, by a purely synthetical method.

M. Chasles' method is founded on Ivory's well-known property of corresponding points on two confocal ellipsoids, and on some elementary propositions in the theory of the Potential.

Struck by the simplicity and beauty of Steiner's method of finding the *direction* of the attraction of a shell on an external point, the author of the present paper was induced to think that by means of the same method of decomposing the shell into pairs of elements employed by Steiner, a correspondingly simple mode of determining the *intensity* of the attraction might probably be found. The author has been fortunate enough to succeed in realizing this idea, and the result is the method contained in the first part of the present paper.

This method is throughout quite elementary. It requires the knowledge of only the most simple properties of ellipsoids, including Ivory's well-known property respecting corresponding points on two confocal ellipsoids.

The proof of the theorem respecting the direction of the attraction differs from that given by Steiner, and harmonizes better with the method employed for determining the intensity of the force. No use is made in this method of the properties of the Potential.

The second part of the present paper is devoted to what the author considers to be an improvement on M. Chasles' method of determining the attraction of a shell on an external point. Its novelty consists in the mode in which the *intensity* of the attraction of the shell is found. M. Chasles first compares the attractions of two confocal shells on the same external point. He then takes the outer surface of one of these shells to pass through the attracted point, and having found the attraction of this shell by a method applicable to this particular case, he deduces from it the attraction of the general confocal shell. Now it may be remarked on this that the method of finding the attraction of the shell contiguous to the attracted point does not seem free from objection, and also that it may be doubted whether it is legitimate to include this limiting case under the general one without a special examination. If, in order to remove these objections, special considerations are introduced, the proof is thereby deprived of its simple and elementary character. Whether these criticisms on M. Chasles' method are well founded or not, the author thinks that mathematicians will not be displeased to see a direct determination of the attraction of a shell on an external point without the intervention of another shell whose outer surface passes through that point. In order to make the paper more complete, the author briefly shews how from the expression for the attraction of a shell, we may pass to the expression the integral of which gives the attraction of a homogeneous ellipsoid on an external point.

ON THE ATTRACTION OF AN INDEFINITELY THIN SHELL BOUNDED BY TWO SIMILAR AND SIMILARLY SITUATED CONCENTRIC ELLIPSOIDS.

We shall find it convenient to consider the relations between two systems of points.

A system of points is said to be related to another system of points when if x, y, z and x', y', z' be corresponding points, then

$$\frac{x}{x_1}=a;\ \frac{y}{y_1}=b;\ \frac{z}{z_1}=c;$$

where a, b, and c are constants.

If $a=b=c$, the systems are similar.

Volumes bounded by corresponding surfaces are in the ratio of $abc:1$; for the ultimate corresponding elements are in this ratio, and therefore, by Newton's fourth Lemma, the whole volumes are in the same ratio.

The shells will be supposed to be contained between two similar and similarly situated concentric surfaces; the ratio of similitude between the inner and outer surfaces being $1 : 1+t$, where t is indefinitely small.

We may without ambiguity designate any shell by the same symbols which denote its inner bounding surface.

If the principal sections of two ellipsoids be confocal the ellipsoids themselves will be said to be confocal.

Let E be an ellipsoid whose principal semi-axes are a, b, c; and let E_1 be a confocal ellipsoid whose principal semi-axes are a_1, b_1, c_1.

Then $$a^2-b^2=a_1^2-b_1^2;\ \&c.$$

or $$a_1^2-a^2=b_1^2-b^2=c_1^2-c^2.$$

First Solution.

Let a, b, c be the semi-axes of E the interior surface of the attracting shell, and let $1+t$ be the ratio of similitude between the inner and outer surfaces.

Let M_1 (whose coordinates are x_1, y_1, z_1) be the attracted point, a_1, b_1, c_1 the semi-axes of a confocal ellipsoid through M_1, then

$$\frac{a}{a_1}x_1,\quad \frac{b}{b_1}y_1,\quad \frac{c}{c_1}z_1$$

will be the coordinates of a point (M' suppose) on the ellipsoid E.

The equations to the normal to the ellipsoid E_1 at M_1 are

$$\frac{a_1^2}{x_1}(x_1-X)=\frac{b_1^2}{y_1}(y_1-Y)=\frac{c_1^2}{z_1}(z_1-Z),$$

or $$a_1^2-\frac{a_1^2X}{x_1}=b_1^2-\frac{b_1^2Y}{y_1}=c_1^2-\frac{c_1^2Z}{z_1}.$$

Take X, Y, Z the coordinates of a point M on this normal such that

$$X=\frac{a^2x_1}{a_1^2},\quad Y=\frac{b^2y_1}{b_1^2},\quad Z=\frac{c^2z_1}{c_1^2}:$$

we see that the relation of M to M' is such that M is a corresponding point to M' in the system of points whose relation is

$$\left(\frac{a}{a_1},\ \frac{b}{b_1},\ \frac{c}{c_1}\right).$$

M is the point in which the normal to the external ellipsoid at M_1 meets the plane of contact of the cone of which M_1 is the vertex and which envelopes the attracting shell E.

Let the attracting shell be divided into pairs of elements by means of double cones of indefinitely small solid angle having their vertices at the point M.

Let one of these cones of solid angle $\delta\omega$ intercept a pair of elements of the shell E at P and Q.

Let P' be the point on the ellipsoid E_1 which corresponds to P on E.

Join $P'M'$ and produce it to Q', so that

$$M'Q' : P'M' :: MQ : PM.$$

Then since M and M' correspond in the above system of points so also do P and P', and the lines joining them both are divided in the same ratio, therefore Q and Q' will be corresponding points in the same system and therefore Q' is also on the ellipsoid E_1.

Now by the property of corresponding points on confocal ellipsoids we have

$$PM_1 = P'M' \text{ and } QM_1 = Q'M'.$$

Since the portions of the line PQ intercepted by the shell at P and Q are equal,

the volumes of elements at P and Q are in the ratio of MP^2 to MQ^2,

i.e. are as $M'P'^2$ to $M'Q'^2$ or as M_1P^2 to M_1Q^2;

therefore the masses of these elements have attractions so that the attraction of the element P on M' = the attraction of the element Q on M',
and therefore the resultant attraction of these elements will bisect the angle between M_1P and M_1Q, i.e. will be in the direction M_1M,

for since $$MP : MQ :: M_1P : M_1Q,$$
the angle PM_1Q is bisected by MM_1.

Hence the attraction of every such pair of elements will be in the direction M_1M, and therefore the resultant attraction of the shell E on M_1 is in this direction.

We have now to find the magnitude of this attraction.

Let p be the perpendicular on the tangent plane at P, then the thickness of the shell at P is pt.

Hence if PN be the normal to the surface at P drawn inwards, the elementary surface intercepted by a cone whose solid angle is $\delta\omega$ will be

$$\delta\omega \, . \, MP^2 \sec MPN,$$

therefore the volume of the element is

$$pt\,\delta\omega\, MP^2 \sec MPN = \frac{pt \, . \, \delta\omega \, . \, MP^3}{MP \cos MPN}.$$

Hence if $\rho = 1$, the attraction of the element on M_1 resolved in the direction

$$M_1M = \frac{pt\,.\,\delta\omega\,.\,MP^3}{MP\cos MPN}\cdot\frac{\cos PM_1M}{M_1P^2} = \frac{pt\,.\,\delta\omega\,.\,MP^3}{M_1P^3}\cdot\frac{M_1P\cos PM_1N}{MP\cos MPN}.$$

Let x, y, z be the coordinates of P, then the direction cosines of PN are $\frac{px}{a^2}$, $\frac{py}{b^2}$, $\frac{pz}{c^2}$ and the projection of MP upon the normal PN will be

$$\frac{px}{a^2}\left(x - \frac{a^2}{a_1^2}x_1\right) + \frac{py}{b^2}\left(y - \frac{b^2}{b_1^2}y_1\right) + \frac{pz}{c^2}\left(z - \frac{c^2}{c_1^2}z_1\right),$$

or

$$MP\cos MPN = p\left[1 - \left(\frac{xx_1}{a_1^2} + \frac{yy_1}{b_1^2} + \frac{zz_1}{c_1^2}\right)\right].$$

Similarly $M_1P\cos PM_1M$ is the projection of M_1P upon M_1M.

The direction cosines of M_1M are $\frac{p_1x_1}{a_1^2}$, $\frac{p_1y_1}{b_1^2}$, $\frac{p_1z_1}{c_1^2}$, where p_1 is the perpendicular from origin on the tangent plane at M_1.

The projection of M_1P upon M_1M is

$$\frac{p_1x_1}{a_1^2}(x_1 - x) + \frac{p_1y_1}{b_1^2}(y_1 - y) + \frac{p_1z_1}{c_1^2}(z_1 - z) = p_1\left[1 - \left(\frac{xx_1}{a_1^2} + \frac{yy_1}{b_1^2} + \frac{zz_1}{c_1^2}\right)\right].$$

Hence attraction of element at P on M_1 resolved in the direction M_1M is

$$t\,.\,\delta\omega\,.\,p_1\frac{MP^3}{M_1P^3} = t\,.\,\delta\omega\,.\,p_1\frac{MP^3}{M'P'^3}\ (\text{since } M_1P = M'P').$$

Let $\delta\omega'$ be the solid angle of a cone whose vertex is M' and base the element of E' which corresponds to the element E at P.

Then the volume of this cone is ultimately $\frac{1}{3}\delta\omega'\,.\,M'P'^3$.

But the volume of the corresponding cone will be $\frac{1}{3}\delta\omega\,.\,MP^3$, and these volumes are as $a_1b_1c_1 : abc$ respectively;

therefore

$$\frac{\delta\omega'\,.\,M'P'^3}{a_1b_1c_1} = \frac{\delta\omega\,.\,MP^3}{abc}.$$

Hence

$$\frac{\delta\omega\,.\,MP^3}{M'P'^3} = \delta\omega'\,.\,\frac{abc}{a_1b_1c_1};$$

therefore the resolved part of the attraction of the element E at P along M_1M is $p_1 t \dfrac{abc}{a_1b_1c_1} . \delta\omega'$, therefore the attraction of the whole shell on M_1 along M_1M will be $4\pi t p_1 . \dfrac{abc}{a_1b_1c_1}$.

Hence if the shell be of uniform density ρ, the attraction of the whole shell on M_1 in the direction of the normal will be $4\pi\rho t p_1 . \dfrac{abc}{a_1b_1c_1}$, where p_1 is the perpendicular from the origin on the tangent plane at M_1.

Hence the attraction of the shell has been determined in direction and magnitude.

Second Solution.

Imagine a shell of which E is the inner boundary to be composed of matter of uniform density, and another shell of which E_1 is the inner boundary to contain the same quantity of matter, also of uniform density. The quantity of matter contained in any portion of E will be equal to that in the corresponding part of E_1,

also since vol. of E : vol. of E_1 :: abc : $a_1b_1c_1$;

therefore density of E : density of E_1 :: $a_1b_1c_1$: abc.

Now let M' and M_1 be two fixed corresponding points on E and E_1, and let P and P_1 be any two corresponding points; then by the property of corresponding points on confocal ellipsoids, $M'P_1 = M_1P$.

Also the same quantity of matter is contained in corresponding elements of the two shells at P and P_1, and since the same is true for all corresponding elements, therefore the potential of shell E_1 at the point M'

= the potential of shell E at the point M_1.

But since, by Newton's Theorem, the shell E_1 exerts no attraction on an internal point, its potential is constant at all internal points and is therefore the same at M' as at O, the common centre of E and E_1.

Hence the potential of the shell E at any point M_1 on the surface of E_1 is constant and equal to the potential of the shell E_1 at its centre O; therefore by the theory of the Potential the attraction of the shell E at M_1 is in the direction of the normal to the surface E_1.

We now proceed to find the magnitude of this attraction.

Let E' be another ellipsoid contiguous to E_1 and inside it and confocal with both E and E_1; let its principal semi-axes be a', b', c', and let

$$a' + \delta a' = a_1, \quad b' + \delta b' = b_1, \quad c' + \delta c' = c_1;$$

then since
$$a_1^2 - a'^2 = b_1^2 - b'^2 = c_1^2 - c'^2,$$

we have ultimately
$$a'\delta a' = b'\delta b' = c'\delta c'.$$

Imagine a shell of which E' is the inner boundary and containing the same quantity of matter as E or E_1, and let this matter be of uniform density, then the potential of the shell E at any point on the surface of E' is constant and equal to the potential of shell E' at O the common centre.

Now let S be the sphere whose centre is at O and radius unity. Imagine a shell of which the inner boundary is S; let l, m, n be the co-ordinates of any point p on S, and let $\delta\sigma$ be an element of the surface at p; then if a cone be described with base $\delta\sigma$ and vertex O, the element of the shell S intercepted : whole volume of shell :: $\delta\sigma$: 4π.

At the points P_1 on E_1 and P' on E', which correspond, take elements of the respective shells which correspond to the element at p on this spherical shell.

The volumes of these corresponding elements will be proportional to the whole volumes of the shells to which they belong, hence if M denote the mass of each of the shells E, E_1 and E', the mass of the element at P_1 and also at P' will be $\frac{M}{4\pi} . \delta\sigma$; also the coordinates of P_1 are $a_1 l$, $b_1 m$, $c_1 n$ and those of P' are $a'l$, $b'm$, $c'n$;

therefore
$$OP_1^2 - OP'^2 = l^2 (a_1^2 - a'^2) + m^2 (b_1^2 - b'^2) + n^2 (c_1^2 - c'^2)$$
$$= (a_1^2 - a'^2)(l^2 + m^2 + n^2) = a_1^2 - a'^2.$$

Let $OP_1 = r_1$ and $OP' = r'$ and let $r_1 = r' + \delta r'$; then we have

$$r'\delta r' = a'\delta a'.$$

Now if V be the potential of the shell E_1 at O, and $V' = V + \delta V$ be the potential of the shell E' at the same point, then

$$V = \frac{M}{4\pi}\int\frac{d\sigma}{r_1} \text{ and } V' = \frac{M}{4\pi}\int\frac{d\sigma}{r'};$$

therefore
$$\delta V = \frac{M}{4\pi}\int d\sigma\left(\frac{1}{r'} - \frac{1}{r_1}\right) = \frac{M}{4\pi}\int d\sigma\,\frac{\delta r'}{r'^2}$$
$$= \frac{M}{4\pi}\int a'\delta a'\,\frac{d\sigma}{r'^3} = \frac{Ma'\delta a'}{4\pi}\int\frac{d\sigma}{r'^3}.$$

Now the volume of the cone whose base is $\delta\sigma$ and vertex O and radius unity is $\frac{1}{3}\delta\sigma$; hence the volume of the corresponding cone enveloping the element at P_1 or P' is $\frac{1}{3}a'b'c'\delta\sigma$; therefore if $\delta\omega$ be the solid angle of the cone
$$\frac{1}{3}r'^3\delta\omega = \frac{1}{3}a'b'c'\delta\sigma,$$
or
$$\frac{\delta\sigma}{r'^3} = \frac{\delta\omega}{a'b'c'},$$
and we have
$$\delta V = \frac{M}{4\pi}a'\delta a'\int\frac{d\omega}{a'b'c'} = \frac{M\delta a'}{b'c'}.$$

Hence it follows that the attraction of shell E at P_1 in the direction of P_1P', i.e. $\frac{\delta V}{P_1P'}$, is
$$\frac{M}{b'c'}\frac{\delta a'}{P_1P'} = \frac{M}{a'b'c'}\cdot\frac{a'\delta a'}{P_1P'}.$$

Now if $x = a_1 l$, $y = b_1 m$, $z = c_1 n$ be the coordinates of P_1, those of P' will be $a'l$, $b'm$, $c'n$ and the projections of P_1P' on the axes will be $l\delta a'$, $m\delta b'$, $n\delta c'$.

Putting for l the value $\frac{x}{a'} = \frac{x}{a'^2}\cdot a'$ and so for m and n, we get
$$l\delta a' = \frac{x}{a'^2}\cdot a'\delta a', \qquad m\delta b' = \frac{y}{b'^2}\cdot b'\delta b', \qquad n\delta c' = \frac{z}{c'^2}\cdot c'\delta c';$$
but the direction cosines of the normal are as $\frac{x}{a'^2} : \frac{y}{b'^2} : \frac{z}{c'^2}$.

Hence P_1P' is ultimately in the direction of the normal at P_1.

Hence attraction of shell E at P_1 which has been shewn to act in the direction of this normal $= \frac{Mp_1}{a'b'c'}$, where p_1 is the perpendicular from O on the tangent plane at P_1.

If we call ρ the density of shell E, the volume of the shell is $4\pi tabc$, and we have
$$M = 4\pi\rho tabc,$$
therefore the attraction of the shell $= \frac{4\pi\rho abc}{a_1b_1c_1}\cdot t\cdot p_1$.

We may regard a homogeneous ellipsoid as made up of indefinitely thin shells.

Let X, Y, Z be the components in the direction of the axes of the attraction of an ellipsoid whose semi-axes are a, b, c on the point P_1, and let $X+\delta X$, $Y+\delta Y$, $Z+\delta Z$ be the attractions of a similar ellipsoid whose semi-axes are $a+\delta a$, $b+\delta b$, $c+\delta c$, where

$$\delta a = at, \quad \delta b = bt, \quad \delta c = ct,$$

then
$$\delta X = \frac{4\pi\rho abc}{a_1 b_1 c_1} \cdot tp_1 \cdot \frac{p_1 x}{a_1^2} = \frac{4\pi\rho bc}{b_1 c_1} \cdot \frac{p_1^2}{a_1^3} \cdot x \cdot \delta a.$$

Let $u = \dfrac{a}{a_1}$, then $\delta u = \dfrac{a}{a_1^2} \cdot \delta a_1$ ultimately,

and
$$a_1 \delta a_1 = p_1^2 t,$$

hence
$$\delta u = \frac{1}{a_1^3} \cdot p_1^2 \cdot \delta a;$$

hence
$$\delta X = 4\pi\rho x \cdot \frac{bc}{b_1 c_1} \cdot \delta u,$$

$$\delta Y = 4\pi\rho y \cdot \frac{bc}{b_1 c_1} \cdot \frac{a_1^2}{b_1^2} \cdot \delta u,$$

$$\delta Z = 4\pi\rho z \cdot \frac{bc}{b_1 c_1} \cdot \frac{a_1^2}{c_1^2} \cdot \delta u.$$

We have now to substitute for the quantities $\dfrac{bc}{b_1 c_1}$, $\dfrac{a_1^2}{b_1^2}$, &c.

Since
$$a_1^2 - a^2 = b_1^2 - b^2 = c_1^2 - c^2,$$

the equation to the ellipsoidal shell through the attracted point is

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_1^2 + (b^2 - a^2)} + \frac{z^2}{a_1^2 + (c^2 - a^2)} = 1,$$

and so we get

$$\frac{x^2}{\frac{1}{u^2}} + \frac{y^2}{\frac{1}{u^2} + \left(\frac{b^2}{a^2} - 1\right)} + \frac{z^2}{\frac{1}{u^2} + \left(\frac{c^2}{a^2} - 1\right)} = a^2,$$

where $\dfrac{b}{a}$ and $\dfrac{c}{a}$ are constants; and so a^2 is known in terms of u^2.

Also
$$b_1^2 = a^2\left[\frac{1}{u^2} + \left(\frac{b^2}{a^2} - 1\right)\right] \text{ and } c_1^2 = a^2\left[\frac{1}{u^2} + \left(\frac{c^2}{a^2} - 1\right)\right].$$

Hence
$$\delta X = 4\pi\rho x \cdot \frac{bc}{a^2}\,\frac{u^2\delta u}{\sqrt{\left[1+u^2\left(\frac{b^2}{a^2}-1\right)\right]\left[1+u^2\left(\frac{c^2}{a^2}-1\right)\right]}},$$

$$\therefore\ X = 4\pi\rho x \cdot \frac{bc}{a^2} \cdot \int_0^{\frac{a}{a_1}} \frac{u^2 du}{\sqrt{\left[1+u^2\left(\frac{b^2}{a^2}-1\right)\right]\left[1+u^2\left(\frac{c^2}{a^2}-1\right)\right]}};$$

a_1, b_1, c_1 are the semi-axes of the ellipsoid confocal with the outer given ellipsoid and passing through the attracted point.

$$\frac{a_1^2}{b_1^2} = \frac{1}{1+u^2\left(\frac{b^2}{a^2}-1\right)} \text{ and } \frac{a_1^2}{c_1^2} = \frac{1}{1+u^2\left(\frac{c^2}{a^2}-1\right)},$$

therefore
$$Y = 4\pi\rho y \cdot \frac{bc}{a^2} \cdot \int \frac{u^2 du}{\left[1+u^2\left(\frac{b^2}{a^2}-1\right)\right]^{\frac{3}{2}}\left[1+u^2\left(\frac{c^2}{a^2}-1\right)\right]^{\frac{1}{2}}},$$

and
$$Z = 4\pi\rho z \cdot \frac{bc}{a^2} \cdot \int \frac{u^2 du}{\left[1+u^2\left(\frac{b^2}{a^2}-1\right)\right]^{\frac{1}{2}}\left[1+u^2\left(\frac{c^2}{a^2}-1\right)\right]^{\frac{3}{2}}}.$$

If in place of u we make λ the independent variable where

$$a^2\left(\frac{1}{u^2}-1\right) = \lambda,$$

and so
$$\frac{\delta u}{a} = -\frac{\delta\lambda}{2\left(a^2+\lambda\right)^{\frac{3}{2}}},$$

then
$$X = 2\pi\rho x abc \int_{\lambda_1}^{\infty} \frac{d\lambda}{\sqrt{(a^2+\lambda)^3(b^2+\lambda)(c^2+\lambda)}},$$

with similar expressions for Y and Z, where

$$\lambda_1 = a_1^2 - a^2 = b_1^2 - b^2 = c_1^2 - c^2.$$

54.

ON THE CALCULATION OF THE BERNOULLIAN NUMBERS FROM B_{32} TO B_{62}.

[From *Appendix* I. *to the Cambridge Observations,* Vol. XXII.]

In the year 1877 I communicated to the meeting of the British Association at Plymouth the values of 31 of Bernoulli's numbers which I had obtained in addition to the 31 of those numbers already known, and I stated that it was my intention to publish some of the steps of the calculation in an Appendix to the Cambridge Observations.

The following Tables accordingly contain some of the principal steps of the calculations, together with more detailed specimens of the work in the cases of the 32nd and the 62nd Bernoulli's numbers, the first and last of those which I have calculated.

In order to render the Tables intelligible, the substance of my communication to the British Association is here reproduced.

A remarkable theorem, due to Staudt, gives at once the fractional part of any one of Bernoulli's numbers, and thus greatly facilitates the finding of those numbers by reducing all the requisite calculations to operations with integers only.

The theorem may be thus stated:—

If 1, 2, a, a'...$2n$ be all the divisors of $2n$, and if unity be added to each of these divisors so as to form the series 2, 3, $a+1$, $a'+1$...$2n+1$, and if from this series only the prime numbers 2, 3, p, p'... be selected, then the fractional part of the nth number of Bernoulli will be

$$(-1)^n\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{p}+\frac{1}{p'}+\dots\right).$$

Having found, several years ago, a simple and elementary proof of this theorem, I was induced to apply the theorem to the calculation of several additional numbers of Bernoulli, and I ultimately obtained the values of the thirty-one numbers which are given in the present paper.

The method which has been employed affords numerous tests, throughout the course of the work, of the correctness with which the requisite operations have been performed, so that I feel entire confidence in the accuracy of the results.

In making these calculations I have received very efficient aid from my Assistants, Mr Graham and Mr Todd.

The following is an outline of the method employed:—

Bernoulli's numbers B_1, B_2, &c. are defined by the equation

$$\frac{x}{\epsilon^x-1}=1-\frac{1}{2}x+\frac{B_1}{1\,.\,2}x^2-\frac{B_2}{1\,.\,2\,.\,3\,.\,4}x^4+\text{\&c.}+(-1)^{n-1}\frac{B_n}{\underline{|2n}}x^{2n}+\text{\&c.}$$

or

$$\frac{x}{\epsilon^x-1}=1-\frac{1}{2}x+\Sigma\,(-1)^{n-1}\frac{B_n}{\underline{|2n}}x^{2n},$$

where n takes all positive integer values from 1 to ∞.

If we multiply by ϵ^x-1, and equate to zero the coefficient of x^{2n+1} on the right-hand side of the resulting equation, we shall find

$$(-1)^n C_n^n B_n+(-1)^{n-1}C_{n-1}^n B_{n-1}+\text{\&c.}+(-1)\,C_1^n B_1+n-\tfrac{1}{2}=0,$$

in which C_r^n denotes the coefficient of x^{2r} in the expansion of $(1+x)^{2n+1}$.

This equation gives B_n when B_1, $B_2,\dots B_{n-1}$ are known.

Now let $$B_n=I_n+(-1)^n(f_n-1),$$
where $(-1)^n f_n$ is the fractional part of B_n given by Staudt's Theorem, so that I_n is an integer.

Substituting in the above equation, and writing for simplicity C_r instead of C_r^n, as we may do without ambiguity, we have

$$\begin{aligned}(-1)^n C_n I_n + (-1)^{n-1} C_{n-1} I_{n-1} + \&c. + (-1)\, C_1 I_1 \\ + C_1 f_1 + C_2 f_2 + \&c. + C_n f_n \\ - C_1 - C_2 - \&c. - C_n + n - \tfrac{1}{2} = 0.\end{aligned}$$

Now by Staudt's Theorem the fraction $\frac{1}{2}$ occurs in each of the fractions f_n; hence the quantity arising from this fraction in $C_1 f_1 + C_2 f_2 + \&c. + C_n f_n$ will be

$$\tfrac{1}{2}(C_1 + C_2 + \ldots + C_n) = \tfrac{1}{2}(2^{2n} - 1).$$

Also, by the same Theorem, if $2r + 1 = p$ be an odd prime number, the fraction $\dfrac{1}{p}$ will occur in each of the fractions f_r, f_{2r}, f_{3r}, &c.

Hence the part of $C_1 f_1 + C_2 f_2 + \&c.$ which contains $\dfrac{1}{p}$ will be

$$\frac{1}{p}\{C_r + C_{2r} + C_{3r} + \&c.\}.$$

Also $C_n = 2n + 1$; hence by substitution and transposition, we find

$$\begin{aligned}(-1)^{n-1}(2n+1)\, I_n = -\{C_1 I_1 + C_3 I_3 + \&c.\} + \{C_2 I_2 + C_4 I_4 + \&c.\} \\ - 2^{2n-1} + n \\ + \frac{1}{3}(C_1 + C_2 + \&c. + C_n) \\ + \frac{1}{5}(C_2 + C_4 + C_6 + \&c.) \\ + \frac{1}{7}(C_3 + C_6 + C_9 + \&c.) \\ + \frac{1}{11}(C_5 + C_{10} + C_{15} + \&c.) \\ + \&c. \\ + \frac{1}{p}(C_r + C_{2r} + C_{3r} + \&c.) \\ + \&c.,\end{aligned}$$

which gives I_n when I_1, $I_2 \ldots I_{n-1}$ are known.

In the above expression p is supposed to include every odd prime number not exceeding $2n+1$.

It may be easily shewn that all the quantities

$$\frac{1}{3}(C_1+C_2+\&c.+C_n)$$

$$\frac{1}{5}(C_2+C_4+C_6+\&c.)$$

$$\frac{1}{7}(C_3+C_6+C_9+\&c.)$$

&c.

are integers. Hence the right-hand side of the above equation is an integer which must be divisible by $2n+1$; and this supplies a test of the correctness of the work.

If
$$F_n=\Sigma\frac{1}{p}(C_r^n+C_{2r}^n+C_{3r}^n+\&c.)-2^{2n-1}+n$$

where, as before mentioned, $p=2r+1$ is an odd prime number, the above equation for I_n may be written

$$(-1)^{n-1}(2n+1)I_n=-\{C_1^nI_1+C_3^nI_3+\&c.\}+\{C_2^nI_2+C_4^nI_4+\&c.\}+F_n.$$

The reason why we assume

$$B_n=I_n+(-1)^n(f_n-1),$$

instead of taking the simpler form

$$B_n=I_n+(-1)^nf_n,$$

is that with the above assumption the quantities I_1, I_2, I_3, I_4, I_5, I_6 all vanish, so that we have fewer quantities to calculate.

The numbers $C_r^nI_r$, which are required in order to find the value of $(2n+1)I_n$, can be readily derived from the numbers $C_r^{n-1}I_r$, which have been already employed in finding the value of the similar quantity $(2n-1)I_{n-1}$ which immediately precedes it. For since

$$C_r^n=\frac{(2n+1)\,2n}{(2n-2r+1)(2n-2r)}C_r^{n-1}=\frac{n(2n+1)}{(n-r)(2n-2r+1)}C_r^{n-1},$$

we have

$$C_r^nI_r=\frac{n(2n+1)}{(n-r)(2n-2r+1)}C_r^{n-1}I_r,$$

which may be written

$$P_r^n = \frac{n(2n+1)}{(n-r)(2n-2r+1)} P_r^{n-1};$$

and a test of the correctness of the work is supplied by the divisions by $n-r$ and $2n-2r+1$ being performed without leaving any remainder.

I have proved that if n be a prime number, other than 2 or 3, then the numerator of the nth number of Bernoulli will be divisible by n.

This forms another excellent test of the correctness of the work.

I have also observed that if q be a prime factor of n, which is not likewise a factor of the denominator of B_n, then the numerator of B_n will be divisible by q. I have not succeeded, however, in obtaining a general proof of this proposition, though I have no doubt of its truth.

Table I.

Formation of the quantities f_n.

f_n	n	f_n	n
$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	1	$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{11}=\frac{371}{330}$	10
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}=\frac{31}{30}$	2	$\frac{1}{2}+\frac{1}{3}+\frac{1}{23}=\frac{121}{138}$	11
$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}=\frac{41}{42}$	3	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}=\frac{3421}{2730}$	12
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}=\frac{31}{30}$	4	$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	13
$\frac{1}{2}+\frac{1}{3}+\frac{1}{11}=\frac{61}{66}$	5	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{29}=\frac{929}{870}$	14
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}=\frac{3421}{2730}$	6	$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{1}{31}=\frac{15745}{14322}$	15
$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	7	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{17}=\frac{557}{510}$	16
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{17}=\frac{557}{510}$	8	$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	17
$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{19}=\frac{821}{798}$	9	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}+\frac{1}{19}+\frac{1}{37}=\frac{2557843}{1919190}$	18

TABLE I.—(*continued*).

f_n	n	f_n	n
$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	19	$\frac{1}{2}+\frac{1}{3}+\frac{1}{83}=\frac{421}{498}$	41
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{11}+\frac{1}{41}=\frac{15541}{13530}$	20	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}+\frac{1}{29}+\frac{1}{43}=\frac{4462547}{3404310}$	42
$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}=\frac{1805}{1806}$	21	$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	43
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{23}=\frac{743}{690}$	22	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{23}+\frac{1}{89}=\frac{66817}{61410}$	44
$\frac{1}{2}+\frac{1}{3}+\frac{1}{47}=\frac{241}{282}$	23	$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{1}{19}+\frac{1}{31}=\frac{313477}{272118}$	45
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}+\frac{1}{17}=\frac{60887}{46410}$	24	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{47}=\frac{1487}{1410}$	46
$\frac{1}{2}+\frac{1}{3}+\frac{1}{11}=\frac{61}{66}$	25	$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	47
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{53}=\frac{1673}{1590}$	26	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}+\frac{1}{17}+\frac{1}{97}=\frac{5952449}{4501770}$	48
$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{19}=\frac{821}{798}$	27	$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	49
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{29}=\frac{929}{870}$	28	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{11}+\frac{1}{101}=\frac{37801}{33330}$	50
$\frac{1}{2}+\frac{1}{3}+\frac{1}{59}=\frac{301}{354}$	29	$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{103}=\frac{4265}{4326}$	51
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{31}+\frac{1}{61}=\frac{79085411}{56786730}$	30	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{53}=\frac{1673}{1590}$	52
$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	31	$\frac{1}{2}+\frac{1}{3}+\frac{1}{107}=\frac{541}{642}$	53
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{17}=\frac{557}{510}$	32	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}+\frac{1}{19}+\frac{1}{37}+\frac{1}{109}=\frac{280724077}{209191710}$	54
$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{23}+\frac{1}{67}=\frac{66961}{64722}$	33	$\frac{1}{2}+\frac{1}{3}+\frac{1}{11}+\frac{1}{23}=\frac{1469}{1518}$	55
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}=\frac{31}{30}$	34	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{17}+\frac{1}{29}+\frac{1}{113}=\frac{1897709}{1671270}$	56
$\frac{1}{2}+\frac{1}{3}+\frac{1}{11}+\frac{1}{71}=\frac{4397}{4686}$	35	$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}=\frac{41}{42}$	57
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}+\frac{1}{19}+\frac{1}{37}+\frac{1}{73}=\frac{188641729}{140100870}$	36	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{59}=\frac{1859}{1770}$	58
$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	37	$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	59
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}=\frac{31}{30}$	38	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{31}+\frac{1}{41}+\frac{1}{61}=\frac{3299288581}{2328255930}$	60
$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{79}=\frac{3281}{3318}$	39	$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	61
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{11}+\frac{1}{17}+\frac{1}{41}=\frac{277727}{230010}$	40	$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}=\frac{31}{30}$	62

Table II.

Values of I_n, or the Integral parts of Bernoulli's numbers.

The values of I_1 to I_6 are zero.

I_n	n
1	7
7	8
55	9
529	10
6192	11
86580	12
14 25517	13
272 98231	14
6015 80874	15
1 51163 15767	16
42 96146 43061	17
1371 16552 05088	18
48833 23189 73593	19
19 29657 93419 40068	20
841 69304 75736 82615	21
40338 07185 40594 55413	22
21 15074 86380 81991 60560	23
1208 66265 22296 52593 46027	24
75008 66746 07696 43668 55720	25
50 38778 10148 10689 14137 89303	26
3652 87764 84818 12333 51104 30843	27
2 84987 69302 45088 22262 69146 43291	28
238 65427 49968 36276 44645 98191 92192	29
21399 94925 72253 33665 81074 47651 91097	30
20 50097 57234 78097 56992 17330 95672 31025	31
2093 80059 11346 37840 90951 85290 02797 01847	32
2 27526 96488 46351 55596 49260 35276 92645 81470	33
262 57710 28623 95760 47303 04973 61582 02081 44900	34
32125 08210 27180 32518 20479 23042 64985 24352 19411	35
41 59827 81667 94710 91391 70744 95262 35893 66896 03011	36
5692 06954 82035 28002 38834 56219 12105 86444 80512 97181	37
8 21836 29419 78457 56922 90653 46861 73330 14550 89276 28860	38
1250 29043 27166 99301 67323 39829 70289 55241 77196 36444 84775	39
2 00155 83233 24837 02749 25329 19881 32987 68724 22013 28259 15915	40
336 74982 91536 43742 33396 67690 33387 53016 21959 89471 93843 67232	41
59470 97050 31354 47718 66049 68440 51540 84057 90715 65106 90499 04704	42
110 11910 32362 79775 59564 13079 04376 91604 63051 14442 23148 86269 99497	43
21355 25954 52535 01188 65838 50190 41065 67897 32987 39163 46921 18045 90304	44
43 32889 69866 41192 41961 66130 59379 20621 84513 68511 80910 91449 86557 88033	45
9188 55282 41669 32822 62005 55215 50189 71389 60388 91627 19959 59100 44871 13437	46
20 34689 67763 29074 49345 50279 90220 02006 59751 40253 37827 70239 36918 42141 08241	47
4700 38339 58035 73107 85752 55535 00606 06545 96737 36975 90579 15139 76356 41204 83354	48
11 31804 34454 84249 27067 51862 57733 93426 78903 65954 75074 79181 78993 54166 54911 76373	49
2838 22495 70693 70695 92641 56336 48176 47382 84680 92801 28821 28228 53171 44648 65111 07028	50
7 40642 48979 67885 06297 50827 14092 09841 76879 73178 80887 06673 11610 03487 48532 84412 10855	51
2009 64548 02756 60448 34656 19672 71536 31868 67270 82253 28766 24346 13019 89213 56500 97796 98883	52
5 66571 70050 80594 14457 19346 03051 93569 61419 46828 75104 20621 38756 44521 52460 86197 22777 98400	53
1658 45111 54136 21691 58237 13374 31991 23014 94962 61472 54647 27402 46681 55898 78137 71265 07431 49939	54
5 03688 59950 49237 74192 89421 91518 01548 12442 37426 49032 14141 52565 13225 28310 97674 29893 27917 85387	55
1586 14682 37658 18636 93634 01572 96643 87827 40978 41277 89638 80472 86451 42973 11365 09885 00683 12009 45121	56
5 17567 43617 54562 69840 73240 68250 71225 61240 84923 59305 50859 06216 69403 18108 29579 66515 49771 87766 32444	57
1748 89218 40217 11733 96900 25877 61815 91451 41476 16182 65448 72627 34721 58762 12289 52384 00153 32666 64382 79521	58
6 11605 19994 95218 52558 24525 26426 41677 80767 72684 67832 00716 84324 01127 35747 50763 44103 14895 29605 90861 82633	59
2212 27769 12707 83494 22883 23456 71293 24455 73185 05498 77801 50566 55269 30277 36635 00257 26591 02528 03139 11549 56836	60
8 27227 76798 77096 98542 21062 45998 45957 31204 65051 84335 66283 84885 29885 84472 02350 07188 81721 85613 01633 96614 27405	61
3195 89251 11415 70958 35916 34369 18081 48735 26276 67109 91122 73184 50424 31195 31118 14531 48045 43981 20342 28242 29698 20300	62

Table III.

Formation of the quantities $\Sigma \frac{1}{p}(C_r^n + C_{2r}^n + C_{3r}^n + \&c.)$ *for the several values of* n.

$n=31,\ 2n+1=63$

1537 22867 28091 29301
461 16860 20574 87155
219 60409 61155 89900
88 02459 79873 52292
49 71096 10967 95355
53 92944 69063 74991
25 89740 74035 40869
2 55909 46673 02320
21 70028 58743 95452
27 76703 24385 25380
13 22870 27886 76823
2 29252 23061 24995
64231 24512 17745
2155 26589 88805
1 16186 84091
1 19133
651
2503 77529 91803 05258

$n=32,\ 2n+1=65$

6148 91469 12365 17205
1844 67440 78004 51891
878 28402 58774 41612
335 39464 88292 66520
223 60238 81321 77796
212 48412 70622 02255
132 24055 48295 17680
7 92100 73035 54800
67 77266 56354 24360
97 06794 53315 77584
67 77266 45331 22640
15 89482 13224 66632
5 28067 15357 03200
26216 09982 26400
30 98315 75760
117 99840
1 35408
10037 56709 02711 61583

$n=33,\ 2n+1=67$

24595 65876 49460 68821
7378 69762 86248 27187
3513 26846 20946 84437
1286 91831 91837 73500
993 86927 65468 89636
790 52070 97251 51119
628 02997 66186 40508
34 86773 84299 47509
202 22050 48830 20584
322 24808 88081 26624
322 24808 87800 73456
100 12378 90141 70152
38 91854 92181 32584
2 76018 07956 12240
652 41677 52432
7247 06840
142 56528
1
40210 35660 25758 34158

$n=34,\ 2n+1=69$

98382 63505 97842 75285
29514 79051 62172 95667
14054 66215 13977 53612
5046 32955 07736 99020
4273 25378 16639 40010
2795 37334 71972 94479
2789 19219 50266 58946
206 03495 01161 12302
578 54913 82703 51040
1020 23484 16095 78584
1431 80684 90114 62742
578 54780 54858 20632
260 12226 91331 59664
25 59440 37411 31680
11254 18937 29452
3 09120 40848
9290 50408
782
1 60957 23943 31633 55153

$n=35,\ 2n+1=71$

3 93530 54023 91371 01141
1 18059 16207 51771 04179
56222 22228 73838 00142
20307 54192 77668 84501
17631 27285 87532 09166
9476 97131 44797 08047
11647 24748 56938 93318
1331 56135 49522 29223
1592 15922 43305 24496
3091 80932 26676 88224
5979 89919 29302 26746
3091 80924 00693 85528
1592 12768 17633 04840
212 00697 76557 07416
1 63547 72275 88820
98 48259 16760
4 19761 86616
1 94327
1
6 43767 96768 67906 53491

$n=36,\ 2n+1=73$

15 74122 16095 65484 04565
4 72236 64829 38364 69043
2 24885 31546 77424 14875
83272 34533 10558 40526
69855 72448 08896 72825
31238 21785 82848 25359
45962 07685 79339 39887
8380 47061 12155 96540
4226 98481 44061 64436
8998 09213 27639 13424
23596 35897 75084 62297
15388 77780 85271 68878
8998 08934 98364 84128
1587 33429 42427 32448
20 46682 93052 54376
2464 87858 02336
141 42745 95216
243 18636
876
1
25 72769 09012 71820 60672

$n=37,\ 2n+1=75$

62 96488 64382 61936 18261
18 88946 59313 41141 90131
8 99498 37768 94562 31180
3 43444 74927 37166 00353
2 67607 15346 51546 66685
1 02597 07084 29344 47375
1 72198 71642 27205 86775
49571 71709 72401 36225
10859 28203 57538 24900
25221 99066 57622 48160
88366 93139 35033 52525
71771 19061 96015 01910
47291 09459 43110 67150
10849 39080 42452 77200
224 48794 98896 47800
50294 39750 10900
3737 72571 59280
18745 61525
2 43090
925
102 74937 93013 77043 72350

$n=38,\ 2n+1=77$

251 85954 57530 47744 73045
75 55786 37256 39445 51219
35 97897 02134 94197 11000
14 07667 80088 60778 65828
10 03408 77399 51531 07890
3 49129 02996 82888 65295
6 15331 42731 58309 56115
2 73908 21952 34485 77900
27132 13542 19627 04320
68271 08236 61443 48460
3 15319 07470 41356 61005
3 15319 07470 41351 27010
2 32560 91224 02591 30220
68269 49998 54229 70080
2189 50580 45903 64876
8 60593 02390 75400
80416 06944 70980
9 97266 73130
338 70540
2 70655
410 18153 91632 42827 74968

$n=39,\ 2n+1=79$

1007 43818 30121 90978 92181
302 23145 49042 07049 49043
143 91684 57480 60840 57721
56 93504 74720 88405 43478
37 45017 22130 58920 48100
12 88839 38881 13420 56975
21 08252 54612 02295 24587
14 16901 66512 98796 73665
66909 30148 11143 06968
1 78887 63430 03495 63160
10 75856 11203 04663 02965
13 11063 52113 82460 54410
10 75856 11203 03879 58270
3 98368 04821 03942 62910
19218 99539 58487 58356
126 26129 07932 91940
14 48900 05828 36780
393 92035 88635
28987 53715
397 08955
1
1637 37464 41384 23961 32815

TABLE III.—*continued.*

$n=40,\ 2n+1=81$

	4029	75273	20487	63915	68725
	1208	92581	96157	28686	33395
	575	67896	17212	51987	95532
	226	98241	27456	80596	49567
	141	76943	58308	47224	55020
	52	94048	29982	14478	06223
	69	78059	51623	76241	31890
	68	90498	33065	47086	33520
	1	71499	19492	05921	89432
	4	54952	47470	69994	82928
	35	20983	63937	92297	95070
	51	80299	76644	86795	32061
	47	04148	18215	71619	22800
	21	69264	66588	51721	20720
	1	53373	26375	01230	96240
		1616	94301	25306	95200
		223	54458	04209	10320
			12155	25678	77880
			17	07629	46120
				35738	05950
					1080
	6538	29904	13950	88360	49673

$n=41,\ 2n+1=83$

	16119	01092	81950	55662	74901
	4835	70327	84563	17675	56915
	2302	74189	90252	77359	52586
	895	26860	35488	24264	03267
	552	56703	26238	03053	36700
	237	79729	89500	47040	52495
	226	02262	92052	24854	53175
	316	39806	70248	92075	74305
	5	27201	22882	99685	82328
	11	28345	04317	73975	79328
	110	84095	58302	97205	39990
	195	22214	95595	22219	81861
	195	22214	95595	22219	79620
	110	84095	58259	34545	44760
	11	22428	42267	02556	91408
		18341	52357	21398	52552
		3006	80714	29737	46320
		3	04149	55035	91365
			745	00808	41620
			22	11211	20870
				3	67524
					1
	26125	62920	85503	12590	23891

$n=42,\ 2n+1=85$

	64476	04371	27802	22650	99605
	19342	81311	38296	68748	78771
	9210	94154	39608	40030	39887
	3516	87500	60249	60409	35310
	2228	32141	19003	23160	11921
	1117	20270	21381	48307	10031
	733	36454	62045	71240	08370
	1377	41466	64740	74261	26650
	23	31422	25211	55900	40425
	27	68771	06277	76233	60744
	336	48147	32426	91174	51900
	703	98290	29570	64982	41697
	771	80849	82585	76217	79895
	534	01108	27241	38093	45200
	75	89146	72146	36606	40770
	1	86550	53889	61232	86640
		35781	00500	13875	81208
		63	49788	85837	49550
			25330	27486	15080
			1012	05436	09050
				624	79080
					1190
1	04478	37801	39109	42509	92974

$n=43,\ 2n+1=87$

2	57904	17485	11208	90603	98421
	77371	25245	53450	63274	21747
	36843	45355	01601	27229	14060
	13857	40525	25853	97563	75705
	9232	15088	46612	49587	75815
	5267	91263	99989	70454	09039
	2476	02233	35946	23361	14089
	5706	32862	79462	29356	38680
	141	50675	07117	85610	51623
	70	75337	53558	92805	23984
	987	27623	50780	15869	38688
	2436	26275	66719	53385	70137
	2916	50059	79649	83465	46485
	2436	26275	66719	51960	49260
	477	15962	83192	53016	08606
	17	18929	96554	28502	84040
	3	81358	23279	25667	84328
		1131	16952	94847	95555
		6	96768	81071	25105
			36058	05108	71010
				64926	17730
				4	45179
4	18145	43696	31477	87672	59286

$n=44,\ 2n+1=89$

10	31616	69940	44835	62415	93685
3	09485	00982	13626	60910	82547
1	47373	11079	28532	34908	53892
	55040	08418	38412	01106	33185
	38738	58806	16065	67132	80456
	24298	88226	22690	02645	66543
	9106	03076	89093	28574	66705
	22571	86365	74402	44302	47263
	958	80431	79852	81358	87972
	217	13966	91956	71022	97744
	2805	64150	28911	83105	85971
	8112	58924	75574	37239	62197
	10565	23250	84466	93118	50014
	10565	23250	84466	93108	84720
	2805	64129	80453	37554	04356
	144	75977	94637	80681	98496
	36	78322	26506	32796	24208
		17508	53706	50690	09460
		159	56413	23245	72580
		10	38259	76512	56435
			46	22743	82376
				830	15284
					1
16	74442	26979	22910	86006	56090

$n=45,\ 2n+1=91$

41	26466	79761	79342	49663	74741
12	37940	03928	53450	90527	03539
5	89493	14657	92002	13642	17917
2	20630	12287	04558	50794	81101
1	62213	29752	88085	98248	78116
1	08277	69370	36146	90754	98767
	37462	25916	39891	87356	06005
	85510	48319	92768	37134	47486
	6422	80070	93125	63558	81296
	961	33966	71106	06982	74309
	7736	77333	34621	36509	50687
	26055	72781	86366	96212	20581
	36789	64891	33411	65386	34480
	43701	64355	76658	66950	23160
	15504	85980	49873	91746	03020
	1122	71078	95912	53584	71290
	323	92967	04394	43657	22864
	2	38991	53093	81919	79129
		3111	50058	03291	65310
		248	63589	11221	93575
			2426	94050	74740
				94429	88555
					1365
67	06615	69773	00887	27623	92033

Table III.—*continued.*

$n=46$, $2n+1=93$

165	05867	19047	17369	98654	98965
49	51760	15714	14507	30852	31923
23	57981	02721	02481	42664	99852
8	90369	77988	61544	18866	67976
6	70564	32843	34174	44613	08710
4	63925	85063	06490	32488	70159
1	69682	61924	61491	41683	50961
3	11125	13732	78697	65961	33429
	40919	58249	34213	63953	09380
	5874	02986	26061	66881	74584
	20738	12959	07875	86518	47431
	80889	98810	51892	49346	42181
1	23440	09258	92343	39345	65723
1	72946	92982	39542	81973	25699
	80889	98810	45561	73035	99658
	8072	19623	17166	09807	39796
	2624	56653	43559	46904	59114
	29	12836	94972	53142	07162
		52612	64617	64749	77060
		5065	06829	62892	57685
			98880	49039	02264
			73	44928	18878
				5	83947
268	57701	29884	00347	68309	72537

$n=47$, $2n+1=95$

660	23468	76188	69479	94619	95861
198	07040	62856	62251	35874	34291
94	31943	10085	12489	68876	62822
36	01280	10129	91093	31574	44388
27	21489	31383	23348	79466	93346
19	10138	38551	08342	38240	73487
8	09432	93570	24179	80247	10221
10	90183	09753	51631	01527	39355
2	45931	22208	25376	36023	35540
	40278	38824	55964	39178	70344
	54119	41554	34542	83174	01615
2	43214	67806	51991	41425	57861
3	99970	98215	59908	19665	45535
6	56639	49121	08468	27389	98895
3	99970	98215	59726	60692	93990
	54117	65191	38358	30015	06290
	19695	27659	81500	88956	30158
	320	34031	97173	28274	26055
	7	83051	55059	32025	75243
		89389	44641	47491	52425
		32	46333	74700	26535
			4204	50055	93465
				1241	58255
1075	09243	47821	06065	90738	25977

$n=48$, $2n+1=97$

2640	93875	04754	77919	78479	83445
792	28162	51426	46190	68520	66099
377	27753	41139	47394	77289	87875
145	35370	87283	90942	28670	39798
108	37153	23049	46940	95462	69225
75	70844	36360	72332	41153	59503
38	98331	48805	90928	36984	12732
36	89710	59446	81610	89003	60540
13	95226	33260	05743	38174	84136
2	74144	94233	12122	46552	60144
1	37716	87279	44213	57261	99972
7	09528	53534	91006	40803	82586
12	54050	43294	26386	36622	12744
23	97892	91849	23159	43800	10520
18	81075	64941	23320	28471	03856
3	40042	89785	53571	18151	32640
1	37689	50126	27429	64235	04528
	3207	53662	06535	06333	24112
	103	87145	35487	73538	18608
	13	87324	20835	69068	45636
		883	91403	07628	27760
		1	86439	56766	01648
			1	60578	00980
					1
4301	11894	89589	01915	63549	89088

$n=49$, $2n+1=99$

10563	75500	19019	11679	13919	33781
3169	12650	05705	67874	24222	38003
1509	10785	74145	58811	30559	90540
583	82981	17671	80971	16015	20138
424	86750	38753	08304	66471	82105
289	86148	74931	92631	73108	59023
184	54457	78596	26980	02288	77522
121	08374	64846	89911	76890	37535
74	93111	34372	18060	80202	11496
17	80340	09554	71787	07391	82844
3	42381	24335	38721	18019	20820
20	11644	04748	25270	68663	98986
38	11653	28964	87513	29567	72032
84	41348	72830	64039	53318	24728
84	41348	72830	64039	50150	79376
20	11644	02133	69968	05063	51752
9	01392	40300	34630	49263	43408
	29469	24270	22540	89436	65279
	1241	08478	11948	28654	53368
	191	73532	00780	44305	07636
		20418	41411	06213	21256
		66	50134	87293	72018
			141	62980	46436
					1617
17198	83414	90506	38151	84000	91699

$n=50$, $2n+1=101$

42255	02000	76076	46716	55677	35125
12676	50600	22822	82755	96796	36275
6036	42630	18154	66017	70390	50000
2332	40675	80026	76219	30376	28267
1650	55774	27959	74112	30704	43270
1077	80364	70990	16504	89535	42415
847	23245	74107	40633	42330	51665
387	69809	84777	64144	31298	43990
382	18490	75383	05882	81257	01390
109	36921	78929	24186	20447	07920
8	35049	02617	48321	32155	53435
55	51258	72040	54203	56059	39490
112	50057	92849	50087	71857	59350
287	06270	08615	98253	25307	52180
362	48989	01185	99829	49201	96300
112	50057	92663	54749	34186	89200
55	51258	09166	76687	79000	41720
2	50117	10192	66943	72529	57410
	13478	44762	37287	84312	67760
	2384	88513	89017	82612	40300
	4	07561	23027	12951	47600
		1963	92871	97855	47900
			9169	62196	73100
				8	16585
					1
68751	79439	41361	97624	09049	22648

$n=51$, $2n+1=103$

1	69020	08003	04305	86866	22709	40501
	50706	02400	91291	98577	86626	50675
	24145	71033	51046	33298	33411	12161
	9280	96457	79446	49859	14416	04892
	6406	10079	41121	18390	88586	55300
	3923	60246	19656	42730	74075	78895
	3751	26391	25299	20662	00059	62105
	1225	38382	04060	70387	72538	64195
	1857	11456	43367	11227	50681	68146
	635	71623	53966	52292	03250	10920
	20	25925	34400	09747	61103	75445
	149	31277	13811	78700	65674	81490
	322	93199	12194	11427	69940	87170
	944	82479	17455	99171	86172	69300
	1493	45834	72886	31297	50712	08756
	596	93489	18042	03533	62912	87240
	322	93199	06481	75682	12723	35720
	19	72770	47510	64948	03149	92905
	1	34095	23743	83471	66656	24135
		26941	50889	17227	18414	95260
		71	36397	14205	03780	34476
			49126	27411	92070	62470
			4	58743	08756	45660
					2042	62905
						1751
						1
2	74823	95356	96505	59860	50467	12474

Table III.—*continued.*

$n=52,\ 2n+1=105$

6	76080	32012	17223	47464	90837	62005
2	02824	09603	65167	49275	46878	65651
	96582	90287	45317	63923	55833	94572
	36877	10835	43044	91277	34623	62825
	25037	84627	05650	79797	10589	94360
	14148	00851	62482	34271	68151	67247
	15995	40243	03127	46512	74215	01615
	3908	04814	66848	96974	01424	03350
	8622	19227	76628	76968	63879	63560
	3505	11558	62111	89576	63660	09904
	51	41028	06172	61397	83649	06225
	391	94805	80960	60322	78441	30429
	902	82063	38685	34517	70537	88100
	3015	04579	94921	21559	32578	38600
	5917	47647	03889	16461	82066	76205
	3015	04579	94920	91853	48292	58400
	1781	01885	75141	81034	76231	84880
	145	36203	50078	46985	49525	79300
	12	30521	00237	54210	58727	86180
	2	78599	69422	12235	65427	35075
		1110	10622	20967	25472	02960
		10	60195	48099	15832	45400
			184	17185	72134	21350
				3	09798	73925
					9	56046
						1820
10	98816	27096	33035	42876	78820	09984

$n=53,\ 2n+1=107$

27	04321	28048	68893	89859	63350	48021
8	11296	38414	60667	26885	89750	40307
3	86331	74994	98818	24837	23262	06626
1	46700	93377	65344	64131	19517	91810
	99091	51201	98154	97282	10154	89600
	51330	37629	94150	95429	53653	65519
	65721	28746	11152	84484	09096	86395
	13046	86125	03658	67371	53114	10605
	38349	93950	88648	71806	52040	85888
	18386	32280	12099	25600	44582	80934
	157	16857	21727	70559	10012	84745
	1005	34094	82822	29033	40776	47809
	2461	48866	48540	59022	86704	02755
	9343	34684	64057	86311	59743	50120
	22597	98590	13976	73572	38047	58457
	14539	39177	62922	21854	67829	28900
	9343	34684	64041	81913	17031	28080
	1005	30256	16213	41774	07757	03915
	104	77904	81151	80372	73341	89830
	26	55359	43853	53930	07627	74135
		15505	94183	62328	32886	40360
		200	41228	57567	75619	48878
			6107	82808	33760	96350
				319	43066	36885
					2581	77946
					10	32122
						.1
43	95161	50952	42417	49056	15321	06993

$n=54,\ 2n+1=109$

108	17285	12194	75575	59438	53401	92085
32	45185	53658	42670	87687	57511	09427
15	45326	86134	77725	30205	93121	98167
5	85306	54848	03542	89787	20121	51950
3	97684	55721	46477	68242	03102	62561
1	90771	17282	31079	72954	44149	45039
2	60640	80664	90582	29871	74052	12025
	47721	74052	16645	40096	68666	89825
1	63807	85912	94599	42478	04060	63196
	92022	24231	94899	90221	11892	42904
	697	68647	92119	65569	64477	61607
	2522	81178	81227	67512	38778	39679
	6552	84249	81581	25696	70770	86148
	28159	21123	30110	73232	20531	52250
	83340	69236	56808	93889	11768	82399
	67120	67450	60831	51244	42073	09416
	46764	40436	89923	59473	57182	07380
	6552	83596	64487	46048	97074	12230
	832	29349	15060	07657	09703	66180
	234	67636	11894	79327	97142	47085
	1	96275	21214	63794	71762	08944
		3360	75986	88443	91157	58108
		1	71193	69570	83414	43410
				24104	72931	37245
				4	22120	94171
					2892	89052
						1962
						1
175	88532	57245	26240	26553	83862	64446

$n=55,\ 2n+1=111$

432	69140	48779	02302	37754	13607	68341
129	80742	14633	70694	31614	21101	26899
61	81305	78397	00328	91107	73372	52620
23	43031	81800	13614	61240	53224	97101
16	14023	64612	93655	98961	29865	39475
7	37429	25606	25888	90494	36307	45359
10	00236	96391	16728	82415	35982	64441
1	96775	01827	94590	29161	24819	18297
6	73431	78568	19304	83954	85615	72068
4	40618	44879	54858	88694	95723	38504
	4330	96649	73527	22949	89855	27907
	6203	83300	79206	64609	69871	50403
	17052	57037	39914	05781	24102	59665
	82649	99258	76274	44754	68852	98085
2	97366	99409	26252	81819	50425	82595
2	97366	99409	26252	81819	44017	59060
2	23918	97150	79987	09479	33801	22396
	40409	15512	64339	33968	65290	42085
	6196	52654	34685	08227	53342	50645
	1933	46718	63181	79888	34756	79425
	22	69432	14044	25126	42249	15915
		50535	56403	76724	33539	48890
		41	30978	30948	38913	51850
			14	01517	83295	51245
				468	55424	52981
				4	90586	01735
					11	97801
						2035
704	14188	02606	57029	63487	47957	17823

TABLE III.—*continued.*

$n=56,\ 2n+1=113$

1730	76561	95116	09209	51016	54430	73365
519	22968	58534	82770	05880	90367	14803
247	25219	39768	27527	97569	95480	45372
94	02296	00824	42074	54503	71544	45401
65	88250	79051	91915	76250	78660	19116
29	82642	34192	84858	27229	59659	45615
37	27607	22031	41051	89638	98004	53826
9	02333	46348	71041	47755	48453	26400
26	70098	80697	09858	43818	91752	43757
20	23381	43671	16553	12549	33027	50704
	30554	71855	21946	50609	92436	90702
	15008	27461	94451	74918	85874	79703
	43425	49159	72479	20287	51794	37840
2	36548	68984	35027	19128	07988	95840
10	28272	31618	47720	12772	69454	50226
12	67163	86438	93150	05625	19692	46688
10	28272	31618	47720	12761	43029	13296
2	36548	68976	88010	49170	80071	95480
	43423	72089	37638	09594	49780	04520
	14920	70775	00261	49187	15049	99270
	241	36078	29364	74285	71685	17496
	6	87718	38544	17014	18145	99088
		871	36102	47471	35482	48356
			652	11800	34514	72635
				38012	96492	62356
				564	44151	23256
					3609	37368
					12	87748
						1
2819	65747	03883	19929	99419	40647	80228

$n=57,\ 2n+1=115$

6923	06247	80464	36838	04066	17722	93461
2076	91874	34139	31037	00067	97241	02451
989	00881	32892	83899	57140	79931	46597
377	62158	96919	80460	14312	76549	04698
268	82951	56468	33201	44962	82169	68996
125	45289	96465	61707	74050	50221	64751
135	59805	29162	82285	56224	52331	42361
44	03434	20285	79570	22979	61971	83291
102	29421	74735	87144	21201	61941	77525
89	31474	09963	89906	69240	10236	03804
2	16071	11886	10502	62595	97502	85510
	36209	83855	54918	57461	69645	18231
1	08336	41115	33823	47381	78434	31590
6	60944	87040	71023	48314	22970	47500
34	51267	29779	37432	39259	12278	63690
52	04423	01445	61152	01674	91594	06040
45	38939	41588	63168	64411	56603	34448
13	18517	56924	70160	52138	26421	48275
2	87517	66712	99209	81709	02331	51140
1	08311	44994	62031	09547	91974	19950
	2375	55545	36765	59974	28523	00580
	85	37867	45562	56681	74141	98905
		16272	85332	54059	06232	70580
			24997	85679	89731	17675
			23	73095	09610	92796
				47434	79632	42860
				6	57206	79090
					4019	61340
						2185
11291	46539	06528	22156	05672	89171	60320

$n=58,\ 2n+1=117$

27692	24991	21857	47352	16264	70891	73845
8307	67497	36557	24176	82575	65115	80979
3956	03570	17408	21050	30894	95841	67692
1515	49252	86452	11467	23346	87475	82993
1091	16739	64229	52861	61660	74961	61410
540	69111	85293	15933	88023	92323	80687
485	15774	28013	03401	85137	61203	49077
218	95202	78106	05354	40258	45696	27180
379	33043	51955	65173	29082	11780	60486
379	75523	24136	66131	42396	56709	82304
14	69761	42173	29659	52645	21410	39037
	91570	34395	47784	68210	31527	91311
2	65186	21340	80116	13206	56033	52003
18	04898	18359	82892	12601	48864	80732
112	59759	55905	20874	30499	99914	34977
206	41270	93869	03552	06642	88695	08703
192	99024	35852	41517	80637	14730	76168
70	17615	86267	47066	11615	89095	04466
18	04898	13981	83568	75002	24996	88840
7	42425	75690	39740	41810	28623	14930
	21755	08678	63011	28185	55947	52680
	973	74737	06533	74524	86937	03814
	2	71989	11986	74987	18460	93980
		8	07787	88684	68170	31155
			1184	10465	60439	39071
			30	65643	12244	38552
				810	87368	78268
				7	57697	12590
					14	82741
45210	69849	27257	80209	35823	13172	90671

$n=59,\ 2n+1=119$

1	10768	99964	87429	89408	65058	83566	95381
	33230	69989	46228	96880	24125	17373	50963
	15824	14381	62765	76468	28826	29627	28702
	6071	25206	40499	81747	88410	47263	40323
	4393	30863	08378	40276	48576	91485	63926
	2348	32141	33172	63673	10119	10778	18127
	1726	49475	09181	95153	57885	26004	92901
	1078	31781	94381	86723	02780	58227	37368
	1363	72509	50719	56412	91686	59494	48932
	1558	30552	88619	49099	97920	90071	21304
	95	23921	32150	40527	83982	31394	28815
	2	79406	94693	89394	28641	73123	63231
	6	39335	77979	76304	07197	86255	16407
	48	21990	92125	78826	08430	35042	80662
	357	55211	15518	07880	13131	60500	88819
	791	92548	22980	60021	34338	63895	24089
	791	92548	22980	60021	34338	63895	19448
	357	55211	15518	07802	03305	63887	01637
	107	75671	63236	79282	47781	28999	28015
	48	21990	04553	45067	04486	60465	43130
	1	86271	29795	92929	52427	82082	42154
		10265	28571	97557	69578	24001	41859
		41	06743	24858	00182	84546	79212
			224	16912	06937	35271	76835
				48617	53678	41198	60921
				1582	63826	19616	40247
					72989	02772	09226
					967	23482	19898
						4957	29741
1	80973	11279	28451	63429	01611	99280	92273

Table III.—*continued.*

$n=60, \ 2n+1=121$

4	43075	99859	49719	57634	60235	34267	81525
1	32922	79957	84915	87405	67285	64887	19155
	63296	57425	57930	13606	10058	72248	56875
	24273	81412	00370	26578	31642	67473	53817
	17537	60713	24716	69781	15415	48645	77625
	10137	05071	52329	47854	56603	27665	37487
	6204	23779	06828	94196	96000	75526	57365
	5183	30430	01421	41639	04478	54545	85975
	4760	34096	29195	28268	48740	95473	39656
	6182	13082	39396	39322	38734	86709	07479
	587	53139	92560	62591	55473	89583	90507
	12	41976	91981	71104	62393	62737	44663
	15	38862	24844	64897	87591	35153	41070
	126	15384	21104	75154	04676	88336	73152
	1106	49118	92012	50819	86169	56844	83215
	2943	87045	64689	78881	18432	42163	29526
	3141	73715	92808	28281	39507	38731	75517
	1748	03254	53643	93698	82827	56780	96892
	613	57942	00077	73796	69719	34537	07756
	297	68407	93416	71077	16473	42669	24425
	14	97596	48193	18569	59718	69234	08680
	1	00574	86413	68783	90199	76046	28740
		564	67719	66797	52514	12518	39165
			5424	89272	07883	93576	79407
			16	80777	70025	10009	06126
				67192	71217	45117	43820
				50	46669	91670	37912
					90027	24112	36660
					9	99721	64435
7	24182	73411	81733	68035	40727	96988	28627

$n=61, \ 2n+1=123$

17	72303	99437	98878	30538	40941	37071	26101
5	31691	19831	39663	48930	93852	31907	68435
2	53186	28491	14125	47219	77277	33867	32300
	96908	22257	13794	41643	91001	02179	40292
	69521	24129	43529	60542	32025	95149	36725
	43087	87536	32518	60993	25317	26629	31215
	22914	64302	15945	60828	59575	36644	72261
	24160	92864	33451	62038	48569	69036	83900
	16158	27469	28051	77738	72920	43387	46656
	23750	41878	23710	79199	97963	29403	90604
	3456	63969	65595	28115	53771	86303	75827
	77	71763	91169	06828	61040	82569	55063
	38	85881	95584	53414	30520	41284	73841
	323	49230	88336	48107	71201	80821	19420
	3340	84643	56448	32271	46499	64905	48791
	10619	16347	83109	36800	73364	76316	34069
	12069	86887	15228	13412	85572	92731	41617
	8217	72442	84893	77144	30360	42686	47231
	3340	84643	56446	49344	42601	05102	82506
	1751	78325	27847	51444	68000	09448	89381
	113	49966	06458	05280	49261	96831	66996
	9	20260	00685	24372	70327	80823	52971
		7120	62690	19969	46409	21219	31521
		1	15962	87773	24083	10560	35871
			498	45356	06712	75090	85626
			24	00699	61640	64838	75626
				2784	20326	36050	40101
				64	33089	43000	82476
					1363	80209	04651
28	97042	79681	94647	52859	15592	96072	72074

$n=62, \ 2n+1=125$

70	89215	97751	95513	22153	63765	48285	04405
21	26764	79325	58653	96184	92269	46058	12531
10	12745	11239	41912	97668	67454	86433	65000
3	86684	50786	22868	24914	40706	84138	33440
2	74514	07044	94746	36352	19313	79031	90650
1	79389	57698	25603	72166	99699	04739	16175
	88326	29369	48517	19801	79218	73622	89925
1	08964	57287	53990	41133	75824	81250	50100
	53416	87275	74718	42927	19308	21871	00000
	88493	42679	27412	45701	65763	91256	30000
	19438	93661	66785	70819	10300	64844	85875
	556	33273	68635	58511	83442	97916	61575
	118	47201	08489	43336	29635	40502	25125
	813	74880	81117	96046	51541	36028	20375
	9852	19173	37381	17857	24498	00057	40750
	37222	30527	22341	74674	66591	26240	46625
	44971	86718	95681	74975	78456	82090	41450
	37222	30527	22341	74674	66565	34669	87750
	17435	39385	60579	34289	09197	40435	61900
	9852	19173	37313	66978	42525	93054	37375
	813	71172	06336	64129	34116	79412	99000
	78	98134	05659	62224	19756	93668	16750
		82860	14788	35981	00107	19894	43375
		22	13577	09957	18335	14391	08375
			12876	71698	40079	39847	12005
			735	39217	49861	75890	75500
			1	26184	65083	59009	40250
				3665	91493	25414	64625
				1	35505	97693	72375
115	86892	47169	98580	24228	40420	37749	33281

Table IV.

Values of odd powers of 2.

Power	Index
2	1
8	3
32	5
128	7
512	9
2048	11
8192	13
32768	15
1 31072	17
5 24288	19
20 97152	21
83 88608	23
335 54432	25
1342 17728	27
5368 70912	29
21474 83648	31
85899 34592	33
3 43597 38368	35
13 74389 53472	37
54 97558 13888	39
219 90232 55552	41
879 60930 22208	43
3518 43720 88832	45
14073 74883 55328	47
56294 99534 21312	49
2 25179 98136 85248	51
9 00719 92547 40992	53
36 02879 70189 63968	55
144 11518 80758 55872	57
576 46075 23034 23488	59
2305 84300 92136 93952	61
9223 37203 68547 75808	63
36893 48814 74191 03232	65
1 47573 95258 96764 12928	67
5 90295 81035 87056 51712	69
23 61183 24143 48226 06848	71
94 44732 96573 92904 27392	73
377 78931 86295 71617 09568	75
1511 15727 45182 86468 38272	77
6044 62909 80731 45873 53088	79
24178 51639 22925 83494 12352	81
96714 06556 91703 33976 49408	83
3 86856 26227 66813 35905 97632	85
15 47425 04910 67253 43623 90528	87
61 89700 19642 69013 74495 62112	89
247 58800 78570 76054 97982 48448	91
990 35203 14283 04219 91929 93792	93
3961 40812 57132 16879 67719 75168	95
15845 63250 28528 67518 70879 00672	97
63382 53001 14114 70074 83516 02688	99
2 53530 12004 56458 80299 34064 10752	101
10 14120 48018 25835 21197 36256 43008	103
40 56481 92073 03340 84789 45025 72032	105
162 25927 68292 13363 39157 80102 88128	107
649 03710 73168 53453 56631 20411 52512	109
2596 14842 92674 13814 26524 81646 10048	111
10384 59371 70696 55257 06099 26584 40192	113
41538 37486 82786 21028 24397 06337 60768	115
1 66153 49947 31144 84112 97588 25350 43072	117
6 64613 99789 24579 36451 90353 01401 72288	119
26 58455 99156 98317 45807 61412 05606 89152	121
106 33823 96627 93269 83230 45648 22427 56608	123
425 35295 86511 73079 32921 82592 89710 26432	125

Table V.

Values of $F_n = \Sigma \frac{1}{p}(C_r^n + C_{2r}^n + C_{3r}^n + \&c.) - 2^{2n-1} + n.$

F_n	n
197 93228 99666 11337	31
814 19505 34163 85807	32
3316 86845 51567 30959	33
13383 28684 34869 42259	34
53472 15732 80850 01814	35
2 11585 84869 23594 53860	36
8 30204 96439 84139 44995	37
32 39222 05336 71210 65438	38
126 21736 96201 37492 94582	39
493 66994 33219 42486 96625	40
1947 11281 62577 29096 11580	41
7764 31244 47406 08533 43608	42
31289 17468 64664 51766 61697	43
1 27017 22068 55657 42382 65606	44
5 16915 50130 31873 53128 29966	45
20 98900 51313 24292 70327 24135	46
84 74040 33538 01845 98808 32232	47
339 71082 32456 85035 95830 13968	48
1353 20164 61977 70633 13121 91076	49
5369 26438 27247 27549 25533 20010	50
21293 83352 40046 79561 16403 01773	51
84695 79078 07200 21679 42563 67028	52
3 38679 58879 39076 64266 70295 35014	53
13 62604 88953 12876 87396 03759 76372	54
55 10477 29438 03576 06856 27545 65366	55
223 50904 11209 06115 72894 59001 70236	56
906 87167 35831 66898 99573 62587 20185	57
3672 32362 44471 59181 11426 06835 29961	58
14819 61331 97306 79316 04023 73930 49260	59
59568 73622 57154 31583 50374 95586 56399	60
2 38586 80524 96330 07051 54180 90465 82983	61
9 53068 50542 05310 40997 94772 15321 76735	62

CALCULATION OF BERNOULLI'S NUMBER FOR $n = 32$.

Table of the values of the alternate binomial coefficients for the index $2n+1=65$, or of the values of C_r^n for $n=32$.

$n=32$, $2n+1=65$

C_r	r
2080	1
6 77040	2
825 98880	3
50473 81560	4
17 90137 99328	5
402 78104 84880	6
6099 25587 71040	7
64804 59369 42300	8
4 98105 89663 01600	9
28 33960 39082 73840	10
121 45544 53211 73600	11
397 37053 30616 65800	12
1002 59642 18786 64480	13
1965 40727 14605 56560	14
3009 10630 52706 45216	15
3609 71421 70081 32870	16
3397 37808 65958 89760	17
2507 58858 77255 37680	18
1448 19483 16025 15360	19
651 68767 42211 31912	20
227 06887 60352 37600	21
60 72772 26605 86800	22
12 32156 69166 40800	23
1 86789 71123 63100	24
20737 46998 21536	25
1642 10735 15280	26
89 50689 96640	27
3 19667 49880	28
6961 90560	29
82 59888	30
43680	31
65	32
18446 74407 37095 51615	Sum $=2^{64}-1$

Formation of the several values of $\frac{1}{p}(C^n_r + C^n_{2r} + C^n_{3r} + \&c.)$, *when* $n=32$.

$n=32,\ 2n+1=65$

$r=1,\ p=2r+1=3$

3) 18446 74407 37095 51615
6148 91469 12365 17205

$r=2,\ p=2r+1=5$

6 77040
50473 81560
402 78104 84880
64804 59369 42300
28 33960 39082 73840
397 37053 30616 65800
1965 40727 14605 56560
3609 71421 70081 32870
2507 58858 77255 37680
651 68767 42211 31912
60 72772 26605 86800
1 86789 71123 63100
1642 10735 15280
3 19667 49880
82 59888
65
5) 9223 37203 90022 59455
1844 67440 78004 51891

$r=3,\ p=2r+1=7$

825 98880
402 78104 84880
4 98105 89663 01600
397 37053 30616 65800
3009 10630 52706 45216
2507 58858 77255 37680
227 06887 60352 37600
1 86789 71123 63100
89 50689 96640
82 59888
7) 6147 98818 11420 91284
878 28402 58774 41612

$r=5,\ p=2r+1=11$

17 90137 99328
28 33960 39082 73840
3009 10630 52706 45216
651 68767 42211 31912
20737 46998 21536
82 59888
11) 3689 34113 71219 31720
335 39464 88292 66520

$r=6,\ p=2r+1=13$

402 78104 84880
397 37053 30616 65800
2507 58858 77255 37680
1 86789 71123 63100
82 59888
13) 2906 83104 57183 11348
223 60238 81321 77796

$r=8,\ p=2r+1=17$

64804 59369 42300
3609 71421 70081 32870
1 86789 71123 63100
65
17) 3612 23016 00574 38335
212 48412 70622 02255

$r=9,\ p=2r+1=19$

4 98105 89663 01600
2507 58858 77255 37680
89 50689 96640
19) 2512 57054 17608 35920
132 24055 48295 17680

$r=11,\ p=2r+1=23$

121 45544 53211 73600
60 72772 26605 86800
23) 182 18316 79817 60400
7 92100 73035 54800

$r=14,\ p=2r+1=29$

1965 40727 14605 56560
3 19667 49880
29) 1965 40730 34273 06440
67 77266 56354 24360

$r=15,\ p=2r+1=31$

3009 10630 52706 45216
82 59888
31) 3609 10630 52789 05104
97 06794 53315 77584

$r=18,\ p=2r+1=37$

37) 2507 58858 77255 37680
67 77266 45331 22640

$r=20,\ p=2r+1=41$

41) 651 68767 42211 31912
15 89482 13224 66632

$r=21,\ p=2r+1=43$

43) 227 06887 60352 37600
5 28067 15357 03200

$r=23,\ p=2r+1=47$

47) 12 32156 69166 40800
26216 09982 26400

$r=26,\ p=2r+1=53$

53) 1642 10735 15280
30 98315 75760

$r=29,\ p=2r+1=59$

59) 6961 90560
117 99840

$r=30,\ p=2r+1=61$

61) 82 59888
1 35408

The following extract from the calculations for B_{31} supplies the further data which are required in making the similar calculations for B_{32}.

Table of the products P_r^n for $n=31$, and calculation of the quantities I_{31} and B_{31}.

P_r	r
3738 72655 92825	7
142 37926 00199 33475	9
3 26491 85674 90972 87920	11
5091 58999 25656 79046 97257	13
51 78275 84925 36404 11409 58306	15
32629 66196 71881 10212 66003 86655	17
119 34005 74495 92280 86701 49365 99931	19
23247 08675 60794 03855 76884 80981 36525	21
21 42517 90194 62108 56131 24755 35639 47600	23
785 22331 21935 11816 58235 15793 23718 18520	25
8645 51751 16414 13997 85806 97596 31329 64045	27
16774 64384 84868 76711 91869 76074 95811 62624	29
26227 04351 88775 49492 30619 77419 37134 85683	Sum

P_r	r
2 56476 64196 67795	8
7135 44902 05779 29205	10
135 63249 71972 73079 18500	12
1 71790 03065 28992 64048 33947	14
1385 12625 98357 65576 54645 74789	16
6 71133 42236 10139 93686 00391 90688	18
1813 75138 15056 43708 74010 68787 86060	20
2 47319 34636 79761 76272 79085 95545 35175	22
147 61606 58640 58443 98042 19219 20139 57165	24
3102 83046 11851 35783 30953 48845 79801 64369	26
15767 53321 46412 01553 86990 44735 33372 18261	28
8498 13384 95367 52252 03010 48537 05036 52967	30
27518 60498 94566 69639 20928 76040 64824 28921	Sum
197 93228 99666 11337	$=F_{31}$

27518 60498 94566 69639 21126 69269 64490 40258	
26227 04351 88775 49492 30619 77419 37134 85683	
63) 1291 56147 05791 20146 90506 91850 27355 54575	P_{31}
20 50097 57234 78097 56992 17330 95672 31025	$=I_{31}$

$$\text{Also } B_{31}=I_{31}+1-\tfrac{5}{6}$$
$$=I_{31}+\tfrac{1}{6}.$$

Hence the numerator of B_{31} is 123 00585 43408 68585 41953 03985 74033 86151
and the denominator is 6

As a test, this numerator should be divisible by 31.

By actual division we find the quotient to be 3 96793 07851 89309 20708 16257 60452 70521 without any remainder. Hence the test is satisfied.

Table of the factors by which the quantities P_r^n for $n=31$ must be multiplied in order to find the corresponding quantities for $n=32$.

$r=7$, Factor $=\frac{65 \cdot 64}{51 \cdot 50}=\frac{13 \cdot 32}{51 \cdot 5}$	$r=8$, Factor $=\frac{65 \cdot 64}{49 \cdot 48}=\frac{65 \cdot 4}{49 \cdot 3}$	$r=9$, Factor $=\frac{65 \cdot 64}{47 \cdot 46}=\frac{65 \cdot 32}{47 \cdot 23}$
$r=10$, Factor $=\frac{65 \cdot 64}{45 \cdot 44}=\frac{13 \cdot 16}{9 \cdot 11}$	$r=11$, Factor $=\frac{65 \cdot 64}{43 \cdot 42}=\frac{65 \cdot 32}{43 \cdot 21}$	$r=12$, Factor $=\frac{65 \cdot 64}{41 \cdot 40}=\frac{13 \cdot 8}{41}$
$r=13$, Factor $=\frac{65 \cdot 64}{39 \cdot 38}=\frac{5 \cdot 32}{3 \cdot 19}$	$r=14$, Factor $=\frac{65 \cdot 64}{37 \cdot 36}=\frac{65 \cdot 16}{37 \cdot 9}$	$r=15$, Factor $=\frac{65 \cdot 64}{35 \cdot 34}=\frac{13 \cdot 32}{7 \cdot 17}$
$r=16$, Factor $=\frac{65 \cdot 64}{33 \cdot 32}=\frac{65 \cdot 2}{33}$	$r=17$, Factor $=\frac{65 \cdot 64}{31 \cdot 30}=\frac{13 \cdot 32}{31 \cdot 3}$	$r=18$, Factor $=\frac{65 \cdot 64}{29 \cdot 28}=\frac{65 \cdot 16}{29 \cdot 7}$
$r=19$, Factor $=\frac{65 \cdot 64}{27 \cdot 26}=\frac{5 \cdot 32}{27}$	$r=20$, Factor $=\frac{65 \cdot 64}{25 \cdot 24}=\frac{13 \cdot 8}{5 \cdot 3}$	$r=21$, Factor $=\frac{65 \cdot 64}{23 \cdot 22}=\frac{65 \cdot 32}{23 \cdot 11}$
$r=22$, Factor $=\frac{65 \cdot 64}{21 \cdot 20}=\frac{13 \cdot 16}{21}$	$r=23$, Factor $=\frac{65 \cdot 64}{19 \cdot 18}=\frac{65 \cdot 32}{19 \cdot 9}$	$r=24$, Factor $=\frac{65 \cdot 64}{17 \cdot 16}=\frac{65 \cdot 4}{17}$
$r=25$, Factor $=\frac{65 \cdot 64}{15 \cdot 14}=\frac{13 \cdot 32}{3 \cdot 7}$	$r=26$, Factor $=\frac{65 \cdot 64}{13 \cdot 12}=\frac{5 \cdot 16}{3}$	$r=27$, Factor $=\frac{65 \cdot 64}{11 \cdot 10}=\frac{13 \cdot 32}{11}$
$r=28$, Factor $=\frac{65 \cdot 64}{9 \cdot 8}=\frac{65 \cdot 8}{9}$	$r=29$, Factor $=\frac{65 \cdot 64}{7 \cdot 6}=\frac{65 \cdot 32}{7 \cdot 3}$	$r=30$, Factor $=\frac{65 \cdot 64}{5 \cdot 4}=13 \cdot 16$
$r=31$, Factor $=\frac{65 \cdot 64}{3 \cdot 2}=\frac{65 \cdot 32}{3}$		

The general equation for finding I_n is

$$(-1)^{n-1}(2n+1)I_n = -(C_1^n I_1 + C_3^n I_3 + \&c.) + (C_2^n I_2 + C_4^n I_4 + \&c.) + F_n.$$

Hence putting $n=32$, the equation for finding I_{32} is

$$P_{32} = 65 I_{32} = (C_1^{32} I_1 + C_3^{32} I_3 + \&c. + C_{31}^{32} I_{31}) - (C_2^{32} I_2 + C_4^{32} I_4 + \&c. + C_{30}^{32} I_{30}) - F_{32}.$$

Table of the products $P_r^n = C_r^n I_r$ for $n = 32$, and calculation of the quantities I_{32} and B_{32}.

P_r	r
6099 25587 71040	7
273 95824 31465 88000	9
7 52052 11742 87069 31200	11
14292 18243 52720 81535 36160	13
181 02208 01083 62555 55851 98784	15
1 45956 33740 16156 32779 21049 55360	17
707 20034 04420 28331 06379 22168 88480	19
1 91122 29427 92298 81501 97313 85143 24000	21
260 61036 46811 76525 19023 36205 50468 48000	23
15554 89989 86905 19795 15515 50951 74607 85920	25
3 26957 75316 75298 38464 45063 81824 21193 67520	27
16 61488 53356 44144 55275 75671 54091 08961 07520	29
8 95482 61960 15233 01854 18130 16189 66511 72000	31
28 99746 33490 63992 38881 71047 04677 31614 73984	Sum
4 53632 15585 96100	8
14991 65046 74768 61360	10
344 04340 75247 90249 64000	12
5 36521 41705 40998 03634 45360	14
5456 55799 32924 09847 00119 61290	16
34 38319 01111 06135 63711 54717 15840	18
12575 34291 17724 63047 26474 10262 50016	20
24 49639 24021 61449 84035 26184 70163 48400	22
2257 65747 79208 93849 11233 52764 25664 03700	24
82742 14563 16036 20888 25426 35887 94710 49840	26
9 11013 03017 92694 23112 48336 95819 28170 55080	28
17 67611 84070 36444 68422 26180 95706 47598 17136	30
27 63649 29648 26477 79222 58362 41320 15644 68122	Sum
814 19505 34163 85807	F_{32}
27 63649 29648 26477 79222 59176 60825 49808 53929	
28 99746 33490 63992 38881 71047 04677 31614 73984	
65) 1 36097 03842 37514 59659 11870 43851 81806 20055	P_{32}
2093 80059 11346 37840 90951 85290 02797 01847	I_{32}

$$\text{Also } B_{32} = I_{32} - 1 + \tfrac{557}{510}$$
$$= I_{32} + \tfrac{47}{510}.$$

Hence the numerator of B_{32} is 10 67838 30147 86652 98863 85444 97914 26479 42017

and the denominator is 510

CALCULATION OF BERNOULLI'S NUMBER FOR $n=62$.

Table of the values of the alternate binomial coefficients for the Index $2n+1=125$, or of the several values of the quantities C_r^n when $n=62$.

$n=62,\ 2n+1=125$

C_r	r
7750	1
96 91375	2
46906 25500	3
117 61743 44125	4
17736 70910 94050	5
17 61577 70018 40875	6
1224 97403 15126 27000	7
62320 55385 32048 98625	8
23 97508 36588 21178 64750	9
715 59315 48903 94232 15775	10
16914 02002 46820 45487 36500	11
3 21917 92460 01984 96178 00125	12
50 02109 28994 15458 63688 94250	13
641 93735 88758 31719 17341 42875	14
6870 94331 70709 71228 66992 39600	15
61852 34256 19392 87370 99034 37125	16
4 71665 45718 35584 15994 82476 00750	17
30 65825 47169 31297 03966 36094 04875	18
170 77912 58485 10610 53673 21400 13500	19
819 08296 12811 25889 76655 76099 87825	20
3396 19764 43363 75640 49548 27731 20250	21
12216 97736 11804 29497 46947 97853 36375	22
38244 45086 97822 14079 03489 32410 53000	23
1 04460 24213 63466 32604 17243 44642 59125	24
2 49510 74978 85308 13877 39472 91774 87510	25
5 22166 16188 77624 49856 53874 31881 80875	26
9 58946 66208 31863 85900 05857 23959 04500	27
15 47391 20472 51416 68156 91269 63661 18625	28
21 96116 01106 18163 05805 27355 45522 77250	29
27 43283 89856 36586 73522 85866 05169 97175	30
30 17467 21788 07033 53213 93231 82841 64000	31
29 23171 36732 19313 73425 99693 33377 83875	32
24 93894 45323 96897 03202 59878 22881 79250	33
18 73157 77414 09609 66716 26273 77063 54125	34
12 37912 96378 01133 34525 53015 70928 94900	35
7 19209 99656 23897 89425 04392 92969 28375	36
3 66927 57321 84276 67466 75695 46727 75750	37
1 64151 80907 14018 51235 12811 13009 78625	38
64283 22593 00594 66217 95226 73626 21000	39
21990 55925 01247 73095 44506 36136 05475	40
6555 45126 69748 64608 39825 74457 90250	41
1698 09882 21681 87820 24774 13865 60125	42
380 96881 92005 23669 65886 40046 45500	43
73 74553 16164 02309 09540 70604 60375	44
12 26330 18867 72518 81586 54437 61950	45
1 74311 14722 00107 18954 60915 04625	46
21056 11661 68303 95700 76267 02000	47
2147 16978 65846 78508 95935 12375	48
183 41067 39645 23348 33526 12250	49
13 00548 41538 48019 24559 12505	50
75745 39402 35761 16747 76500	51
3577 96577 44519 71160 78875	52
135 01757 63944 14006 06750	53
3 99584 72764 70196 44125	54
9064 80783 31934 39800	55
153 12175 39390 78375	56
1 85429 23159 83250	57
1529 02664 73625	58
7 97406 33500	59
2345 31275	60
3 17750	61
125	62
212 67647 93255 86539 66460 91296 44855 13215	Sum $=2^{124}-1$

Formation of the several values of $\frac{1}{p}(C^n_r + C^n_{2r} + C^n_{3r} + \&c.)$ *when* $n = 62$.

$n=62,\ 2n+1=125$

$r=1,\ p=2r+1=3$

3) 212 67647 93255 86539 66460 91296 44855 13215
70 89215 97751 95513 22153 63765 48285 04405

$r=2,\ p=2r+1=5$

96 91375
117 61743 44125
17 61577 70018 40875
62320 55385 32048 98625
715 59315 48903 94232 15775
3 21917 92460 01984 96178 00125
641 93735 88758 31719 17341 42875
61852 34256 19392 87370 99034 37125
30 65825 47169 31297 03966 36094 04875
819 08296 12811 25889 76655 76099 87825
12216 97736 11804 29497 46947 97853 36375
1 04460 24213 63466 32604 17243 44642 59125
5 22166 16188 77624 49856 53874 31881 80875
15 47391 20472 51416 68156 91269 63661 18625
27 43283 89856 36586 73522 85866 05169 97175
29 23171 36732 19313 73425 99693 33377 83875
18 73157 77414 09609 66716 26273 77063 54125
7 19209 99656 23897 89425 04392 92969 28375
1 64151 80907 14018 51235 12811 13009 78625
21990 55925 01247 73095 44506 36136 05475
1698 09882 21681 87820 24774 13865 60125
73 74553 16164 02309 09540 70604 60375
1 74311 14722 00107 18954 60915 04625
2147 16978 65846 78508 95935 12375
13 00548 41538 48019 24559 12505
3577 96577 44519 71160 78875
3 99584 72764 70196 44125
153 12175 39390 78375
1529 02664 73625
2345 31275
125
5) 106 33823 96627 93269 80924 61347 30290 62655
21 26764 79325 58653 96184 92269 46058 12531

$r=3,\ p=2r+1=7$

46906 25500
17 61577 70018 40875
23 97508 36588 21178 64750
3 21917 92460 01984 96178 00125
6870 94331 70709 71228 66992 39600
30 65825 47169 31297 03966 36094 04875
3396 19764 43363 75640 49548 27731 20250
1 04460 24213 63466 32604 17243 44642 59125
9 58946 66208 31863 85900 05857 23959 04500
27 43283 89856 36586 73522 85866 05169 97175
24 93894 45323 96897 03202 59878 22881 79250
7 19209 99656 23897 89425 04392 92969 28375
64283 22593 00594 66217 95226 73626 21000
1698 09882 21681 87820 24774 13865 60125
12 26330 18867 72518 81586 54437 61950
2147 16978 65846 78508 95935 12375
75745 39402 35761 16747 76500
3 99584 72764 70196 44125
1 85429 23159 83250
2345 31275
7) 70 89215 78675 93390 83680 72184 05035 55000
10 12745 11239 41912 97668 67454 86433 65000

$r=5,\ p=2r+1=11$

17736 70910 94050
715 59315 48903 94232 15775
6870 94331 70709 71228 66992 39600
819 08296 12811 25889 76655 76099 87825
2 49510 74978 85308 13877 39472 91774 87510
27 43283 89856 36586 73522 85866 05169 97175
12 37912 96378 01133 34525 53015 70928 94900
21990 55925 01247 73095 44506 36136 05475
12 26330 18867 72518 81586 54437 61950
13 00548 41538 48019 24559 12505
9064 80783 31934 39800
2345 31275
11) 42 53529 58648 51550 74058 47775 25521 67840
3 86684 50786 22868 24914 40706 84138 33440

$r=6,\ p=2r+1=13$

17 61577 70018 40875
3 21917 92460 01984 96178 00125
30 65825 47169 31297 03966 36094 04875
1 04460 24213 63466 32604 17243 44642 59125
27 43283 89856 36586 73522 85866 05169 97175
7 19209 99656 23897 89425 04392 92969 28375
1698 09882 21681 87820 24774 13865 60125
2147 16978 65846 78508 95935 12375
3 99584 72764 70196 44125
2345 31275
13) 35 68682 91584 31702 72578 51079 27414 78450
2 74514 07044 94746 36352 19313 79031 90650

$r=8,\ p=2r+1=17$

62320 55385 32048 98625
61852 34256 19392 87370 99034 37125
1 04460 24213 63466 32604 17243 44642 59125
29 23171 36732 19313 73425 99693 33377 83875
21990 55925 01247 73095 44506 36136 05475
2147 16978 65846 78508 95935 12375
153 12175 39390 78375
17) 30 49622 80870 35263 26838 94883 80565 74975
1 79389 57698 25603 72166 99699 04739 16175

$r=9,\ p=2r+1=19$

23 97508 36588 21178 64750
30 65825 47169 31297 03966 36094 04875
9 58946 66208 31863 85900 05857 23959 04500
7 19209 99656 23897 89425 04392 92969 28375
12 26330 18867 72518 81586 54437 61950
3 99584 72764 70196 44125
19) 16 78199 58020 21826 76234 05155 98835 08575
88326 29369 48517 19801 79218 73622 89925

$r=11,\ p=2r+1=23$

16914 02002 46820 45487 36500
12216 97736 11804 29497 46947 97853 36375
24 93894 45323 96897 03202 59878 22881 79250
73 74553 16164 02309 09540 70604 60375
9064 80783 31934 39800
23) 25 06185 17613 41779 46076 43970 68761 52300
1 08964 57287 53990 41133 75824 81250 50100

$n=62,\ 2n+1=125$

$r=14,\ p=2r+1=29$

641 93735 88758 31719 17341 42875
15 47391 20472 51416 68156 91269 63661 18625
1698 09882 21681 87820 24774 13865 60125
153 12175 39390 78375
29) 15 49089 30996 66834 44888 59938 34259 00000
53416 87275 74718 42927 19308 21871 00000

$r=15,\ p=2r+1=31$

6870 94331 70709 71228 66992 39600
27 43283 89856 36586 73522 85866 05169 97175
12 26330 18867 72518 81586 54437 61950
2345 31275
31) 27 43296 23057 49786 16751 38681 28945 30000
88493 42679 27412 45701 65763 91256 30000

$r=18,\ p=2r+1=37$

30 65825 47169 31297 03966 36094 04875
7 19209 99656 23897 89425 04392 92969 28375
3 99584 72764 70196 44125
37) 7 19240 65481 71071 20306 81123 99259 77375
19438 93661 66785 70819 10300 64844 85875

$r=20,\ p=2r+1=41$

819 08296 12811 25889 76655 76099 87825
21990 55925 01247 73095 44506 36136 05475
2345 31275
41) 22809 64221 14058 98985 21162 14581 24575
556 33273 68635 58511 83442 97916 61575

$r=21,\ p=2r+1=43$

3396 19764 43363 75640 49548 27731 20250
1698 09882 21681 87820 24774 13865 60125
43) 5094 29646 65045 63460 74322 41596 80375
118 47201 08489 43336 29635 40502 25125

$r=23,\ p=2r+1=47$

38244 45086 97822 14079 03489 32410 53000
1 74311 14722 00107 18954 60915 04625
47) 38246 19398 12544 14186 22443 93325 57625
813 74880 81117 96046 51541 36028 20375

$r=26,\ p=2r+1=53$

5 22166 16188 77624 49856 53874 31881 80875
3577 96577 44519 71160 78875
53) 5 22166 16188 81202 46433 98394 03042 59750
9852 19173 37381 17857 24498 00057 40750

$r=29,\ p=2r+1=59$

21 96116 01106 18163 05805 27355 45522 77250
1529 02664 73625
59) 21 96116 01106 18163 05805 28884 48187 50875
37222 30527 22341 74674 66591 26240 46625

$r=30,\ p=2r+1=61$

27 43283 89856 36586 73522 85866 05169 97175
2345 31275
61) 27 43283 89856 36586 73522 85866 07515 28450
44971 86718 95681 74975 78456 82090 41450

$r=33,\ p=2r+1=67$

67) 24 93894 45323 96897 03202 59878 22881 79250
37222 30527 22341 74674 66565 34669 87750

$r=35,\ p=2r+1=71$

71) 12 37912 96378 01133 34525 53015 70928 94900
17435 39385 60579 34289 09197 40435 61900

$r=36,\ p=2r+1=73$

73) 7 19209 99656 23897 89425 04392 92969 28375
9852 19173 37313 66978 42525 93054 37375

$r=39,\ p=2r+1=79$

79) 64283 22593 00594 66217 95226 73626 21000
813 71172 06336 64129 34116 79412 99000

$r=41,\ p=2r+1=83$

83) 6555 45126 69748 64608 39825 74457 90250
78 98134 05659 62224 19756 93668 16750

$r=44,\ p=2r+1=89$

89) 73 74553 16164 02309 09540 70604 60375
82860 14788 35981 00107 19894 43375

$r=48,\ p=2r+1=97$

97) 2147 16978 65846 78508 95935 12375
22 13577 09957 18335 14391 08375

$r=50,\ p=2r+1=101$

101) 13 00548 41538 48019 24559 12505
12876 71698 40079 39847 12005

$r=51,\ p=2r+1=103$

103) 75745 39402 35761 16747 76500
735 39217 49861 75890 75500

$r=53,\ p=2r+1=107$

107) 135 01757 63944 14006 06750
1 26184 65083 59009 40250

$r=54,\ p=2r+1=109$

109) 3 99584 72764 70196 44125
3665 91493 25414 64625

$r=56,\ p=2r+1=113$

113) 153 12175 39390 78375
1 35505 97693 72375

Table of the Factors by which the quantities P_r^n *for* $n=61$ *must be multiplied in order to find the corresponding quantities for* $n=62$.

$r=7$, Factor $=\frac{125\cdot124}{111\cdot110}=\frac{3100}{22\cdot111}$

$r=8$, Factor $=\frac{125\cdot124}{109\cdot108}=\frac{31000}{216\cdot109}$

$r=9$, Factor $=\frac{125\cdot124}{107\cdot106}=\frac{31000}{212\cdot107}$

$r=10$, Factor $=\frac{125\cdot124}{105\cdot104}=\frac{3100}{52\cdot42}$

$r=11$, Factor $=\frac{125\cdot124}{103\cdot102}=\frac{31000}{204\cdot103}$

$r=12$, Factor $=\frac{125\cdot124}{101\cdot100}=\frac{310}{202}$

$r=13$, Factor $=\frac{125\cdot124}{99\cdot98}=\frac{31000}{196\cdot99}$

$r=14$, Factor $=\frac{125\cdot124}{97\cdot96}=\frac{31000}{192\cdot97}$

$r=15$, Factor $=\frac{125\cdot124}{95\cdot94}=\frac{31000}{188\cdot95}$

$r=16$, Factor $=\frac{125\cdot124}{93\cdot92}=\frac{31000}{184\cdot93}$

$r=17$, Factor $=\frac{125\cdot124}{91\cdot90}=\frac{3100}{18\cdot91}$

$r=18$, Factor $=\frac{125\cdot124}{89\cdot88}=\frac{31000}{176\cdot89}$

$r=19$, Factor $=\frac{125\cdot124}{87\cdot86}=\frac{31000}{172\cdot87}$

$r=20$, Factor $=\frac{125\cdot124}{85\cdot84}=\frac{3100}{42\cdot34}$

$r=21$, Factor $=\frac{125\cdot124}{83\cdot82}=\frac{31000}{164\cdot83}$

$r=22$, Factor $=\frac{125\cdot124}{81\cdot80}=\frac{1550}{648}$

$r=23$, Factor $=\frac{125\cdot124}{79\cdot78}=\frac{31000}{156\cdot79}$

$r=24$, Factor $=\frac{125\cdot124}{77\cdot76}=\frac{31000}{152\cdot77}$

$r=25$, Factor $=\frac{125\cdot124}{75\cdot74}=\frac{310}{111}$

$r=26$, Factor $=\frac{125\cdot124}{73\cdot72}=\frac{31000}{144\cdot73}$

$r=27$, Factor $=\frac{125\cdot124}{71\cdot70}=\frac{3100}{994}$

$r=28$, Factor $=\frac{125\cdot124}{69\cdot68}=\frac{31000}{136\cdot69}$

$r=29$, Factor $=\frac{125\cdot124}{67\cdot66}=\frac{31000}{132\cdot67}$

$r=30$, Factor $=\frac{125\cdot124}{65\cdot64}=\frac{3100}{832}$

$r=31$, Factor $=\frac{125\cdot124}{63\cdot62}=\frac{1000}{252}$

$r=32$, Factor $=\frac{125\cdot124}{61\cdot60}=\frac{3100}{732}$

$r=33$, Factor $=\frac{125\cdot124}{59\cdot58}=\frac{31000}{116\cdot59}$

$r=34$, Factor $=\frac{125\cdot124}{57\cdot56}=\frac{31000}{112\cdot57}$

$r=35$, Factor $=\frac{125\cdot124}{55\cdot54}=\frac{3100}{594}$

$r=36$, Factor $=\frac{125\cdot124}{53\cdot52}=\frac{31000}{104\cdot53}$

$r=37$, Factor $=\frac{125\cdot124}{51\cdot50}=\frac{310}{51}$

$r=38$, Factor $=\frac{125\cdot124}{49\cdot48}=\frac{31000}{96\cdot49}$

$r=39$, Factor $=\frac{125\cdot124}{47\cdot46}=\frac{31000}{92\cdot47}$

$r=40$, Factor $=\frac{125\cdot124}{45\cdot44}=\frac{3100}{396}$

$r=41$, Factor $=\frac{125\cdot124}{43\cdot42}=\frac{31000}{84\cdot43}$

$r=42$, Factor $=\frac{125\cdot124}{41\cdot40}=\frac{3100}{328}$

$r=43$, Factor $=\frac{125\cdot124}{39\cdot38}=\frac{31000}{76\cdot39}$

$r=44$, Factor $=\frac{125\cdot124}{37\cdot36}=\frac{31000}{72\cdot37}$

$r=45$, Factor $=\frac{125\cdot124}{35\cdot34}=\frac{3100}{238}$

$r=46$, Factor $=\frac{125\cdot124}{33\cdot32}=\frac{31000}{64\cdot33}$

$r=47$, Factor $=\frac{125\cdot124}{31\cdot30}=\frac{100}{6}$

$r=48$, Factor $=\frac{125\cdot124}{29\cdot28}=\frac{31000}{56\cdot29}$

$r=49$, Factor $=\frac{125\cdot124}{27\cdot26}=\frac{31000}{52\cdot27}$

$r=50$, Factor $=\frac{125\cdot124}{25\cdot24}=\frac{310}{12}$

$r=51$, Factor $=\frac{125\cdot124}{23\cdot22}=\frac{31000}{44\cdot23}$

$r=52$, Factor $=\frac{125\cdot124}{21\cdot20}=\frac{3100}{84}$

$r=53$, Factor $=\frac{125\cdot124}{19\cdot18}=\frac{31000}{684}$

$r=54$, Factor $=\frac{125\cdot124}{17\cdot16}=\frac{31000}{544}$

$r=55$, Factor $=\frac{125\cdot124}{15\cdot14}=\frac{3100}{42}$

$r=56$, Factor $=\frac{125\cdot124}{13\cdot12}=\frac{31000}{312}$

$r=57$, Factor $=\frac{125\cdot124}{11\cdot10}=\frac{3100}{22}$

$r=58$, Factor $=\frac{125\cdot124}{9\cdot8}=\frac{31000}{144}$

$r=59$, Factor $=\frac{125\cdot124}{7\cdot6}=\frac{31000}{84}$

$r=60$, Factor $=\frac{125\cdot124}{5\cdot4}=\frac{3100}{4}$

$r=61$, Factor $=\frac{125\cdot124}{3\cdot2}=\frac{31000}{12}$

The following extract from the calculations for B_{61} supplies the further data which are required in making the similar calculations for B_{62}.

Table of the products P_r^n for $n=61$, and calculation of the quantities I_{61} and B_{61}.

$n=61$

P_r	r
964 96341 45012 37140	7
964 89657 65941 44480 78045	9
709 87762 29656 03133 39385 12416	11
446 32904 46564 67643 64169 13414 48289	13
238 13879 23030 96548 04209 62125 83207 39024	15
107 06939 54705 19032 58969 74701 13695 04939 02335	17
40 25652 28249 10002 12635 23598 54178 27001 58963 45042	19
12 55182 69448 74874 88060 27995 11438 03444 11435 21755 26595	21
3 21576 40020 43518 49711 48609 93756 17416 30520 44444 26034 94720	23
67013 45301 04531 62370 59301 99455 73390 66066 99036 42526 75425 75320	25
11231 92690 01350 11344 11384 88426 10111 02042 63460 59712 86463 90646 44690	27
1495 24216 89836 15461 98946 23048 42766 90639 91405 60304 98194 27552 08337 44768	29
155 88977 58940 11502 64044 87592 68543 79442 43154 07109 74300 17982 62640 39210 01200	31
12 52735 11130 65823 80947 92554 38160 79807 93537 05108 42554 57225 79654 88308 53904 41190	33
76200 72588 67366 51664 72332 20669 16734 05384 05765 11897 06029 79475 60838 23923 30496 01586	35
3436 04646 97275 63168 89528 79432 85080 48564 67069 85420 40166 55083 75038 77057 44549 30465 42575	37
112 10695 64594 13504 42139 70913 71749 31071 09518 52483 41211 65312 62173 47670 44666 56845 56543 86100	39
2 57214 84207 01260 94423 04144 67701 47558 15044 83117 16659 84555 02098 71789 07406 75788 13556 63504 40576	41
4011 14723 17325 72699 40249 05126 47523 41384 39220 48875 47894 38625 57212 13956 40395 37641 30013 25696 79394	43
40 79437 80361 27635 97401 30647 51948 92803 78672 37883 93735 67400 74226 47306 81895 83279 14464 97593 68619 21663	45
25705 59787 87797 26191 94794 17115 04630 03419 37495 12225 69774 98915 60970 10378 11301 69297 37355 62468 49910 70920	47
94 01591 50554 71453 90027 46633 07601 64638 39147 39457 95157 51563 74099 01857 13757 96469 23049 28627 81245 98033 10917	49
18314 01945 38397 81142 51756 35540 10858 11425 38938 64976 87217 55946 03250 40950 35753 00762 30167 14245 14457 60953 83690	51
16 87872 33205 42521 24918 73620 52220 43451 02967 59560 37380 91077 17414 07192 86078 91797 33570 62616 16951 13720 73193 08800	53
618 59772 65027 85766 57268 29020 42958 36182 03055 01314 74215 62227 37323 11466 33589 83982 33494 20456 03407 40119 24748 31132	55
6810 92549 60879 33598 99438 14748 05752 18187 49591 64300 62439 20770 82079 77657 98879 69766 67926 76502 85769 43176 51024 62060	57
13215 03881 30741 91631 57225 17932 37777 11664 03691 29995 67059 46701 18170 63553 01176 06592 67910 74055 83529 45784 76509 00402	59
20661 62484 19137 85710 31655 46598 32491 46638 75498 17026 16147 84648 68636 76909 76794 00194 28931 54031 89899 63234 14415 30589	Sum

P_r	r
3 31320 18836 89997 72019	8
2 66693 72038 65962 93743 29234	10
1 81615 29323 22841 13916 65975 29500	12
1 05278 19363 60383 00590 50922 84521 85704	14
51610 87066 84400 26912 81306 11708 43921 18731	16
21241 16304 99659 31737 15459 55819 22481 50881 09376	18
7280 72679 73513 70244 59338 72132 63345 64463 36534 23148	20
2060 26085 99438 35927 83133 80731 79420 81422 03431 02817 20660	22
476 68199 69471 46106 12987 81829 82540 47419 26264 52866 50466 79083	24
89 21905 44597 87914 19855 84927 97629 57046 17662 79931 87388 00141 65854	26
13 34911 68626 82125 54426 67263 91301 27021 31250 59921 40779 48817 63530 01597	28
1 57559 69464 90906 02362 51676 66589 32518 95763 16754 35575 45346 98938 05686 49552	30
14452 36699 93578 17658 78109 10421 03333 08814 63116 74540 96649 31618 72384 39026 45205	32
1012 89026 20578 31483 39325 62887 20505 88928 48887 08759 34066 15643 40587 25845 19619 98600	34
53 19595 19372 08958 57180 51919 28203 98358 57851 14947 21065 36321 97658 24055 41400 85654 11943	36
2 04708 84567 33692 18114 19357 71372 08907 73807 56096 74038 64311 32708 35093 50374 72353 40000 91120	38
5622 61071 38888 61163 17544 00695 22634 93248 12401 70831 89557 90085 04790 54505 86023 95045 20787 48165	40
106 85138 02205 80828 46327 85065 45356 25477 14586 95071 39929 55833 42462 87931 96338 41268 67903 68058 56640	42
1 35335 92369 91826 49507 34982 43435 24030 17278 74067 66018 60929 96796 89030 65303 15111 96691 82485 08229 42176	44
1091 20035 25033 87814 43607 29215 15664 30791 00522 59675 08269 86975 14649 40960 64551 39142 55673 78451 76458 52536	46
5 28717 88547 23215 68582 24341 28297 28291 56152 30058 71218 77129 87692 21121 55475 27071 50118 75363 73234 51919 59398	48
1428 87056 91956 77753 33416 02807 48407 75074 49519 52390 64650 96383 18658 97527 67379 12344 56280 09008 07936 35403 32328	50
1 94837 80347 27439 30797 92842 05773 52990 57895 78308 19504 95599 03524 06683 41684 70820 08271 58896 19768 45591 52241 61715	52
116 29171 13142 05062 45676 86666 84610 98972 68332 81333 06936 16842 53516 90349 91744 44337 00067 76452 46815 30223 10462 17076	54
2444 40510 00611 34277 34348 19912 07637 33051 80859 05140 55988 60047 79577 24735 84269 57353 60541 04881 16412 80092 85863 28123	56
12421 63859 33413 08160 25365 40853 53199 76473 81555 06485 50749 40725 16944 15022 61000 66767 25271 99148 83192 35078 83636 21782	58
6694 81687 21005 57718 07023 47327 93924 32961 18028 34525 45701 69446 00742 51674 66882 21113 54067 02661 35387 62270 41926 67156	60
21679 11499 65386 68639 52347 37280 90301 99388 13670 18402 89434 37562 09528 52868 66850 50666 32631 09490 13249 10031 07505 18421	Sum
2 38586 80524 96330 07051 54180 90465 82983	F_{61}
21679 11499 65386 68639 52347 37280 90301 99388 13670 18402 89434 37562 09528 52868 66852 89253 13156 05820 20300 64211 97971 01404	
20661 62484 19137 85710 31655 46598 32491 46638 75498 17026 16147 84648 68636 76909 76794 00194 28931 54031 89899 63234 14415 30589	
123)1017 49015 46248 82929 20691 90682 57810 52749 38172 01376 73286 52913 40891 75958 90058 89058 84224 51788 30401 00977 83555 70815	P_{61}
8 27227 76798 77096 98542 21062 45998 45957 31204 65051 84335 66283 84885 29885 84472 02350 07188 81721 85613 01633 96614 27405	I_{61}

Also $$B_{61} = I_{61} + 1 - \tfrac{5}{6} = I_{61} + \tfrac{1}{6}.$$

Hence the numerator of B_{61} is

49 63366 60792 62581 91253 26374 75990 75743 87227 90311 06013 97703 09311 79315 06832 14100 43132 90331 13678 09803 79685 64431

and the denominator is 6.

The numerator should be divisible by 61.

By actual division we find the quotient to be

81366 66570 37091 50676 28301 22557 22553 17823 40824 77147 77011 52611 66874 01751 34657 38412 01480 83830 78849 24257 14171

without any remainder. Hence the test is satisfied.

Putting $n=62$ in the general formula for I_n, the equation for finding I_{62} is

$$P_{62} = 125 I_{62} = (C_1^{62} I_1 + C_3^{62} I_3 + \&\text{c.} + C_{61}^{62} I_{61}) - (C_2^{62} I_2 + C_4^{62} I_4 + \&\text{c.} + C_{60}^{62} I_{60}) - F_{62}.$$

Table of the products P_r^n or $C_r^n I_r$ for $n=62$, and calculation of the quantities I_{62} and B_{62}.

$n=62$

P_r	r
1224 97403 15126 27000	7
1318 62960 12351 64825 61250	9
1047 31611 99283 12256 57764 08000	11
713 05918 28669 60263 49682 70245 77250	13
413 34280 85888 01399 17721 06713 37034 10400	15
202 63438 70321 17827 24545 91925 22866 08858 95750	17
83 39696 65578 86264 76322 66209 22181 66068 51635 05500	19
28 58555 94542 40458 51444 95140 21053 41372 87282 67294 53750	21
8 08898 76714 82398 03720 87545 28273 40141 63107 25233 04696 80000	23
1 87154 68858 77520 75089 04356 92173 67127 07033 93705 33182 82720 57200	25
35029 14828 00991 29946 43152 05353 03163 14217 47211 11780 56376 36824 93500	27
5241 12474 42890 18466 94632 87483 18156 27525 70508 10657 44462 06932 90192 32000	29
618 61022 18016 32946 98590 77748 75173 78739 80770 12340 25000 71359 62858 69881 00000	31
56 74282 35688 25327 01546 71125 92487 54244 00883 77901 98596 10461 67343 85967 95884 97500	33
3 97680 55597 45515 49091 99040 13593 29756 84664 27393 71853 34498 92886 17169 26199 06629 03900	35
20885 77265 91283 25144 26547 57336 93626 48138 19444 21182 83365 30901 22784 68388 39417 34201 60750	37
803 72702 36451 93949 36709 29307 41033 44866 77861 76453 69463 74812 96803 37137 79986 96163 85952 75000	39
22 07547 09417 88230 69522 22725 63329 38621 99997 16675 57158 14287 27868 28754 51165 41924 75153 84450 88000	41
41951 94473 13460 70742 73859 84791 07026 25815 16813 47887 93767 19768 12947 47857 12637 20270 00813 41633 13500	43
531 35534 41680 49039 99764 91627 35469 23074 53295 69076 49498 27488 65975 06937 55786 05736 75804 30842 13107 44350	45
4 28426 63131 29954 36532 46569 51917 43833 90322 91585 37094 96249 81926 82835 06301 88361 54956 22593 74474 98511 82000	47
2075 84997 62960 22130 27671 97738 85791 33753 65790 05125 71141 72703 68283 17358 45083 26599 81857 46055 68821 50303 69250	49
5 61002 57220 26019 91519 80678 87098 18776 22714 49701 72216 44016 14947 63599 50060 35912 28884 73499 42291 97812 14989 07500	51
764 97137 85625 99647 25849 15549 98294 54651 92975 82414 60245 95602 92158 23068 25214 11867 55393 87574 93399 49331 41792 00000	53
45658 40362 28246 63723 22183 32460 27879 08673 68346 20850 01629 26306 11944 17753 36392 93934 24572 24135 84831 99277 79042 02600	55
9 59721 31990 32997 34403 75375 32680 83261 99147 15186 05997 07342 90433 83968 51807 51230 12577 53316 89039 03874 47599 17105 63000	57
48 76978 60958 69040 54508 80720 94091 79650 18870 76551 17449 86231 87340 86782 15992 43548 13964 90868 53939 21585 63425 21180 05500	59
26 28516 23278 09475 67117 87425 96660 10529 35902 77702 23226 56866 92973 03712 27159 85467 35342 46671 19785 35941 92741 85579 38750	61
85 11645 16809 96562 49020 79663 81837 01215 24471 83370 44978 75762 49844 39975 60548 91277 44873 82476 22450 02809 87250 27463 53200	Sum

P_r	r
4 36243 87697 24342 90375	8
3 78548 77893 70185 48811 44975	10
2 78716 53911 88518 57990 91348 22500	12
1 75237 54310 12235 45871 22992 27887 54125	14
93497 95410 94928 02378 28453 11066 01306 49875	16
42037 54178 68324 74709 63945 75484 93164 37520 04000	18
15805 49934 99924 70418 93522 43425 18467 43582 93596 72100	20
4928 09310 63471 38407 62125 61935 61886 20685 42157 55195 47875	22
1262 57193 31306 84320 74728 50027 73304 40020 26162 02910 25672 46375	24
263 10794 21854 47616 07261 35156 70330 73480 92422 63878 24298 71041 80125	26
44 09874 49641 03355 91136 70628 86864 80995 38445 07413 00528 99973 00663 94875	28
5 87061 36227 42558 50148 80045 26955 41837 46232 95599 16206 61749 59985 54841 50975	30
61205 37936 88650 74784 45544 02056 28869 63559 23035 39722 67230 71062 90152 47243 17125	32
4918 48341 53810 73932 51737 85949 77393 88593 85259 98048 17677 13807 26535 87280 87127 12500	34
299 17897 49734 17582 67887 53537 32642 14280 82979 97707 46194 89474 10632 33983 64192 04513 37125	36
13 49059 14453 11321 77198 12944 11678 30812 04939 28358 62074 39126 51776 97257 35887 84641 87931 17500	38
44015 38690 16552 26277 38349 54937 37798 71386 82942 66613 32397 70867 79925 98404 46147 08687 23336 34625	40
1009 87584 96457 33439 74440 05191 78671 92009 61035 20491 88358 63059 80594 28625 26369 14429 58845 76163 28000	42
15 74854 96797 09692 69792 73444 24359 02753 51216 57694 24390 72383 26089 93975 31680 81257 87329 79368 45012 04000	44
16016 67184 07220 74927 80220 67078 53027 24678 58049 47882 36915 70184 44191 14479 17184 24914 42181 49623 43851 46125	46
100 92521 21283 37245 23429 52327 44590 99161 58079 62943 39767 18612 18262 65251 35303 81290 97094 43519 52136 75805 05750	48
36912 48970 42216 75294 46580 72526 67200 22757 79254 36758 36816 56565 65356 86131 57294 02234 53902 32708 71689 14585 85140	50
71 90442 74720 84069 69923 54885 46404 07985 65201 51850 05540 04250 11007 22840 38364 23122 10022 92597 77169 20639 51773 96625	52
6626 91737 26844 79662 05850 85794 53935 07634 52788 99494 76141 95070 93794 13322 50142 91262 90626 28725 20724 94331 32954 58375	54
2 42873 58365 99203 92941 17930 04084 51145 01942 52021 13324 86046 84236 11841 88497 19092 24236 43501 64474 64092 39995 56928 58375	56
26 74102 75273 31982 84499 05053 22635 36060 46446 40326 46185 64108 50557 31032 34034 09865 95728 01609 27873 53908 85027 27241 33625	58
51 88483 07587 79322 31504 43191 79152 91355 44914 71967 57229 18813 20655 75450 47868 33713 62994 01945 62549 25407 59574 93170 45900	60
81 12158 60420 69598 79225 90120 85689 41029 32563 98786 56239 85421 01781 36936 61135 01499 75370 26254 19489 19026 62190 99866 38965	Sum
9 53068 50542 05310 40997 94772 15321 76735	F_{62}
81 12158 60420 69598 79225 90120 85689 41029 32563 98786 56239 85421 01781 36936 61135 01509 28438 76796 24799 60024 56963 15188 15700	
85 11645 16809 96562 49020 79663 81837 01215 24471 83370 44978 75762 49844 39975 60548 91277 44873 82476 22450 02809 87250 27463 53200	
125)3 99486 56389 26963 69794 89542 96147 60185 91907 84583 88738 90341 48063 03038 99413 89768 16435 05679 97650 42785 30287 12275 37500	P_{62}
3195 89251 11415 70958 35916 34369 18081 48735 26276 67109 91122 73184 50424 31195 31118 14531 48045 43981 20342 28242 29698 20300	I_{62}

Also $$B_{62} = I_{62} - 1 + \tfrac{31}{30} = I_{62} + \tfrac{1}{30}.$$

Hence the numerator of B_{62} is

95876 77533 42471 28750 77490 31075 42444 62057 88300 13297 33681 95535 12729 35859 33544 35944 41363 19436 10268 47268 90946 09001

and the denominator is 30.

This numerator should be divisible by 31.

By actual division we find the quotient to be

3092 79920 43305 52540 34757 75195 98143 37485 73816 13332 17215 54694 68152 55995 46243 36643 36818 16756 00331 24105 44869 22871

without any remainder. Hence the test is satisfied.

Table of Bernoulli's Numbers expressed as Vulgar Fractions.

Numerator	*Denominator*	*n*
1	6	1
1	30	2
1	42	3
1	30	4
5	66	5
691	2730	6
7	6	7
3617	510	8
43867	798	9
1 74611	330	10
8 54513	138	11
2363 64091	2730	12
85 53103	6	13
2 37494 61029	870	14
861 58412 76005	14322	15
770 93210 41217	510	16
257 76878 58367	6	17
26315 27155 30534 77373	19 19190	18
2 92999 39138 41559	6	19
2 61082 71849 64491 22051	13530	20
15 20097 64391 80708 02691	1806	21
278 33269 57930 10242 35023	690	22
5964 51111 59391 21632 77961	282	23
560 94033 68997 81768 62491 27547	46410	24
49 50572 05241 07964 82124 77525	66	25
80116 57181 35489 95734 79249 91853	1590	26
29 14996 36348 84862 42141 81238 12691	798	27
2479 39292 93132 26753 68541 57396 63229	870	28
84483 61334 88800 41862 04677 59940 36021	354	29
121 52331 40483 75557 20403 04994 07982 02460 41491	567 86730	30
123 00585 43408 68585 41953 03985 74033 86151	6	31
10 67838 30147 86652 98863 85444 97914 26479 42017	510	32
1 47260 00221 26335 65405 16194 28551 93234 22418 99101	64722	33
7877 31308 58718 72814 19091 49208 47460 62443 47001	30	34
1505 38134 73333 67003 80307 65673 77857 20851 14381 60235	4686	35
58279 54961 66994 41104 38277 24464 10673 65282 48830 18442 60429	1401 00870	36
34152 41728 92211 68014 33007 37314 72635 18668 83077 83087	6	37
246 55088 82593 53727 07687 19604 05851 99904 36526 78288 65801	30	38
41 48463 65575 40082 82951 79035 54954 20734 92199 37537 24004 83487	3318	39
4 60378 42994 79457 64693 55749 69019 04684 97942 57872 75128 89196 56867	2 30010	40
1 67701 41491 85145 83682 31545 09786 26990 02077 36027 57025 34148 81613	498	41
20 24576 19593 52903 60231 13116 01117 31009 98991 73911 98090 87728 10839 32477	34 04310	42
660 71461 94176 78653 57384 78474 26261 49627 78306 86653 38893 17619 96983	6	43
13114 26488 67401 75079 95511 42401 93118 43345 75027 55720 28644 29691 98905 74047	61410	44
117 90572 79021 08279 98841 23351 24921 50837 75254 94966 96471 16231 54521 57279 22535	2 72118	45
129 55859 48207 53752 79894 27828 53857 67496 59341 48371 94351 43023 31632 68299 46247	1410	46
122 08138 06579 74446 96073 01679 41320 12039 58508 41520 26966 21436 21510 52846 49447	6	47
2 11600 44959 72665 13097 59772 81098 24233 67304 39543 89060 23415 06387 33420 05066 83499 87259	45 01770	48
67 90826 06729 05495 62405 11175 46403 60560 73421 95728 50448 75090 73961 24999 29470 58239	6	49
945 98037 81912 21252 95227 43306 94937 21872 70284 15330 66936 13338 56962 04311 39541 51972 47711	33330	50
32040 19410 86090 70782 43020 78211 62417 75491 81719 71527 17450 67900 25010 86861 53083 66781 58791	4326	51
31 95336 31363 83001 12871 03352 79617 42746 71189 60607 82727 38327 10347 01628 49568 36554 97212 24053	1590	52
3637 39031 72617 41440 81518 20151 59342 71692 31298 64058 16900 38930 81637 82818 79873 38620 23465 72901	642	53
34 69342 24784 78287 89552 08865 93238 52541 39976 67857 60491 14687 00058 91371 50126 63197 24897 59230 65973 38057	2091 91710	54
7645 99294 04847 42892 24813 42467 24347 50052 87524 13412 30790 66835 93870 75979 76062 69585 77997 79302 17515	1518	55
26508 79602 15509 97133 52597 21468 51620 14443 15149 91925 09896 45178 84276 80966 75651 48755 15366 78120 35526 00109	16 71270	56
217 37832 31936 91633 33310 76108 66529 91475 72115 66790 90831 36080 61101 14933 60548 42345 93650 90418 86185 62649	42	57
30 95539 16571 84297 69125 13458 03384 14168 69004 12806 43298 44245 50404 57210 08957 52457 19682 71388 19959 57547 52259	1770	58
36 69631 19969 71311 15349 47151 58558 50066 84606 36108 06992 04301 05944 06764 14485 04580 64618 89371 77635 45170 95799	6	59
51507486 53507 91090 61843 99685 78499 83274 09517 03532 62675 21309 28691 67199 29747 49229 85358 81132 93670 77682 67780 32820 70131	23282 55930	60
49 63366 60792 62581 91253 26374 75990 75743 87227 90311 06013 97703 09311 79315 06832 14100 43132 90331 13678 09803 79685 64431	6	61
95876 77533 42471 28750 77490 31075 42444 62057 88300 13297 33681 95535 12729 35859 33544 35944 41363 19436 10268 47268 90946 09001	30	62

55.

ON SOME PROPERTIES OF BERNOULLI'S NUMBERS.

[In 1872 a paper on this subject was communicated to the Cambridge Philosophical Society. The paper contained a comparatively simple proof of the theorem given above as Staudt's theorem, which was there attributed to Clausen: another property of Bernoulli's numbers was also established, viz.: "That if n be a prime number other than 2 or 3, then the numerator of the nth number of Bernoulli will be divisible by n."]

ON THE CALCULATION OF BERNOULLI'S NUMBERS.

[A table of the values of the first sixty-two numbers of Bernoulli, as given above, was printed in Vol. 85 of *Crelle's Journal.* A paper on this subject was also published in the *Report* of the British Association in 1877, of which the greater part is contained in the above paper, and the remainder is given below.]

Thirty-one of the numbers of Bernoulli are at present known to Mathematicians, and are to be found in a communication by Ohm in *Crelle's Journal,* Vol. xx. p. 11. Of these numbers the first fifteen are given in Euler's *Institutiones Calculi Differentialis,* Part 2, Chap. 5, and Ohm states that the sixteen following numbers were calculated and communicated to him by Professor Rothe of Erlangen. I find, however, that the first two of these had been already given by Euler in a memoir contained in the *Acta Petropolitana* for 1781.

....

It may be sometimes useful to have the values of Bernoulli's numbers expressed in integers and repeating decimals.

It readily follows from Staudt's theorem that if the fractional part of the nth number of Bernoulli be converted into a repeating decimal, then the number of figures in the repeating part will be either $2n$ or a divisor of $2n$, and the first figure of the repeating part will occupy the second place of decimals.

Table of Bernoulli's Numbers expressed in Integers and Repeating Decimals.

No.		No.
1	$\cdot 1\dot{6}$	1
2	$\cdot 0\dot{3}$	2
3	$\cdot 0\dot{2}380\ 9\dot{5}$	3
4	$\cdot 0\dot{3}$	4
5	$\cdot 0\dot{7}\dot{5}$	5
6	$\cdot 2\dot{5}311\ 3\dot{5}$	6
7	$1\ \cdot 1\dot{6}$	7
8	$7\ \cdot 0\dot{9}215\ 68627\ 45098\ 0\dot{3}$	8
9	$54\ \cdot 9\dot{7}117\ 79448\ 62155\ 388\dot{4}$	9
10	$529\ \cdot 1\dot{2}\dot{4}$	10
11	$6192\ \cdot 1\dot{2}318\ 84057\ 97101\ 44927\ 53\dot{6}$	11
12	$86580\ \cdot 2\dot{5}311\ 3\dot{5}$	12
13	$14\ 25517\ \cdot 1\dot{6}$	13
14	$272\ 98231\ \cdot 0\dot{6}781\ 60919\ 54022\ 98850\ 57471\ 264\dot{3}$	14
15	$6015\ 80873\ \cdot 9\dot{0}064\ 23683\ 84303\ 86817\ 48359\ 16771\ \dot{4}$	15
16	$1\ 51163\ 15767\ \cdot 0\dot{9}215\ 68627\ 45098\ 0\dot{3}$	16
17	$42\ 96146\ 43061\ \cdot 1\dot{6}$	17
18	$1371\ 16552\ 05088\ \cdot 3\dot{3}277\ 21590\ 87948\ 561\dot{6}$	18
19	$48833\ 23189\ 73593\ \cdot 1\dot{6}$	19
20	$19\ 29657\ 93419\ 40068\ \cdot 1\dot{4}863\ 26681\ \dot{4}$	20
21	$841\ 69304\ 75736\ 82615\ \cdot 0\dot{0}055\ 37098\ 56035\ 43743\ 07862\ 67995\ 57032\ 11517\ 16\dot{5}$	21
22	$40338\ 07185\ 40594\ 55413\ \cdot 0\dot{7}681\ 15942\ 02898\ 55072\ 46\dot{3}$	22
23	$21\ 15074\ 86380\ 81991\ 60560\ \cdot 1\dot{4}539\ 00709\ 21985\ 81560\ 28368\ 79432\ 62411\ 34751\ 77304\ 9\dot{6}$	23
24	$1208\ 66265\ 22296\ 52593\ 46027\ \cdot 3\dot{1}193\ 70825\ 25317\ 81943\ 54664\ 94290\ 02370\ 17884\ 07670\ 760\dot{6}$	24
25	$75008\ 66746\ 07696\ 43668\ 55720\ \cdot 0\dot{7}\dot{5}$	25
26	$50\ 38778\ 10148\ 10689\ 14137\ 89303\ \cdot 0\dot{5}220\ 12578\ 616\dot{3}$	26
27	$3652\ 87764\ 84818\ 12333\ 51104\ 30842\ \cdot 9\dot{7}117\ 79448\ 62155\ 388\dot{4}$	27
28	$2\ 84987\ 69302\ 45088\ 22262\ 69146\ 43291\ \cdot 0\dot{6}781\ 60919\ 54022\ 98850\ 57471\ 264\dot{3}$	28

No. No.

29 238 65427 49968 36276 44645 98191 92192 ·1$\dot{4}$971 75141 24293 29
78531 07344 63276 83615 81920 90395 48022 59887 005$\dot{6}$

30 21399 94925 72253 33665 81074 47651 91097 ·3$\dot{9}$267 41511 61723 30
87457 42183 07692 65988 72659 15822 23522 99560 12610 $\dot{6}$

31 20 50097 57234 78097 56992 17330 95672 31025 ·1$\dot{6}$ 31

32 2093 80059 11346 37840 90951 85290 02797 01847 ·0$\dot{9}$215 68627 32
45098 0$\dot{3}$

33 2 27526 96488 46351 55596 49260 35276 92645 81469 ·9$\dot{6}$540 33
58898 05630 23392 35499 52102 83983 80766 97259 04638
29918 72933 46929 9$\dot{4}$

34 262 57710 28623 95760 47303 04973 61582 02081 44900 ·0$\dot{3}$ 34

35 32125 08210 27180 32518 20479 23042 64985 24352 19411 ·0$\dot{6}$167 35
30687 15322 23644 89970 12377 29406 74349 12505 33504
05463 08151 94195 47588 $\dot{5}$

36 41 59827 81667 94710 91391 70744 95262 35893 66896 03011 36
·3$\dot{4}$647 07892 24934 86300 26351 72786 57869 86190 73528
95096 22602 62909 14538 93184 24$\dot{6}$

37 5692 06954 82035 28002 38834 56219 12105 86444 80512 97181 37
·1$\dot{6}$

38 8 21836 29419 78457 56922 90653 46861 73330 14550 89276 38
28860 ·0$\dot{3}$

39 1250 29043 27166 99301 67323 39829 70289 55241 77196 36444 39
84775 ·0$\dot{1}$115 12959 61422 54370 10247 13682 94153 10427
96865 58167 57082 57986 73899 93972 27245 328$\dot{5}$

40 2 00155 83233 24837 02749 25329 19881 32987 68724 22013 40
28259 15915 ·2$\dot{0}$745 61975 56627 97269 68392 67857 91922
09034 38980 91387 33098 56093 21333 85504 97804 44328 $\dot{5}$

41 336 74982 91536 43742 33396 67690 33387 53016 21959 89471 41
93843 67232 ·1$\dot{5}$461 84738 95582 32931 72690 76305 22088
35341 3$\dot{6}$

42 59470 97050 31354 47718 66049 68440 51540 84057 90715 65106 42
90499 04704 ·3$\dot{1}$085 21256 87731 14081 85506 02030 95487
77872 75541 88660 84463 51830 47372 30158 24058 32606
3137$\dot{6}$

43 110 11910 32362 79775 59564 13079 04376 91604 63051 14442 43
23148 86269 99497 ·1$\dot{6}$

44 21355 25954 52535 01188 65838 50190 41065 67897 32987 39163 44
46921 18045 90304 ·0$\dot{8}$804 75492 59078 32600 55365 57563
91467 18775 4437$\dot{3}$

No. | No.

45 43 32889 69866 41192 41961 66130 59379 20621 84513 68511 45
80910 91449 86557 88032 ·84̇801 07894 36935 44712 22043
37824 03222 13157 52724 92080 64148 64139 82169 49999
63251 23659 58885 48350 3̇

46 9188 55282 41669 32822 62005 55215 50189 71389 60388 91627 46
19959 59100 44871 13437 ·05̇460 99290 78014 18439 71631
20567 37588 65248 22695 03̇

47 20 34689 67763 29074 49345 50279 90220 02006 59751 40253 47
37827 70239 36918 42141 08241 ·16̇

48 4700 38339 58035 73107 85752 55535 00606 06545 96737 36975 48
90579 15139 76356 41204 83354 ·32̇224 63608 75833 28335
29922 67485 89999 04482 01485 19360 16278 04174 80235
55179 40721 09414 74131 28613 85632 76̇

49 11 31804 34454 84249 27067 51862 57733 93426 78903 65954 49
75074 79181 78993 54166 54911 76373 ·16̇

50 2838 22495 70693 70695 92641 56336 48176 47382 84680 92801 50
28821 28228 53171 44648 65111 07028 ·13̇414̇

51 7 40642 48979 67885 06297 50827 14092 09841 76879 73178 51
80887 06673 11610 03487 48532 84412 10855 ·01̇410 07859
45446 13962 08969 02450 30050 85529 35737 40175 68192
32547 38788 71937 12436 43088 30328 24780 39759 59315
765̇

52 2009 64548 02756 60448 34656 19672 71536 31868 67270 82253 52
28766 24346 13019 89213 56500 97796 98883 ·05̇220 12578
6163̇

53 5 66571 70050 80594 14457 19346 03051 93569 61419 46828 53
75104 20621 38756 44521 52460 86197 22777 98400 ·15̇732
08722 74143 30218 06853 58255 45171 33956 38629 28348
9096̇

54 1658 45111 54136 21691 58237 13374 31991 23014 94962 61472 54
54647 27402 46681 55898 78137 71265 07431 49939 ·34̇194
64710 14554 06621 99281 22390 70085 52107 53810 46409
53506 23597 84716 13430 57045 61619 57851 96268 05479
05077 11801 7726̇

55 5 03688 59950 49237 74192 89421 91518 01548 12442 37426 55
49032 14141 52565 13225 28310 97674 29893 27917 85387
·03̇227 93148 88010 54018 445̇

No. No.

56 1586 14682 37658 18636 93634 01572 96643 87827 40978 41277 56
89638 80472 86451 42973 11365 09885 00683 12009 45121
·1$\dot{3}$548 91788 87911 58819 34098 02126 52653 37138 82257
20559 81379 43001 43005 02013 43888 18084 45074 70366
84676 92234 04955 51287 34$\dot{4}$

57 5 17567 43617 54562 69840 73240 68250 71225 61240 84923 57
59305 50859 06216 69403 18108 29579 66515 49771 87766
32444 ·0$\dot{2}$380 9$\dot{5}$

58 1748 89218 40217 11733 96900 25877 61815 91451 41476 16182 58
65448 72627 34721 58762 12289 52384 00153 32666 64382
79521 ·0$\dot{5}$028 24858 75706 21468 92655 36723 16384 18079
09604 51977 40112 994$\dot{3}$

59 6 11605 19994 95218 52558 24525 26426 41677 80767 72684 59
67832 00716 84324 01127 35747 50763 44103 14895 29605
90861 82633 ·1$\dot{6}$

60 2212 27769 12707 83494 22883 23456 71293 24455 73185 05498 60
77801 50566 55269 30277 36635 00257 26591 02528 03139
11549 56836 ·4$\dot{1}$706 43950 64162 89896 44622 10131 68427
75098 18261 25962 01999 15049 $\dot{7}$

61 8 27227 76798 77096 98542 21062 45998 45957 31204 65051 61
84335 66283 84885 29885 84472 02350 07188 81721 85613
01633 96614 27405 ·1$\dot{6}$

62 3195 89251 11415 70958 35916 34369 18081 48735 26276 67109 62
91122 73184 50424 31195 31118 14531 48045 43981 20342
28242 29698 20300 ·0$\dot{3}$

56.

NOTE ON THE VALUE OF EULER'S CONSTANT; LIKEWISE ON THE VALUES OF THE NAPIERIAN LOGARITHMS OF 2, 3, 5, 7 AND 10, AND OF THE MODULUS OF COMMON LOGARITHMS, ALL CARRIED TO 260 PLACES OF DECIMALS.

[From the *Proceedings of the Royal Society*, Vol. XXVII. (1878).]

IN the *Proceedings of the Royal Society*, Vol. XIX., pp. 521, 522, Mr Glaisher has given the values of the logarithms of 2, 3, 5, and 10, and of Euler's constant to 100 places of decimals, in correction of some previous results given by Mr Shanks.

In Vol. XX., pp. 28 and 31, Mr Shanks gives the results of his re-calculation of the above-mentioned logarithms and of the modulus of common logarithms to 205 places, and of Euler's constant to 110 places of decimals.

Having calculated the value of 31 Bernoulli's numbers, in addition to the 31 previously known, I was induced to carry the approximation to Euler's constant to a much greater extent than had been before practicable. For this purpose I likewise re-calculated the values of the above-mentioned logarithms, and found the sum of the reciprocals of the first 500 and of the first 1000 integers, all to upwards of 260 places of decimals. I also found two independent relations between the logarithms just mentioned and the logarithm of 7, which furnished a test of the accuracy of the work.

On comparing my results with those of Mr Shanks, I found that the latter were all affected by an error in the 103rd and 104th places of decimals, in consequence of an error in the 104th place in the determination of $\log \frac{81}{80}$. With this exception, the logarithms given by Mr Shanks were found to be correct to 202 places of decimals.

The error in the determination of $\log_e 10$, of course entirely vitiated Mr Shanks' value of the modulus from the 103rd place onwards. As he gives the complete remainder, however, after the division by his value of $\log_e 10$, I was enabled readily to find the correction to be applied to the erroneous value of the modulus. Afterwards I tested the accuracy of the entire work by multiplying the corrected modulus by my value of $\log_e 10$.

Mr Shanks' values of the sum of the reciprocals of the first 500 and of the first 1000 integers, as well as his value of Euler's constant, were found to be incorrect from the 102nd place onwards.

Let S_n, or S simply, when we are concerned with a given value of n, denote the sum of the harmonic series,

$$1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}.$$

Also let R_n, or R simply, denote the value of the semi-convergent series,

$$\frac{B_1}{2n^2}-\frac{B_2}{4n^4}+\frac{B_3}{6n^6}-\ldots$$

where B_1, B_2, B_3, &c., are the successive Bernoulli's numbers.

Then if Euler's constant be denoted by E, we shall have

$$E=S_n+R_n-\frac{1}{2n}-\log_e n,$$

and the error committed by stopping at any term in the convergent part of R_n will be less than the value of the next term of the series.

I have calculated accurately the values of the Bernoulli's numbers as far as B_{62}, and approximately as far as B_{100}, retaining a number of significant figures varying from 35 to 20.

When $n=1000$, the employment of the numbers up to B_{61} suffices to give the value of R_{1000} to 265 places of decimals. When $n=500$, it is necessary to employ the approximate values up to B_{74}, in order to determine R_{500} with an equal degree of exactness.

In order to reduce as much as possible the number of quantities which must be added together to find S_{500} and S_{1000}, I have resolved the reciprocal of every integer up to 1000 into fractions whose denominators are primes or powers of primes.

Thus S_{500} and S_{1000} may be expressed by means of such fractions, and by adding or subtracting one or more integers, each of these fractions may be reduced to a positive proper fraction, the value of which in decimals may be taken from Gauss' Table, in the second volume of his collected works, or calculated independently.

Thus I have found that:—

$$S_{500}=\frac{249}{256}+\frac{2}{81}+\frac{3}{5}+\frac{120}{343}+\frac{3}{121}+\frac{86}{169}+\frac{205}{289}+\frac{58}{361}+\frac{1}{23}+\frac{3}{29}+\frac{21}{31}+\frac{30}{37}+\frac{11}{41}$$
$$+\frac{15}{43}+\frac{26}{47}+\frac{32}{53}+\frac{24}{59}+\frac{33}{61}+\frac{27}{67}+\frac{67}{71}+\frac{28}{73}+\frac{38}{79}+\frac{73}{83}+\frac{72}{89}+\frac{33}{97}+\frac{61}{101}$$
$$+\frac{45}{103}+\frac{11}{107}+\frac{102}{109}+\frac{68}{113}+\frac{23}{127}+\frac{111}{131}+\frac{116}{137}+\frac{25}{139}+\frac{126}{149}+\frac{27}{151}+\frac{28}{157}$$
$$+\frac{29}{163}+\frac{85}{167}+\frac{88}{173}+\frac{91}{179}+\frac{92}{181}+\frac{97}{191}+\frac{98}{193}+\frac{100}{197}+\frac{101}{199}+\frac{107}{211}+\frac{113}{223}$$
$$+\frac{115}{227}+\frac{116}{229}+\frac{118}{233}+\frac{121}{239}+\frac{122}{241}$$

+ (the sum of the reciprocals of the primes from 251 to 499) − 19.

Similarly I have found that:—

$$S_{1000}=\frac{249}{512}+\frac{310}{729}+\frac{181}{625}+\frac{75}{343}+\frac{62}{121}+\frac{35}{169}+\frac{220}{289}+\frac{11}{361}+\frac{300}{529}+\frac{726}{841}+\frac{32}{961}+\frac{34}{37}$$
$$+\frac{21}{41}+\frac{10}{43}+\frac{40}{47}+\frac{48}{53}+\frac{28}{59}+\frac{56}{61}+\frac{7}{67}+\frac{31}{71}+\frac{40}{73}+\frac{45}{79}+\frac{25}{83}+\frac{49}{89}+\frac{44}{97}$$
$$+\frac{69}{101}+\frac{82}{103}+\frac{90}{107}+\frac{104}{109}+\frac{12}{113}+\frac{67}{127}+\frac{84}{131}+\frac{121}{137}+\frac{85}{139}+\frac{144}{149}+\frac{10}{151}$$
$$+\frac{26}{157}+\frac{141}{163}+\frac{83}{167}+\frac{34}{173}+\frac{53}{179}+\frac{132}{181}+\frac{171}{191}+\frac{102}{193}+\frac{196}{197}+\frac{125}{199}+\frac{90}{211}$$
$$+\frac{95}{223}+\frac{21}{227}+\frac{212}{229}+\frac{138}{233}+\frac{22}{239}+\frac{223}{241}+\frac{211}{251}+\frac{216}{257}+\frac{221}{263}+\frac{226}{269}+\frac{47}{271}$$
$$+\frac{48}{277}+\frac{236}{281}+\frac{49}{283}+\frac{246}{293}+\frac{53}{307}+\frac{261}{311}+\frac{54}{313}+\frac{266}{317}+\frac{57}{331}+\frac{170}{337}+\frac{175}{347}$$
$$+\frac{176}{349}+\frac{178}{353}+\frac{181}{359}+\frac{185}{367}+\frac{188}{373}+\frac{191}{379}+\frac{193}{383}+\frac{196}{389}+\frac{200}{397}+\frac{202}{401}+\frac{206}{409}$$
$$+\frac{211}{419}+\frac{212}{421}+\frac{217}{431}+\frac{218}{433}+\frac{221}{439}+\frac{223}{443}+\frac{226}{449}+\frac{230}{457}+\frac{232}{461}+\frac{233}{463}+\frac{235}{467}$$
$$+\frac{241}{479}+\frac{245}{487}+\frac{247}{491}+\frac{251}{499}$$

+ (the sum of the reciprocals of the primes from 503 to 997) − 43.

This mode of finding S_{500} and S_{1000} is attended with the advantage that if an error were made in the calculation of the former of these quantities, it would not affect the latter.

The logarithms required have been found in the following manner:—

Let $\log\frac{10}{9}=a$, $\log\frac{25}{24}=b$, $\log\frac{81}{80}=c$, $\log\frac{50}{49}=d$, and $\log\frac{126}{125}=e$.

Then we have

$$\log 2=7a-2b+3c,\quad \log 3=11a-3b+5c,\quad \log 5=16a-4b+7c.$$

Also $$\log 7=\frac{1}{2}(39a-10b+17c-d);$$

or again, $$\log 7=19a-4b+8c+e,$$

and we have the equation of condition

$$a-2b+c=d+2e,$$

which supplies a sufficient test of the accuracy of the calculations by which a, b, c, d, and e have been found.

Since
$$\log\frac{10}{9}=-\log\left(1-\frac{1}{10}\right)$$
$$\log\frac{25}{24}=-\log\left(1-\frac{4}{100}\right)$$
$$\log\frac{81}{80}=\quad\log\left(1+\frac{1}{80}\right)$$
$$\log\frac{50}{49}=-\log\left(1-\frac{2}{100}\right)$$
$$\log\frac{126}{125}=\quad\log\left(1+\frac{8}{1000}\right).$$

If we have settled beforehand on the number of decimal places which we wish to retain, and have already formed the decimal values of the reciprocals of the successive integers to the extent required, then the formation of the values of a, b, c, d, and e, will only involve operations which, though numerous, are of extreme simplicity.

In this way have been found the following results:—

Log 10 ÷ 9 =

·10536 05156 57826 30122 75009 80839 31279 83061 20372 98327
40725 63939 23369 25840 23240 13454 64887 65695 46213 41207
66027 72591 03705 17148 67351 70132 21767 11456 06836 27564
22686 82765 81669 95879 19464 85052 49713 75112 78720 90836
46753 73554 69033 76623 27864 87959 35883 39553 19538 32230
68063 73738 05700 33668 65

Log 25 ÷ 24 =

·04082 19945 20255 12955 45770 65155 31987 01772 11747 63352
02297 28561 42083 06828 16287 62241 55690 62020 38337 10701
85958 13391 57612 02856 02344 55254 44440 90711 64191 09254
90615 87090 13793 32587 08185 56690 89768 86470 69797 42768
97243 12354 16791 64980 33118 36535 36811 73829 09383 64151
16223 48133 67972 69296

Log 81 ÷ 80 =

·01242 25199 98557 15331 12931 28631 20890 67623 60339 58145
90685 43409 40510 22236 97287 99924 04408 75833 17607 39941
83907 88915 98331 57135 00593 07313 64880 85644 69078 59065
10006 71375 61155 92285 64823 02773 78467 95356 20673 20672
56121 24774 48623 61600 82118 41837 57253 45313 78157 48027
60627 91715 42041 36587 2

Log 50 ÷ 49 =

·02020 27073 17519 44840 80453 01024 19238 78525 33383 73356
83210 27195 49256 65918 71880 87170 92908 14086 00703 48551
55810 69865 22995 29709 68602 61790 51909 27000 19877 96234
68586 52194 37909 61418 83597 32774 05301 16399 74760 65371
30928 59153 97434 74168 79079 46094 49807 56880 62620 29129
95963 65850 08854 45

Log 126 ÷ 125 =

·00796 81696 49176 87351 07973 39067 84478 84307 61916 78206
21803 11515 15228 34251 08036 00862 32503 51700 93221 55597
11104 32429 31908 69430 97326 52573 22928 44338 63827 35942
41437 63883 38664 80785 92159 70835 21671 40563 92519 30299
88730 07233 43319 67047 32333 55315 84852 90164 08154 11413
00140 51668 01463 4832

All these are Napierian logarithms.

The above-mentioned equation of condition is satisfied to 263 places of decimals.

Whence have been deduced the following:—

$\mathrm{Log}_\epsilon 2 =$ ·69314 71805 59945 30941 72321 21458 17656 80755 00134 36025
52541 20680 00949 33936 21969 69471 56058 63326 99641 86875
42001 48102 05706 85733 68552 02357 58130 55703 26707 51635
07596 19307 27570 82837 14351 90307 03862 38916 73471 12335
01153 64497 95523 91204 75172 68157 49320 65155 52473 41395
25882 95045 30081 06850 15

$\mathrm{Log}_\epsilon 3 =$ 1·09861 22886 68109 69139 52452 36922 52570 46474 90557 82274
94517 34694 33363 74942 93218 60896 68736 15754 81373 20887
87970 02906 59578 65742 36800 42259 30519 82105 28018 70767
27741 06031 62769 18338 13671 79373 69884 43609 59903 74257
03167 95911 52114 55919 17750 67134 70549 40166 77558 02222
03170 25294 68992 45403 15

$\mathrm{Log}_\epsilon 5 =$ 1·60943 79124 34100 37460 07593 33226 18763 95256 01354 26851
77219 12647 89147 41789 87707 65776 46301 33878 09317 96107
99966 30302 17155 62899 72400 52293 24676 19963 36166 17463
70572 75521 79637 49718 32456 53492 85620 23415 25057 27015
51936 00879 77738 97256 88193 54071 27661 54731 22180 95279
48521 29282 13604 17624 80

$\mathrm{Log}_\epsilon 7 =$ 1·94591 01490 55313 30510 53527 43443 17972 96370 84729 58186
11884 59390 14993 75798 62752 06926 77876 58498 58787 15269
93061 69420 58511 40911 72375 22576 77786 84314 89580 95163
90077 59078 24468 10427 47833 82259 34900 84673 74412 50497
37048 53551 76783 55774 86240 15102 77418 08868 67107 51412
13480 93879 74210 03537 95

$\mathrm{Log}_\epsilon 10 =$ 2·30258 50929 94045 68401 79914 54684 36420 76011 01488 62877
29760 33327 90096 75726 09677 35248 02359 97205 08959 82983
41967 78404 22862 48633 40952 54650 82806 75666 62873 69098
78168 94829 07208 32555 46808 43799 89482 62331 98528 39350
53089 65377 73262 88461 63366 22228 76982 19886 74654 36674
74404 24327 43685 24474 95

$M =$ ·43429 44819 03251 82765 11289 18916 60508 22943 97005 80366
65661 14453 78316 58646 49208 87077 47292 24949 33843 17483
18706 10674 47663 03733 64167 92871 58963 90656 92210 64662
81226 58521 27086 56867 03295 93370 86965 88266 88331 16360
77384 90514 28443 48666 76864 65860 85135 56148 21234 87653
43543 43573 17247 48049 05993 55353 05

where M denotes the modulus of common logarithms.

In these calculations the value of $\log \frac{50}{49}$ has been determined with less accuracy than that of $\log \frac{126}{125}$, and therefore the value of $\log 7$ found by means of the latter quantity has been preferred.

If now in the formula which gives Euler's constant we take $n = 500$, we find the following results:—

$$\frac{1}{2n} = 0{\cdot}001$$

$R_{500} =$ ·00000 03333 33200 00025 39671 87309 34479 09501 49853 06920
81561 41982 03143 98353 10049 47690 35814 25947 82825 73530
80967 33251 23444 83365 27221 32891 79715 39888 78668 70158
11997 43277 84264 18919 84678 56672 58294 26067 37401 94207
08483 64907 04495 03811 66583 11699 18899 16275 81704 82573
08004 99446 91635

$S_{500} =$ 6·79282 34299 90524 60298 92871 45367 97369 48198 13814 39677
91166 43088 89685 43566 23790 55049 24576 49403 73586 56039
17565 98584 37506 59282 23134 68847 97117 15030 24984 83148
07266 84437 10123 70203 14772 22094 00570 47964 42959 21001
09719 01932 14586 27077 01576 02007 28842 06850 09735 01135
74118 52998 6631

$\text{Log}_e 500 =$

6·21460 80984 22191 74263 67422 42594 91605 47278 04331 52606
36739 79303 69340 93242 07062 36272 51021 28288 27237 62074
83901 87110 62880 60166 54305 61594 90289 71296 61913 55661
26910 65179 94054 14829 26073 41092 64585 48079 22114 05716
58115 31635 24264 74180 14925 98528 81625 94504 71489 68628
97329 77937 00975

$E =$ ·57721 56649 01532 86060 65120 90082 40243 10421 59335 93992
35988 05767 23488 48677 26777 66467 09369 47063 29174 67495
14631 44724 98070 82480 96050 40144 86542 83622 41739 97644
92353 62535 00333 74293 73377 37673 94279 25952 58247 09491
60087 35203 94816 56708 53233 15177 66115 28621 19950 15079
84793 74508 5697

Again, if in the same formula we take $n=1000$, we find the following:—

$$\frac{1}{2n}=0{\cdot}0005$$

$R_{1000}=$ ·00000 00833 33325 00000 39682 49801 59487 73237 84632 11743
88611 32124 18782 98862 06644 51967 06850 04241 14869 65631
43736 78499 44114 24665 37423 82138 50259 70190 89962 61572
33894 07843 88131 36054 55889 69002 08034 44545 27898 47738
31546 74821 27649 54293 18527 10448 88349 55931 43201 82238
86978 52223 81562

$S_{1000}=$ 7·48547 08605 50344 91265 65182 04333 90017 65216 79169 70880
36657 73626 74995 76993 49165 20244 09599 34437 41184 50813
96798 01438 22544 03715 81484 21958 84703 40431 40398 43368
92966 39178 33827 35905 57913 00071 54692 68403 25933 79804
87809 56515 86955 67800 24804 71415 08712 32350 00711 42865
21027 95267 06455

$\text{Log}_e 1000=$

6·90775 52789 82137 05205 39743 64053 09262 28033 04465 88631
89280 99983 70290 27178 29032 05744 07079 91615 26879 48950
25903 35212 68587 45900 22857 63952 48420 26999 88621 07296
34506 84487 21624 97666 40425 31399 68447 86995 95585 18051
59268 96133 19788 65384 90098 66686 30946 59660 23963 10024
23212 72982 31056

$E=$ ·57721 56649 01532 86060 65120 90082 40243 10421 59335 93992
35988 05767 23488 48677 26777 66467 09369 47063 29174 67495
14631 44724 98070 82480 96050 40144 86542 83622 41739 97644
92353 62535 00333 74293 73377 37673 94279 25952 58247 09491
60087 35203 94816 56708 53233 15177 66115 28621 19950 15079
84793 74508 56961

It will be seen that the two values found for E agree to 263 places of decimals, which supplies another independent verification of the value obtained for $\log_e 2$.

57.

SUPPLEMENTARY NOTE ON THE VALUES OF THE NAPIERIAN LOGARITHMS OF 2, 3, 5, 7, AND 10, AND OF THE MODULUS OF COMMON LOGARITHMS.

[From the *Proceedings of the Royal Society*. Vol. XLII. (1886).]

IN Vol. XXVII. of the *Proceedings of the Royal Society*, pp. 88—94, I have given the values of the logarithms referred to, and of the Modulus, all carried to 260 places of decimals.

These logarithms were derived from the five quantities a, b, c, d, e, which were calculated independently, where

$$a = \log\frac{10}{9},\quad b = \log\frac{25}{24},\quad c = \log\frac{81}{80},\quad d = \log\frac{50}{49},\ \text{and}\ e = \log\frac{126}{125},$$

and a complete test of the accuracy of these latter calculations is afforded by the equation of condition

$$a - 2b + c = d + 2e.$$

In the actual case the values found for a, b, c, d, e satisfied this equation to 263 places of decimals.

Although this proved that the values of the logarithms found in the above paper had been determined with a greater degree of accuracy than was there claimed for them, yet I was not entirely satisfied with the result, since the calculation of the fundamental quantities had been carried to 269 places of decimals, and therefore the above-cited equation of condition shewed that some errors, which I had not succeeded in tracing, had crept into the calculations so as to vitiate the results beyond the 263rd place of decimals.

Of course in working with such a large number of interminable decimals, the necessary neglect of decimals of higher orders causes an uncertainty in a few of the last decimal places, but when due care is taken, this uncertainty ought not to affect more than two or three of the last figures.

The Napierian logarithm of 10 is equal to $23a-6b+10c$, and the Modulus of common logarithms is the reciprocal of this quantity.

Since the value found for the logarithm of 10 cannot be depended upon beyond 262 places of decimals, a corresponding uncertainty will affect the value of the Modulus found from it.

In the operation of dividing unity by the assumed value of log 10, however, the quotient was carried to 282 places of decimals.

This was done for the purpose of supplying the means of correcting the value found for the Modulus, without the necessity of repeating the division, when I should have succeeded in tracing the errors of calculation alluded to above, and thus finding a value of log 10 which might be depended upon to a larger number of decimal places.

Through inadvertence, the values of the logarithms concerned, and the resulting value of the Modulus, were printed in my paper in the *Proceedings* above referred to exactly as they resulted from the calculations, without the suppression of the decimals of higher orders, which in the case of the logarithms were uncertain, and in the case of the Modulus were known to be incorrect.

Although it was unlikely that this oversight would lead to any misapprehension as to the degree of accuracy claimed for my results in the mind of a reader of the paper itself, there might be a danger of such misapprehension if my printed results were quoted in full unaccompanied by the statement that the later decimal places were not to be depended on.

My attention has been recalled to this subject by the circumstance that in the excellent article on Logarithms which Mr Glaisher has contributed to the new edition of the *Encyclopædia Britannica*, he has quoted my value of the Modulus, and has given the whole of the 282 decimals as printed in the *Proceedings of the Royal Society*, without expressly stating that this value does not claim to be accurate beyond 262 or 263 places of decimals.

I have now succeeded in tracing and correcting the errors which vitiated the later decimals in my former calculations, and have extended the computations to a few more decimal places. The computations of the fundamental logarithms a, b, c, d, e have now been carried to 276 decimal places, of which only the last two or three are uncertain.

The equation of condition, $a-2b+c=d+2e$, by which the accuracy of all this work is tested, is now satisfied to 274 places of decimals.

The parts of the several logarithms concerned which immediately follow the first 260 decimal places as already given in my paper in the *Proceedings*, are as follows:—

	a	05700 33668 72127 8
	b	67972 72775 92889 4
	c	42038 01732 39184 3
	d	08865 93150 99834 1
	e	01463 48349 12851 7
Whence	$a-2b+c=$	11792 89849 25533 3
and	$d+2e=$	11792 89849 25537 5
	Difference $=$	4 2

Also the corresponding parts of the logarithms which are derived from the above are—

	log 2	30070 95326 36668 7
	log 3	68975 60690 10659 1
	log 5	13580 59722 56777 3
	log 7	74183 10810 25196 7
Whence	log 10	43651 55048 93446 0

And the correction to the value of log 10 which was formerly employed in finding the Modulus is

$$-(263)\ 33\ 69426\ 01554\ 0$$

where the number within brackets denotes the number of cyphers which precede the first significant figure.

The corresponding correction of M, the Modulus of common logarithms, will be found by changing the sign of this and multiplying by M^2, the approximate value of which is

$$0{\cdot}18861\ 16970\ 1161$$

Hence this correction is

$$(264)\ 6\ 35513\ 15874\ 7$$

And finally the corrected value of the Modulus is

$M = \cdot 43429$ 44819 03251 82765 11289 18916 60508 22943 97005 80366
65661 14453 78316 58646 49208 87077 47292 24949 33843 17483
18706 10674 47663 03733 64167 92871 58963 90656 92210 64662
81226 58521 27086 56867 03295 93370 86965 88266 88331 16360
77384 90514 28443 48666 76864 65860 85135 56148 21234 87653
43543 43573 17253 83562 21868 25

which is true, certainly to 272 and probably to 273 places of decimals.

58.

NOTE ON SIR WILLIAM THOMSON'S CORRECTION OF THE ORDINARY EQUILIBRIUM THEORY OF THE TIDES.

[From the *Report of the British Association*, 1886, p. 541.]

IN Art. 806 of Thomson and Tait's *Treatise on Natural Philosophy* it is pointed out that if the Earth's surface is supposed to be only partially covered by the Ocean, the rise and fall of the water at any place, according to the equilibrium theory, would be falsely estimated, if, as is usually done, it were taken to be the same as the rise and fall of the spheroidal surface that would bound the water were there no dry land.

In the articles which immediately follow the above, it is shewn that in order to satisfy the condition that the volume of the water remains unchanged, the expression for the radius vector of the spheroid bounding the water must contain, in addition to the terms which would be sufficient if there were no land, a quantity a which depends on the positions of the Sun and Moon at the time considered, and which is the same for all points of the sea at the same time.

This quantity a contains five constant coefficients which depend merely on the configuration of land and water. The values of these coefficients in the case of the actual oceans of our globe have been carefully determined very recently by Mr H. H. Turner of Trinity College, in a joint paper by Professor G. H. Darwin and himself, which is published in Vol. XL. of the *Proceedings of the Royal Society*.

It should be remarked that every inland sea or detached sheet of water on the globe has in the same way a set of five constants, peculiar to itself, which enter into the expression of the height of the tide at any time in that sheet of water.

By taking such constants into account the formulæ which apply to the Oceanic tides are rendered equally applicable to the tides of such a sea as the Caspian, which are thus theoretically shewn to be very small, as they are known to be practically.

In the work above cited reference is made to a passage in a memoir by Sir William Thomson on the Rigidity of the Earth, published in the *Philosophical Transactions* for 1862, as being the only one known to the writers in which any consciousness is shewn that such a correction of the ordinary equilibrium theory as that above mentioned is required.

However just this remark may be in reference to modern writers on the equilibrium theory, it is only fair to Bernoulli, the originator of the equilibrium theory, to point out that in his prize essay on the Tides he distinctly recognises the fact that when the sea is supposed to have only a limited extent the rise and fall of its surface cannot be the same as if the Earth were entirely covered by it. In particular, he shews that the Tides are so much the smaller as the sea has less extent in longitude, and thus explains why they are altogether insensible in the Caspian and in the Black Sea and very small in the Mediterranean, of which the communication with the Ocean is almost entirely cut off at the Straits of Gibraltar (see Bernoulli, *Traité sur le Flux et Reflux de la Mer*, Chap. XI. sect. ii.). It may be as well to mention that this treatise of Bernoulli, as well as the dissertations of Maclaurin and Euler on the same subject, is published in the 3rd volume of the Jesuit's edition of Newton's *Principia* and also appears in the Glasgow reprint of that edition.

59.

ON CERTAIN APPROXIMATE FORMULÆ FOR CALCULATING THE TRAJECTORIES OF SHOT.

[From the *Proceedings of the Royal Society,* Vol. XXVI. (1877) and *Nature,* Vol. XLI. (1890).]

IN the postscript to a paper by Mr W. D. Niven, "On the Calculation of the Trajectories of Shot," which is published in the *Proceedings* of the Royal Society, Vol. XXVI. pp. 268—287, I have given, without demonstration, some convenient and not inelegant formulæ applicable to a limited arc of a trajectory when the resistance is supposed to vary as the nth power of the velocity.

In these formulæ, the angle between the chord of the arc and the tangent at any point is supposed to be always small. The index n is not restricted to integral values, but may take any value whatever.

As the proof of these formulæ is not altogether obvious, and a similar method of treatment may be found useful in other problems, I think it may not be unacceptable to your readers if I shew here how the formulæ may be demonstrated.

Analysis.

Investigation of formulæ applicable to a small arc of a trajectory, when the resistance varies as the nth power of the velocity.

Let x and y denote the horizontal and vertical coordinates at time t, u the horizontal velocity, and ϕ the angle which the direction of motion makes with the horizon at the same time.

Hence the velocity at time t is $u \sec\phi$, and we may denote the resistance by $ku^n(\sec\phi)^n$, where k is constant throughout the small arc in question.

Also let p and q denote the values of u at the beginning and end of the arc, α and β the corresponding values of ϕ, g the force of gravity, T the time taken to describe the arc, X and Y the corresponding total horizontal and vertical motion.

Making ϕ the independent variable, the fundamental formulæ are

$$(1)\quad \frac{du}{d\phi} = \frac{ku^{n+1}}{g}(\sec\phi)^{n+1};$$

$$(2)\quad \frac{dx}{d\phi} = -\frac{u^2}{g}(\sec\phi)^2;$$

$$(3)\quad \frac{dy}{d\phi} = -\frac{u^2}{g}(\sec\phi)^2\tan\phi\,;$$

$$(4)\quad \frac{dt}{d\phi} = -\frac{u}{g}(\sec\phi)^2.$$

From the first of these equations

$$\frac{1}{u^{n+1}}\frac{du}{d\phi} = \frac{k}{g}(\sec\phi)^{n+1};$$

and therefore, by integration between the limits $\phi=\alpha$ and $\phi=\beta$,

$$\frac{1}{q^n} - \frac{1}{p^n} = \frac{kn}{g}\int_\beta^\alpha (\sec\phi)^{n+1}\,d\phi.$$

Also, we have

$$X = \frac{1}{g}\int_\beta^\alpha u^2(\sec\phi)^2\,d\phi\,;$$

$$Y = \frac{1}{g}\int_\beta^\alpha u^2(\sec\phi)^2\tan\phi\,d\phi\,;$$

and

$$T = \frac{1}{g}\int_\beta^\alpha u\,(\sec\phi)^2\,d\phi\,;$$

and we wish to compare the two former of these definite integrals with the following known one, viz.:—

$$\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}} = (n-2)\int_\beta^\alpha \frac{1}{u^{n-1}}\frac{du}{d\phi}\,d\phi = \frac{k(n-2)}{g}\int_\beta^\alpha u^2(\sec\phi)^{n+1}\,d\phi\,;$$

and the last with

$$\frac{1}{q^{n-1}}-\frac{1}{p^{n-1}}=(n-1)\int_\beta^\alpha \frac{1}{u^n}\frac{du}{d\phi}\,d\phi=\frac{k(n-1)}{g}\int_\beta^\alpha u\,(\sec\phi)^{n+1}\,d\phi.$$

This may be done by means of the following lemma, which follows immediately from Taylor's theorem:—

Lemma.

If $F(\phi)$ be any function either of ϕ only, or of ϕ and u, where u is a function of ϕ given by the above differential equation (1), and if α and β be the limiting values of ϕ in the integral and $\gamma=\frac{1}{2}(\alpha+\beta)$, then, putting for a moment $\phi=\gamma+\omega$,

$$\int_\beta^\alpha F(\phi)\,d\phi=\int_{-\frac{1}{2}(\alpha-\beta)}^{\frac{1}{2}(\alpha-\beta)} F(\gamma+\omega)\,d\omega$$

$$=\int_{-\frac{1}{2}(\alpha-\beta)}^{\frac{1}{2}(\alpha-\beta)}\left\{F(\gamma)+F'(\gamma)\,\omega+F''(\gamma)\frac{\omega^2}{2}+F'''(\gamma)\frac{\omega^3}{6}+F''''(\gamma)\frac{\omega^4}{24}+\&\text{c.}\right\}d\omega$$

$$=(\alpha-\beta)\left\{F(\gamma)+\frac{1}{24}(\alpha-\beta)^2F''(\gamma)+\frac{1}{1920}(\alpha-\beta)^4F''''(\gamma)+\&\text{c.}\right\},$$

where $$F'(\phi)=\frac{dF(\phi)}{d\phi},\quad F''(\phi)=\frac{d^2F(\phi)}{d\phi^2},\ \&\text{c.},$$

and $F(\gamma)$, $F'(\gamma)$, $F''(\gamma)$, &c., are what $F(\phi)$, $F'(\phi)$, $F''(\phi)$, &c., become when γ is substituted for ϕ, and the corresponding value of u (u_0 suppose) is put for u.

In what follows, the last of the terms above written, which is of the 5th order in $(\alpha-\beta)$, is neglected, together with all terms of the same order of small quantities.

All the definite integrals with which we are here concerned are included in the two forms

$$\int_\beta^\alpha u^l(\sec\phi)^m\,d\phi,\quad\text{and}\quad\int_\beta^\alpha u^l(\sec\phi)^m\tan\phi\,d\phi.$$

In the first place, we will apply the above formula to the case in which $F(\phi)$ is a function of ϕ only, viz. when $F(\phi)=(\sec\phi)^{n+1}$.

Hence

$$F'(\phi) = (n+1)(\sec\phi)^{n+1}\tan\phi;$$

$$F''(\phi) = (n+1)[(n+1)(\sec\phi)^{n+1}(\tan\phi)^2 + (\sec\phi)^{n+3}]$$

$$= (n+1)[\overline{n+2}\,(\sec\phi)^{n+3} - \overline{n+1}\,(\sec\phi)^{n+1}];$$

and therefore,

$$\int_\beta^\alpha (\sec\phi)^{n+1}\,d\phi = (\alpha-\beta)(\sec\gamma)^{n+1}\left\{1 + \frac{n+1}{24}(\alpha-\beta)^2[\overline{n+2}\,(\sec\gamma)^2 - \overline{n+1}]\right\},$$

to the 4th order inclusive.

Hence

$$\frac{1}{q^n} - \frac{1}{p^n} = \frac{kn}{g}(\alpha-\beta)(\sec\gamma)^{n+1}\left\{1 + \frac{n+1}{24}(\alpha-\beta)^2[\overline{n+2}\,(\sec\gamma)^2 - \overline{n+1}]\right\},$$

which gives q when p is known.

In the next place, let $F(\phi) = u^l(\sec\phi)^m$.

Hence

$$F'(\phi) = \frac{dF\phi}{d\phi} = lu^{l-1}\frac{du}{d\phi}(\sec\phi)^m + mu^l(\sec\phi)^m\tan\phi$$

$$= F(\phi)\left[\frac{l}{u}\frac{du}{d\phi} + m\tan\phi\right],$$

or

$$F'(\phi) = F(\phi)\left[\frac{kl}{g}u^n(\sec\phi)^{n+1} + m\tan\phi\right];$$

and

$$F''(\phi) = F'(\phi)\left[\frac{kl}{g}u^n(\sec\phi)^{n+1} + m\tan\phi\right]$$

$$+ F(\phi)\left[\frac{kln}{g}u^{n-1}\frac{du}{d\phi}(\sec\phi)^{n+1} + \frac{kl}{g}(n+1)u^n(\sec\phi)^{n+1}\tan\phi + m(\sec\phi)^2\right],$$

or

$$F''(\phi) = F(\phi)\left[\frac{k^2l^2}{g^2}u^{2n}(\sec\phi)^{2n+2} + 2\frac{klm}{g}u^n(\sec\phi)^{n+1}\tan\phi + m^2(\sec\phi)^2 - m^2\right]$$

$$+ F(\phi)\left[\frac{k^2ln}{g^2}u^{2n}(\sec\phi)^{2n+2} + \frac{kl}{g}(n+1)u^n(\sec\phi)^{n+1}\tan\phi + m(\sec\phi)^2\right]$$

$$= F(\phi)\left\{\frac{k^2l}{g^2}(l+n)u^{2n}(\sec\phi)^{2n+2}\right.$$

$$\left. + \frac{kl}{g}(2m+n+1)u^n(\sec\phi)^{n+1}\tan\phi + m(m+1)(\sec\phi)^2 - m^2\right\}.$$

Since
$$\frac{du}{d\phi}=\frac{k}{g}u^{n+1}(\sec\phi)^{n+1},$$
this last expression may be put under the form
$$F''(\phi)=F(\phi)\left\{l(l+n)\left(\frac{du}{ud\phi}\right)^2+l(2m+n+1)\left(\frac{du}{ud\phi}\right)\tan\phi+m(m+1)(\sec\phi)^2-m^2\right\}.$$
Also
$$F(\gamma)=u_0^{\,l}(\sec\gamma)^m.$$
Hence, by the above lemma,
$$\int_\beta^\alpha u^l(\sec\phi)^m\,d\phi=(\alpha-\beta)\,u_0^{\,l}(\sec\gamma)^m\left\{1+\frac{1}{24}(\alpha-\beta)^2\left[l(l+n)\left(\frac{du}{ud\phi}\right)_0^2\right.\right.$$
$$\left.\left.+l(2m+n+1)\left(\frac{du}{ud\phi}\right)_0\tan\gamma+m(m+1)(\sec\gamma)^2-m^2\right]\right\}$$
where $\left(\frac{du}{ud\phi}\right)_0$ denotes what $\frac{du}{ud\phi}$ becomes when $\omega=0$, or when γ is substituted for ϕ, and u_0 for u, that is
$$\left(\frac{du}{ud\phi}\right)_0=\frac{k}{g}\,u_0^{\,n}(\sec\gamma)^{n+1}.$$

The factor $u_0^{\,l}$ may be eliminated from this expression, and the expression itself simplified, by means of the formula
$$\frac{1}{q^{n-l}}-\frac{1}{p^{n-l}}=(n-l)\int_\beta^\alpha\frac{1}{u^{n-l+1}}\frac{du}{d\phi}\,d\phi=\frac{k(n-l)}{g}\int_\beta^\alpha u^l(\sec\phi)^{n+1}\,d\phi,$$
for, putting $m=n+1$ in the above expression, we have
$$\int_\beta^\alpha u^l(\sec\phi)^{n+1}\,d\phi=(\alpha-\beta)\,u_0^{\,l}(\sec\gamma)^{n+1}\left\{1+\frac{1}{24}(\alpha-\beta)^2\left[l(l+n)\left(\frac{du}{ud\phi}\right)_0^2\right.\right.$$
$$\left.\left.+3l(n+1)\left(\frac{du}{ud\phi}\right)_0\tan\gamma+\overline{n+1}\,\overline{n+2}\,(\sec\gamma)^2-(n+1)^2\right]\right\}.$$
Hence
$$\int_\beta^\alpha u^l(\sec\phi)^m\,d\phi\div\int_\beta^\alpha u^l(\sec\phi)^{n+1}\,d\phi$$
$$=\int_\beta^\alpha u^l(\sec\phi)^m\,d\phi\div\frac{g}{k(n-l)}\left(\frac{1}{q^{n-l}}-\frac{1}{p^{n-l}}\right)=(\sec\gamma)^{m-n-1}$$
$$\left\{1+\frac{1}{24}(\alpha-\beta)^2\left[2l(m-n-1)\left(\frac{du}{ud\phi}\right)_0\tan\gamma+\overline{m-n-1}\,\overline{m+n+2}\,(\sec\gamma)^2\right.\right.$$
$$\left.\left.-\overline{m-n-1}\,\overline{m+n+1}\right]\right\}.$$

It will be noticed that the term involving $\left(\frac{du}{u\,d\phi}\right)_0^2$ has disappeared by this division.

Now make $m=2$, and this formula becomes

$$\int_\beta^\alpha u^l (\sec\phi)^2\,d\phi = \frac{g}{k(n-l)}\left(\frac{1}{q^{n-l}}-\frac{1}{p^{n-l}}\right)(\cos\gamma)^{n-1}$$

$$\left\{1-\frac{1}{24}(\alpha-\beta)^2\left[2l(n-1)\left(\frac{du}{u\,d\phi}\right)_0\tan\gamma+\overline{n-1}\,\overline{n+4}\,(\sec\gamma)^2-\overline{n-1}\,\overline{n+3}\right]\right\}.$$

Divide throughout by g, and put $l=2$, then, from before,

$$X=\frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}}-\frac{1}{p^{n-2}}\right)(\cos\gamma)^{n-1}$$

$$\left\{1-\frac{n-1}{24}(\alpha-\beta)^2\left[4\left(\frac{du}{u\,d\phi}\right)_0\tan\gamma+(n+4)(\sec\gamma)^2-\overline{n+3}\right]\right\}.$$

Similarly, divide throughout by g, and put $l=1$, then

$$T=\frac{1}{k(n-1)}\left(\frac{1}{q^{n-1}}-\frac{1}{p^{n-1}}\right)(\cos\gamma)^{n-1}$$

$$\left\{1-\frac{n-1}{24}(\alpha-\beta)^2\left[2\left(\frac{du}{u\,d\phi}\right)_0\tan\gamma+(n+4)(\sec\gamma)^2-\overline{n+3}\right]\right\}.$$

Lastly, let

$$F(\phi)=u^l(\sec\phi)^m\tan\phi=f(\phi)\tan\phi \text{ suppose,}$$

so that

$$f(\phi)=u^l(\sec\phi)^m;$$

then $$F'(\phi)=f'(\phi)\tan\phi+f(\phi)(\sec\phi)^2,$$

and $$F''(\phi)=f''(\phi)\tan\phi+2f'(\phi)(\sec\phi)^2+2f(\phi)(\sec\phi)^2\tan\phi.$$

Hence $\displaystyle\int_\beta^\alpha F(\phi)\,d\phi=(\alpha-\beta)\left\{F(\gamma)+\frac{1}{24}(\alpha-\beta)^2F''(\gamma)\right\}$ approximately,

$$=(\alpha-\beta)\left\{f(\gamma)\tan\gamma+\frac{1}{24}(\alpha-\beta)^2[f''(\gamma)\tan\gamma+2f'(\gamma)(\sec\gamma)^2+2f(\gamma)(\sec\gamma)^2\tan\gamma]\right\};$$

also

$$\int_\beta^\alpha f(\phi)\,d\phi=(\alpha-\beta)\left\{f(\gamma)+\frac{1}{24}(\alpha-\beta)^2f''(\gamma)\right\} \text{ approximately;}$$

and therefore

$$\int_\beta^\alpha F(\phi)d\phi \div \int_\beta^\alpha f(\phi)d\phi = \tan\gamma + \frac{1}{12}(\alpha-\beta)^2\left[\frac{f'(\gamma)}{f(\gamma)}(\sec\gamma)^2 + (\sec\gamma)^2\tan\gamma\right];$$

in which the term involving $f''(\gamma)$ has disappeared.

Now, since $f(\phi) = u^l(\sec\phi)^m$, we have, as before

$$f'(\phi) = f(\phi)\left[l\left(\frac{du}{ud\phi}\right)_0 + m\tan\phi\right];$$

and therefore

$$\frac{f'(\gamma)}{f(\gamma)} = l\left(\frac{du}{ud\phi}\right)_0 + m\tan\gamma.$$

Hence

$$\int_\beta^\alpha F(\phi)d\phi \div \int_\beta^\alpha f(\phi)d\phi = \tan\gamma + \frac{1}{12}(\alpha-\beta)^2(\sec\gamma)^2\left[l\left(\frac{du}{ud\phi}\right)_0 + \overline{m+1}\tan\gamma\right];$$

and in the particular case where $l=2$, and $m=2$, we have

$$\frac{Y}{X} = \tan\gamma + \frac{1}{12}(\alpha-\beta)^2(\sec\gamma)^2\left[2\left(\frac{du}{ud\phi}\right)_0 + 3\tan\gamma\right]$$

$$= \tan\left\{\gamma + \frac{1}{12}(\alpha-\beta)^2\left[2\left(\frac{du}{ud\phi}\right)_0 + 3\tan\gamma\right]\right\}.$$

Hence the angle which the chord of the arc makes with the axis of x is

$$\gamma + \frac{1}{12}(\alpha-\beta)^2\left[2\left(\frac{du}{ud\phi}\right)_0 + 3\tan\gamma\right] = \bar{\gamma}, \text{ suppose.}$$

Multiplying by the value of X found above, we have

$$Y = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\gamma)^{n-1}\left\{\tan\gamma - \frac{1}{24}(\alpha-\beta)^2\right.$$
$$\left\{\left(\frac{du}{ud\phi}\right)_0\left[4(n-1)(\tan\gamma)^2 - 4(\sec\gamma)^2\right]\right.$$
$$\left.\left. + \tan\gamma\left[\overline{n-1}\,\overline{n+4}(\sec\gamma)^2 - 6(\sec\gamma)^2 - \overline{n-1}\,\overline{n+3}\right]\right\}\right\};$$

or

$$Y = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\gamma)^{n-1}\left\{\tan\gamma - \frac{1}{24}(\alpha-\beta)^2\right.$$
$$\left.\left\{\left(\frac{du}{ud\phi}\right)_0\left[4(n-2)(\sec\gamma)^2 - 4(n-1)\right] + \tan\gamma\left[\overline{n-2}\,\overline{n+5}(\sec\gamma)^2 - \overline{n-1}\,\overline{n+3}\right]\right\}\right\}.$$

Considering $\frac{1}{q^{n-2}}-\frac{1}{p^{n-2}}$, $\frac{1}{q^{n-1}}-\frac{1}{p^{n-1}}$, and $\alpha-\beta$ to be small quantities of the first order, the above expressions for $\frac{1}{q^n}-\frac{1}{p^n}$, X, Y, and T are true to the fourth order.

The quantity $\left(\frac{du}{u\,d\phi}\right)_0$ which occurs as a factor in some of the terms of the third order may be put under a very convenient form in the following manner.

We have, by Taylor's theorem,

$$u=u_0+\left(\frac{du}{d\phi}\right)_0\omega+\left(\frac{d^2u}{d\phi^2}\right)_0\frac{\omega^2}{2}+\&c.$$

In this make $\omega=\frac{1}{2}(\alpha-\beta)$ and $-\frac{1}{2}(\alpha-\beta)$ successively; therefore

$$p=u_0+\frac{1}{2}(\alpha-\beta)\left(\frac{du}{d\phi}\right)_0+\frac{1}{8}(\alpha-\beta)^2\left(\frac{d^2u}{d\phi^2}\right)_0+\&c.$$

and

$$q=u_0-\frac{1}{2}(\alpha-\beta)\left(\frac{du}{d\phi}\right)_0+\frac{1}{8}(\alpha-\beta)^2\left(\frac{d^2u}{d\phi^2}\right)_0-\&c.$$

Hence we have to the first order of small quantities

$$\frac{p-q}{\alpha-\beta}=\left(\frac{du}{d\phi}\right)_0,$$

and

$$\frac{1}{2}(p+q)=u_0;$$

and therefore $$\left(\frac{du}{u\,d\phi}\right)_0=\frac{2(p-q)}{(p+q)(\alpha-\beta)} \text{ to the first order.}$$

Making this substitution for $\left(\frac{du}{u\,d\phi}\right)_0$ the expressions for X, Y, and T become

$$X=\frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}}-\frac{1}{p^{n-2}}\right)(\cos\gamma)^{n-1}$$
$$\left\{1-\frac{n-1}{3}\cdot\frac{p-q}{p+q}(\alpha-\beta)\tan\gamma-\frac{n-1}{24}(\alpha-\beta)^2\left[\overline{n+4}\,(\sec\gamma)^2-\overline{n+3}\right]\right\};$$

$$Y = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\gamma)^{n-1}$$

$$\left\{\tan\gamma - \frac{1}{3}\cdot\frac{p-q}{p+q}(\alpha-\beta)\left[\overline{n-2}(\sec\gamma)^2 - \overline{n-1}\right]\right.$$

$$\left. - \frac{1}{24}(\alpha-\beta)^2\tan\gamma\left[\overline{n-2}\,\overline{n+5}\,(\sec\gamma)^2 - \overline{n-1}\,\overline{n+3}\right]\right\};$$

$$T = \frac{1}{k(n-1)}\left(\frac{1}{q^{n-1}} - \frac{1}{p^{n-1}}\right)(\cos\gamma)^{n-1}$$

$$\left\{1 - \frac{n-1}{6}\,\frac{p-q}{p+q}(\alpha-\beta)\tan\gamma - \frac{n-1}{24}(\alpha-\beta)^2\left[\overline{n+4}\,(\sec\gamma)^2 - \overline{n+3}\right]\right\};$$

and these values are still true to the fourth order, considering $\dfrac{p-q}{p+q}$ and $\alpha-\beta$ to be small quantities of the first order as before.

The angle which the chord of the arc makes with the axis of x becomes, in like manner,

$$\bar{\gamma} = \gamma + \frac{1}{3}\,\frac{p-q}{p+q}(\alpha-\beta) + \frac{1}{4}(\alpha-\beta)^2\tan\gamma,$$

which is true to the third order.

The above expressions for X and Y may be transformed by introducing this angle $\bar{\gamma}$ into them instead of γ, thus

$$(\cos\bar{\gamma})^{n-1} = (\cos\gamma)^{n-1} - (n-1)(\cos\gamma)^{n-2}\sin\gamma\left[\frac{1}{3}\,\frac{p-q}{p+q}(\alpha-\beta) + \frac{1}{4}(\alpha-\beta)^2\tan\gamma\right]$$

$$= (\cos\gamma)^{n-1}\left\{1 - \frac{n-1}{3}\,\frac{p-q}{p+q}(\alpha-\beta)\tan\gamma - \frac{n-1}{4}(\alpha-\beta)^2(\tan\gamma)^2\right\}.$$

Hence we find

$$X = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\bar{\gamma})^{n-1}\left\{1 - \frac{n-1}{24}(\alpha-\beta)^2\left[\overline{n-2}\,(\sec\gamma)^2 - \overline{n-3}\right]\right\},$$

and

$$Y = X\tan\bar{\gamma} = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\bar{\gamma})^{n-2}\sin\bar{\gamma}$$

$$\left\{1 - \frac{n-1}{24}(\alpha-\beta)^2\left[\overline{n-2}\,(\sec\gamma)^2 - \overline{n-3}\right]\right\};$$

or putting Q for
$$1-\frac{n-1}{24}(\alpha-\beta)^2\left[\overline{n-2}\,(\sec\gamma)^2-\overline{n-3}\right],$$
we have
$$X=\frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}}-\frac{1}{p^{n-2}}\right)(\cos\bar{\gamma})^{n-1}Q;$$
$$Y=\frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}}-\frac{1}{p^{n-2}}\right)(\cos\bar{\gamma})^{n-2}\sin\bar{\gamma}\,Q.$$

Similarly, if
$$\bar{\gamma}'=\gamma+\frac{1}{6}\frac{p-q}{p+q}(\alpha-\beta)+\frac{1}{4}(\alpha-\beta)^2\tan\gamma,$$
we have
$$(\cos\bar{\gamma}')^{n-1}=(\cos\gamma)^{n-1}-(n-1)(\cos\gamma)^{n-2}\sin\gamma\left[\frac{1}{6}\frac{p-q}{p+q}(\alpha-\beta)+\frac{1}{4}(\alpha-\beta)^2\tan\gamma\right];$$
$$=(\cos\gamma)^{n-1}\left\{1-\frac{n-1}{6}\frac{p-q}{p+q}(\alpha-\beta)\tan\gamma-\frac{n-1}{4}(\alpha-\beta)^2(\tan\gamma)^2\right\};$$
and therefore
$$T=\frac{1}{k(n-1)}\left(\frac{1}{q^{n-1}}-\frac{1}{p^{n-1}}\right)(\cos\bar{\gamma}')^{n-1}\left\{1-\frac{n-1}{24}(\alpha-\beta)^2\left[\overline{n-2}\,(\sec\gamma)^2-\overline{n-3}\right]\right\}$$
$$=\frac{1}{k(n-1)}\left(\frac{1}{q^{n-1}}-\frac{1}{p^{n-1}}\right)(\cos\bar{\gamma}')^{n-1}Q,$$
where Q has the same value as before.

Hence the values of X, Y, and T are as stated in my postscript to Mr Niven's paper.

Although the method of finding the expressions for X and T given above, is perhaps the plainest and most straightforward that can be taken, the following leads to simpler operations.

Let
$$f(\phi)=u^l(\sec\phi)^{n+1}.$$

Then $\displaystyle\int f(\phi)\,d\phi=\int u^l(\sec\phi)^{n+1}\,d\phi=\frac{g}{k}\int u^{l-n-1}\frac{du}{d\phi}\,d\phi$ by equation (1)
$$=\frac{g}{k(l-n)}u^{l-n}+\text{const.}$$

Hence
$$\int_\beta^\alpha f(\phi)\,d\phi=\frac{g}{k(l-n)}(p^{l-n}-q^{l-n}).$$

Now let

$$F(\phi) = f(\phi)(\sec\phi)^m = u^l(\sec\phi)^{m+n+1},$$

then

$$F'(\phi) = f'(\phi)(\sec\phi)^m + mf(\phi)(\sec\phi)^m \tan\phi,$$

and

$$\begin{aligned} F''(\phi) &= f''(\phi)(\sec\phi)^m + 2mf'(\phi)(\sec\phi)^m \tan\phi \\ &\qquad + mf(\phi)\left[m(\sec\phi)^m(\tan\phi)^2 + (\sec\phi)^{m+2}\right] \\ &= f''(\phi)(\sec\phi)^m + 2mf'(\phi)(\sec\phi)^m \tan\phi \\ &\qquad + mf(\phi)\left[\overline{m+1}\,(\sec\phi)^{m+2} - m(\sec\phi)^m\right]. \end{aligned}$$

Hence, by the lemma,

$$\begin{aligned} \int_\beta^\alpha F(\phi)\,d\phi &= (\alpha-\beta)\left\{F(\gamma) + \frac{1}{24}(\alpha-\beta)^2 F''(\gamma)\right\} \\ &= (\alpha-\beta)\left\{f(\gamma)(\sec\gamma)^m + \frac{1}{24}(\alpha-\beta)^2(\sec\gamma)^m\left[f''(\gamma) + 2mf'(\gamma)\tan\gamma \right.\right. \\ &\qquad \left.\left. + mf(\gamma)\left[\overline{m+1}\,(\sec\gamma)^2 - m\right]\right]\right\} \\ &= (\alpha-\beta)(\sec\gamma)^m\left\{f(\gamma) + \frac{1}{24}(\alpha-\beta)^2\left[f''(\gamma) + 2mf'(\gamma)\tan\gamma \right.\right. \\ &\qquad \left.\left. + mf(\gamma)\left[\overline{m+1}\,(\sec\gamma)^2 - m\right]\right]\right\}. \end{aligned}$$

But from above

$$\begin{aligned} \frac{g}{k(l-n)}(p^{l-n} - q^{l-n}) &= \int_\beta^\alpha f(\phi)\,d\phi \\ &= (\alpha-\beta)\left\{f(\gamma) + \frac{1}{24}(\alpha-\beta)^2 f''(\gamma)\right\}. \end{aligned}$$

Hence, by division,

$$\begin{aligned} &\int_\beta^\alpha F(\phi)\,d\phi \div \frac{g}{k(l-n)}(p^{l-n} - q^{l-n}) \\ &\qquad = (\sec\gamma)^m\left\{1 + \frac{1}{24}(\alpha-\beta)^2\left[2m\frac{f'(\gamma)}{f(\gamma)}\tan\gamma + m\left[\overline{m+1}\,(\sec\gamma)^2 - m\right]\right]\right\}. \end{aligned}$$

It will be noticed that in this division the quantity $f''(\gamma)$ has disappeared.

Now, from above,

$$f(\phi) = u^l (\sec\phi)^{n+1},$$

and therefore

$$\frac{f'(\phi)}{f(\phi)} = l\frac{du}{ud\phi} + (n+1)\tan\phi,$$

and

$$\frac{f'(\gamma)}{f(\gamma)} = l\left(\frac{du}{ud\phi}\right)_0 + (n+1)\tan\gamma.$$

Hence

$$\int_\beta^\alpha F(\phi)\,d\phi \div \frac{g}{k(l-n)}(p^{l-n} - q^{l-n})$$

$$= (\sec\gamma)^m \left\{1 + \frac{1}{24}(\alpha-\beta)^2 \left[2lm\left(\frac{du}{ud\phi}\right)_0 \tan\gamma + 2m(n+1)(\tan\gamma)^2 \right.\right.$$

$$\left.\left. + m\left[\overline{m+1}\,(\sec\gamma)^2 - m\right]\right]\right\}$$

$$= (\sec\gamma)^m \left\{1 + \frac{1}{24}(\alpha-\beta)^2 \left[2lm\left(\frac{du}{ud\phi}\right)_0 \tan\gamma + m(m+2n+3)(\sec\gamma)^2 \right.\right.$$

$$\left.\left. - m(m+2n+2)\right]\right\}.$$

Now make $m+n+1=2$, or $m=-(n-1)$, and we have

$$\int_\beta^\alpha u^l (\sec\phi)^2 \div \frac{g}{k(l-n)}(p^{l-n} - q^{l-n})$$

$$= (\cos\gamma)^{n-1} \left\{1 - \frac{1}{24}(\alpha-\beta)^2 \left[2l(n-1)\left(\frac{du}{ud\phi}\right)_0 \tan\gamma + (n-1)(n+4)(\sec\gamma)^2 \right.\right.$$

$$\left.\left. - (n-1)(n+3)\right]\right\}.$$

In this make $l=2$, and $l=1$, successively, and we obtain the same expressions for X and T as before.

The case thus treated is not one of mere curiosity, but is practically important. From theoretical considerations, Newton concluded that the resistance of the air to the motion of projectiles is proportional to the square of the velocity, and very little progress has been made in the theory of the subject since his time. Experiments have shewn that the relation between the velocity of a projectile and the resistance offered by the air to its motion is far from being so simple as that given by the theory. The most extensive and accurate series of such experiments which we have are those made by Mr Bashforth by means of his chronograph, which measures with the greatest precision the times taken by the same projectile in passing over several successive arcs in the course of its flight. In a summary of his results for ogival-headed shot, struck with a radius of $1\frac{1}{2}$ diameters, given in *Nature* (Vol. XXXIII. pp. 605, 606), Mr Bashforth concludes that the resistance may be approximately represented by supposing it to vary, as one power of the velocity when that velocity lies between certain limits, as another power when the velocity lies between certain other limits, and so on.

Thus, if v denote the velocity expressed in feet per second,

d the diameter of the shot in inches,

and w its weight in pounds,

and if $\frac{d^2}{w} = c$,

then, when v lies between 430 f.s. and 850 f.s.,

$$\text{the resistance is nearly} = 61\cdot3c\left(\frac{v}{1000}\right)^2;$$

when v lies between 850 f.s. and 1040 f.s.,

$$\text{the resistance is nearly} = 74\cdot4c\left(\frac{v}{1000}\right)^3;$$

when v lies between 1040 f.s. and 1100 f.s.,

$$\text{the resistance is nearly} = 79\cdot2c\left(\frac{v}{1000}\right)^6;$$

when v lies between 1100 f.s. and 1300 f.s.,

$$\text{the resistance is nearly} = 108\cdot8c\left(\frac{v}{1000}\right)^3;$$

and lastly, when v lies between 1300 f.s. and 2700 f.s.,

$$\text{the resistance is nearly} = 141\cdot5c\left(\frac{v}{1000}\right)^2.$$

Hence the resistance varies nearly as the square of the velocity both when the velocity is less than 850 f.s., and when it is greater than 1300 f.s., but the coefficient increases from 61·3 in the former case, to 141·5 in the latter. Also, the resistance varies nearly as the cube of the velocity, both when v lies between 850 f.s. and 1040 f.s., and also when it lies between 1100 f.s. and 1300 f.s., but the coefficient increases from 74·4 in the former to 108·8 in the latter case. Again, for velocities which are nearly equal to that of sound in air, the proportionate increase of the resistance is much greater than that of the velocity.

Mr Bashforth remarks that the points of transition from one law of resistance to another, as stated above, are somewhat arbitrary, but that, if they were changed a little in either direction, the practical error would not be large.

Of course, if we had at our disposal much more numerous and still more accurate observations, it would be possible to represent the experimental results with any degree of exactness that might be desired, by subdividing the observations into a larger number of groups, so that the limiting velocities in any one group should be closer together, and that the change of the index of the power of the velocity in passing from one group to the next should be less abrupt.

60.

ON THE EXPRESSION OF THE PRODUCT OF ANY TWO LEGENDRE'S COEFFICIENTS BY MEANS OF A SERIES OF LEGENDRE'S COEFFICIENTS.

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THE expression for the product of two Legendre's coefficients which is the subject of the present paper, was found by induction on the 13th of February, 1873, and on the following day I succeeded in proving that the observed law of formation of this product held good generally. Having considerably simplified this proof, I now venture to offer it to the Royal Society; and, for the sake of completeness, I have prefixed to it the whole of the inductive process by which the theorem was originally arrived at, although for the proof itself only the first two steps of this process are required. The theorem seems to deserve attention, both on account of its elegance, and because it appears to be capable of useful applications.

As usual let Legendre's nth coefficient be denoted by P_n, then P_n may be defined by the equation

$$P_n = \frac{1}{2^n \lfloor n} \cdot \frac{d^n}{d\mu^n} (\mu^2 - 1)^n.$$

It is well known that the following relation holds good between three consecutive values of the functions P, viz.

$$(n+1) P_{n+1} = (2n+1) \mu P_n - n P_{n-1}.$$

Now $$P_1 = \mu,$$

$$\therefore\ P_1 P_n = \frac{n+1}{2n+1} P_{n+1} + \frac{n}{2n+1} P_{n-1}.$$

Again, we have $$P_2 = \frac{3}{2}\mu P_1 - \frac{1}{2},$$

$$\therefore\ P_2 P_n = \frac{3}{2}\mu P_1 P_n - \frac{1}{2} P_n$$

$$= \frac{3}{2}\frac{n+1}{2n+1}\mu P_{n+1} + \frac{3}{2}\frac{n}{2n+1}\mu P_{n-1} - \frac{1}{2}P_n.$$

Substitute for μP_{n+1} and μP_{n-1} their equivalents obtained by writing $n+1$ and $n-1$ successively for n in the above formula,

$$\therefore\ P_2 P_n = \frac{3}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)} P_{n+2}$$
$$+ \left\{\frac{3}{2}\frac{(n+1)^2}{(2n+1)(2n+3)} - \frac{1}{2} + \frac{3}{2}\frac{n^2}{(2n-1)(2n+1)}\right\} P_n$$
$$+ \frac{3}{2}\frac{(n-1)\,n}{(2n-1)(2n+1)} P_{n-2}.$$

By a slight reduction the coefficient of P_n becomes

$$\frac{n(n+1)}{(2n-1)(2n+3)}.$$

Hence $$P_2 P_n = \frac{3}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)} P_{n+2} + \frac{n(n+1)}{(2n-1)(2n+3)} P_n$$
$$+ \frac{3}{2}\frac{(n-1)\,n}{(2n-1)(2n+1)} P_{n-2}.$$

Again, putting $n=2$ in our original formula, we have

$$P_3 = \frac{5}{3}\mu P_2 - \frac{2}{3} P_1;$$

$$\therefore\ P_3 P_n = \frac{5}{3}\mu P_2 P_n - \frac{2}{3} P_1 P_n$$

$$= \frac{5}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}\mu P_{n+2} + \frac{5}{3}\frac{n(n+1)}{(2n-1)(2n+3)}\mu P_n$$
$$+ \frac{5}{2}\frac{(n-1)\,n}{(2n-1)(2n+1)}\mu P_{n-2} - \frac{2}{3}\frac{n+1}{2n+1} P_{n+1} - \frac{2}{3}\frac{n}{2n+1} P_{n-1}.$$

Substitute for μP_{n+2}, μP_n and μP_{n-2} their equivalents as before,

$$\therefore\ P_3 P_n = \frac{5}{2}\frac{(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)} P_{n+3}$$

$$+\left\{\frac{5}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}\frac{n+2}{2n+5} + \frac{5}{3}\frac{n(n+1)}{(2n-1)(2n+3)}\frac{n+1}{2n+1} - \frac{2}{3}\frac{n+1}{2n+1}\right\} P_{n+1}$$

$$+\left\{\frac{5}{3}\frac{n(n+1)}{(2n-1)(2n+3)}\frac{n}{2n+1} + \frac{5}{2}\frac{(n-1)n}{(2n-1)(2n+1)}\frac{n-1}{2n-3} - \frac{2}{3}\frac{n}{2n+1}\right\} P_{n-1}$$

$$+\frac{5}{2}\frac{(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)} P_{n-3}.$$

By reduction the coefficient of P_{n+1} in this expression becomes

$$\frac{3}{2}\frac{n(n+1)(n+2)}{(2n-1)(2n+1)(2n+5)},$$

and similarly the coefficient of P_{n-1} becomes

$$\frac{3}{2}\frac{(n-1)n(n+1)}{(2n-3)(2n+1)(2n+3)}.$$

Hence we have

$$P_3 P_n = \frac{5}{2}\frac{(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)} P_{n+3}$$

$$+\frac{3}{2}\frac{n(n+1)(n+2)}{(2n-1)(2n+1)(2n+5)} P_{n+1}$$

$$+\frac{3}{2}\frac{(n-1)n(n+1)}{(2n-3)(2n+1)(2n+3)} P_{n-1}$$

$$+\frac{5}{2}\frac{(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)} P_{n-3}.$$

Again, since
$$P_4 = \frac{7}{4}\mu P_3 - \frac{3}{4} P_2,$$

we have
$$P_4 P_n = \frac{7}{4}\mu (P_3 P_n) - \frac{3}{4}(P_2 P_n).$$

Whence by substituting the values found above for $P_3 P_n$ and $P_2 P_n$ and again for μP_{n+3}, μP_{n+1}, &c., we obtain

$$\begin{aligned}
P_4 P_n = &\frac{5 \,.\, 7}{2 \,.\, 4} \frac{(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)} \left\{\frac{n+4}{2n+7} P_{n+4} + \frac{n+3}{2n+7} P_{n+2}\right\} \\
&+ \frac{3 \,.\, 7}{2 \,.\, 4} \frac{n(n+1)(n+2)}{(2n-1)(2n+1)(2n+5)} \left\{\frac{n+2}{2n+3} P_{n+2} + \frac{n+1}{2n+3} P_n\right\} \\
&+ \frac{3 \,.\, 7}{2 \,.\, 4} \frac{(n-1)n(n+1)}{(2n-3)(2n+1)(2n+3)} \left\{\frac{n}{2n-1} P_n + \frac{n-1}{2n-1} P_{n-2}\right\} \\
&+ \frac{5 \,.\, 7}{2 \,.\, 4} \frac{(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)} \left\{\frac{n-2}{2n-5} P_{n-2} + \frac{n-3}{2n-5} P_{n-4}\right\} \\
&- \frac{3 \,.\, 3}{2 \,.\, 4} \frac{(n+1)(n+2)}{(2n+1)(2n+3)} P_{n+2} - \frac{3}{4} \frac{n(n+1)}{(2n-1)(2n+3)} P_n \\
&- \frac{3 \,.\, 3}{2 \,.\, 4} \frac{(n-1)n}{(2n-1)(2n+1)} P_{n-2}.
\end{aligned}$$

By reduction, the coefficient of P_{n+2} in this expression becomes

$$\frac{5}{2} \frac{n(n+1)(n+2)(n+3)}{(2n-1)(2n+1)(2n+3)(2n+7)}.$$

Similarly, the coefficient of P_{n-2} becomes

$$\frac{5}{2} \frac{(n-2)(n-1)n(n+1)}{(2n-5)(2n-1)(2n+1)(2n+3)};$$

and finally, the coefficient of P_n becomes

$$\left(\frac{3}{2}\right)^2 \frac{(n-1)n(n+1)(n+2)}{(2n-3)(2n-1)(2n+3)(2n+5)}.$$

Hence, collecting the terms, we have

$$\begin{aligned}
P_4 P_n = &\frac{1 \,.\, 3 \,.\, 5 \,.\, 7}{1 \,.\, 2 \,.\, 3 \,.\, 4} \frac{(n+1)(n+2)(n+3)(n+4)}{(2n+1)(2n+3)(2n+5)(2n+7)} P_{n+4} \\
&+ \frac{1 \,.\, 3 \,.\, 5}{1 \,.\, 2 \,.\, 3} \cdot \frac{1}{1} \frac{n(n+1)(n+2)(n+3)}{(2n-1)(2n+1)(2n+3)(2n+7)} P_{n+2} \\
&+ \frac{1 \,.\, 3}{1 \,.\, 2} \cdot \frac{1 \,.\, 3}{1 \,.\, 2} \frac{(n-1)n(n+1)(n+2)}{(2n-3)(2n-1)(2n+3)(2n+5)} P_n \\
&+ \frac{1}{1} \cdot \frac{1 \,.\, 3 \,.\, 5}{1 \,.\, 2 \,.\, 3} \frac{(n-2)(n-1)n(n+1)}{(2n-5)(2n-1)(2n+1)(2n+3)} P_{n-2} \\
&+ \frac{1 \,.\, 3 \,.\, 5 \,.\, 7}{1 \,.\, 2 \,.\, 3 \,.\, 4} \frac{(n-3)(n-2)(n-1)n}{(2n-5)(2n-3)(2n-1)(2n+1)} P_{n-4},
\end{aligned}$$

where the law of the terms is obvious, except perhaps as regards the succession of the factors in the several denominators.

With respect to this it may be observed that the factors in the denominator of any term P_p are obtained by omitting the factor $2p+1$ from the regular succession of five factors

$$(n+p-3)(n+p-1)(n+p+1)(n+p+3)(n+p+5).$$

For instance, where $p=n+4$, $2p+1=2n+9$, so that the factor $2n+9$ is to be omitted, and we have $2n+1$, $2n+3$, $2n+5$ and $2n+7$, as the remaining factors, and so of the rest.

Hence by induction we may write, supposing to fix the ideas that m is not greater than n,

$$P_m P_n = \frac{1.3.5\ldots(2m-1)}{1.2.3\ldots m} \cdot \frac{(n+1)(n+2)\ldots(n+m)}{(2n+1)(2n+3)\ldots(2n+2m+1)}$$

$$\times[(2n+2m+1)P_{n+m}]$$

$$+\frac{1.3.5\ldots(2m-3)}{1.2.3\ldots(m-1)} \cdot \frac{1}{1} \cdot \frac{n(n+1)\ldots(n+m-1)}{(2n-1)(2n+1)\ldots(2n+2m-1)}$$

$$\times[(2n+2m-3)P_{n+m-2}]$$

$$+ \text{\&c., \&c.}$$

$$+\frac{1.3.5\ldots(2m-2r-1)}{1.2.3\ldots(m-r)} \cdot \frac{1.3.5\ldots(2r-1)}{1.2.3\ldots r}$$

$$\times \frac{(n-r+1)(n-r+2)\ldots(n-r+m)}{(2n-2r+1)(2n-2r+3)\ldots(2n-2r+2m+1)}$$

$$\times[(2n+2m-4r+1)P_{n+m-2r}]$$

$$+ \text{\&c., \&c.}$$

$$+\frac{1}{1} \cdot \frac{1.3.5\ldots(2m-3)}{1.2.3\ldots(m-1)} \cdot \frac{(n-m+2)(n-m+3)\ldots(n+1)}{(2n-2m+3)(2n-2m+5)\ldots(2n+3)}$$

$$\times[(2n-2m+5)P_{n-m+2}]$$

$$+\frac{1.3.5\ldots(2m-1)}{1.2.3\ldots m} \cdot \frac{(n-m+1)(n-m+2)\ldots n}{(2n-2m+1)(2n-2m+3)\ldots(2n+1)}$$

$$\times[(2n-2m+1)P_{n-m}].$$

And it remains to verify this observed law by proving that if it holds good for two consecutive values of m, it likewise holds good for the next higher value.

If the function $\frac{1 \cdot 3 \cdot 5 \ldots (2m-1)}{1 \cdot 2 \cdot 3 \ldots m}$ be denoted by $A(m)$, the general term of the above expression for $P_m P_n$ may be very conveniently represented by

$$\frac{A(m-r) A(r) A(n-r)}{A(n+m-r)} \left(\frac{2n+2m-4r+1}{2n+2m-2r+1}\right) P_{n+m-2r},$$

r being an integer which varies from 0 to m.

The fundamental property of the function A is that

$$A(m+1) = \frac{2m+1}{m+1} A(m),$$

or

$$A(m) = \frac{m+1}{2m+1} A(m+1).$$

We may interpret $A(m)$ when m is zero or a negative integer, by supposing this relation to hold good generally, so that putting $m=0$, we have

$$A(0) = A(1) = 1.$$

Similarly

$$A(-1) = \frac{0}{-1} A(0) = 0;$$

and hence the value of $A(m)$ when m is a negative integer will be always zero.

We will now proceed to the general proof of the theorem stated above.

Let Q_m denote the quantity of which the general term is

$$\frac{A(m-r) A(r) A(n-r)}{A(n+m-r)} \left(\frac{2n+2m-4r+1}{2n+2m-2r+1}\right) P_{n+m-2r}.$$

In this expression r is supposed to vary from 0 to m, but it may be remarked that if r be taken beyond those limits, for instance if $r=-1$, or $r=m+1$, then in consequence of the property of the function A above stated, the coefficient of the corresponding term will vanish. Hence practically we may consider r to be unrestricted in value.

Similarly, let Q_{m-1} denote the quantity of which the general term is

$$\frac{A(m-r)\,A(r-1)\,A(n-r+1)}{A(n+m-r)}\left(\frac{2n+2m-4r+3}{2n+2m-2r+1}\right)P_{n+m-2r+1},$$

writing $m-1$ for m and $r-1$ for r in the general term given above. Also let Q_{m+1} denote the quantity of which the general term is

$$\frac{A(m-r+1)\,A(r)\,A(n-r)}{A(n+m-r+1)}\left(\frac{2n+2m-4r+3}{2n+2m-2r+3}\right)P_{n+m-2r+1},$$

writing $m+1$ for m in the general term first given. In consequence of the evanescence of $A(m)$ when m is negative, we may in all these general terms suppose r to vary from 0 to $m+1$.

Let us assume that $Q_{m-1}=P_{m-1}P_n$, and also that $Q_m=P_mP_n$, then we have to prove that $Q_{m+1}=P_{m+1}P_n$.

As before, $$(m+1)\,P_{m+1}+mP_{m-1}-(2m+1)\,\mu P_m=0,$$

$$\therefore\ (m+1)\,P_{m+1}P_n+mP_{m-1}P_n-(2m+1)\,\mu P_mP_n=0.$$

Hence our theorem will be established if we prove that

$$(m+1)\,Q_{m+1}+mQ_{m-1}-(2m+1)\,\mu Q_m=0.$$

Now $Q_m=\ldots\ldots$

$$+\frac{A(m-r+1)\,A(r-1)\,A(n-r+1)}{A(n+m-r+1)}\left(\frac{2n+2m-4r+5}{2n+2m-2r+3}\right)P_{n+m-2r+2}$$

$$+\frac{A(m-r)\,A(r)\,A(n-r)}{A(n+m-r)}\left(\frac{2n+2m-4r+1}{2n+2m-2r+1}\right)P_{n+m-2r}$$

$+\ldots\ldots$

Multiplying by μ and substituting for $\mu P_{n+m-2r+2}$ and μP_{n+m-2r}, &c., in terms of $P_{n+m-2r+1}$, &c., we find the coefficient of $P_{n+m-2r+1}$ in μQ_m to be

$$\frac{A(m-r+1)\,A(r-1)\,A(n-r+1)}{A(n+m-r+1)}\left(\frac{n+2m-2r+2}{2n+m-2r+3}\right)$$

$$+\frac{A(m-r)\,A(r)\,A(n-r)}{A(n+m-r)}\left(\frac{n+m-2r+1}{2n+2m-2r+1}\right).$$

Hence the coefficient of $P_{n+m-2r+1}$ in $(m+1)\,Q_{m+1}+mQ_{m-1}-(2m+1)\,\mu Q_m$ will be

$$\frac{A\,(m-r+1)\,A\,(r)\,A\,(n-r)}{A\,(n+m-r+1)}\,(m+1)\left(\frac{2n+2m-4r+3}{2n+2m-2r+3}\right)$$
$$-\frac{A\,(m-r+1)\,A\,(r-1)\,A\,(n-r+1)}{A\,(n+m-r+1)}\,(2m+1)\left(\frac{n+m-2r+2}{2n+2m-2r+3}\right)$$
$$-\frac{A\,(m-r)\,A\,(r)\,A\,(n-r)}{A\,(n+m-r)}\,(2m+1)\left(\frac{n+m-2r+1}{2n+2m-2r+1}\right)$$
$$+\frac{A\,(m-r)\,A\,(r-1)\,A\,(n-r+1)}{A\,(n+m-r)}\,m\left(\frac{2n+2m-4r+3}{2n+2m-2r+1}\right).$$

The sum of the first two lines of this expression is

$$\frac{A\,(m-r+1)\,A\,(r-1)\,A\,(n-r)}{A\,(n+m-r+1)\,(2n+2m-2r+3)}$$
$$\times\left\{\frac{2r-1}{r}\,(m+1)\,(2n+2m-4r+3)-\frac{2n-2r+1}{n-r+1}\,(2m+1)\,(n+m-2r+2)\right\}.$$

Suppose for a moment that $n-r+1=q$, then the quantity within the brackets becomes

$$\frac{2r-1}{r}\,(m+1)\,(2m+1+2q-2r)-\frac{2q-1}{q}\,(2m+1)\,(m+1+q-r).$$

Now this quantity evidently vanishes when $q=r$, and therefore it is divisible by $q-r$. It also vanishes when $m+1=r$, and therefore it is likewise divisible by $m-r+1$.

Hence it is readily found that this quantity

$$=-\frac{q-r}{qr}\,(m-r+1)\,(2m+2q+1),$$

or

$$=-\frac{n-2r+1}{r\,(n-r+1)}\,(m-r+1)\,(2n+2m-2r+3).$$

So that the sum of the first two lines of the expression for the coefficient of $P_{n+m-2r+1}$ is

$$-\frac{A\,(m-r+1)\,A\,(r-1)\,A\,(n-r)}{A\,(n+m-r+1)}\left\{\frac{(m-r+1)\,(n-2r+1)}{r\,(n-r+1)}\right\}.$$

Again, the sum of the other two lines of the expression for the coefficient of $P_{n+m-2r+1}$ is

$$\frac{A(m-r)A(r-1)A(n-r)}{A(n+m-r)(2n+2m-2r+1)}$$
$$\times\left\{-\frac{2r-1}{r}(2m+1)(n+m-2r+1)+\frac{2n-2r+1}{n-r+1}m(2n+2m-4r+3)\right\}.$$

As before suppose $n-r+1=q$, and the quantity within the brackets becomes

$$-\frac{2r-1}{r}(2m+1)(m+q-r)+\frac{2q-1}{q}m(2m+1+2q-2r).$$

Now this quantity evidently vanishes when $q=r$, so that it is divisible by $q-r$. It also vanishes when $m=-q$, and therefore it is likewise divisible by $m+q$.

Hence it is readily found that this quantity

$$=\frac{q-r}{qr}(q+m)(2m-2r+1),$$

or

$$=\frac{n-2r+1}{r(n-r+1)}(n+m-r+1)(2m-2r+1),$$

and therefore the sum of the last two lines of the expression for the coefficient of $P_{n+m-2r+1}$ is

$$\frac{A(m-r)A(r-1)A(n-r)}{A(n+m-r)}\times\left\{\frac{(n-2r+1)}{r(n-r+1)}\cdot\frac{(n+m-r+1)(2m-2r+1)}{2n+2m-2r+1}\right\}.$$

Hence the whole coefficient of $P_{n+m-2r+1}$ is

$$\frac{A(m-r)A(r-1)A(n-r)}{A(n+m-r+1)}\cdot\frac{(n-2r+1)}{r(n-r+1)}$$
$$\times\{(2m-2r+1)-(2m-2r+1)\}=0.$$

And the same holds good for the coefficient of every term. Hence we finally obtain

$$(m+1)Q_{m+1}+mQ_{m-1}-(2m+1)\mu Q_m=0,$$

which establishes the theorem above enunciated.

The principle of the process employed in the above proof may be thus stated:

Every term in the value of Q_m gives rise to two terms in the value of μQ_m or in that of $(2m+1)\mu Q_m$; one of these terms is to be subtracted from the corresponding term in $(m+1)Q_{m+1}$, and the other from the corresponding term in mQ_{m-1}, and it will be found that the two series of terms thus formed identically destroy each other.

Hence we can find at once the value of the definite integral

$$\int_{-1}^{1} P_m P_n P_p \, d\mu,$$

for if $p = n + m - 2r$ we have

$$P_m P_n = \ldots + \frac{A\left(\dfrac{m+p-n}{2}\right) A\left(\dfrac{n+m-p}{2}\right) A\left(\dfrac{n+p-m}{2}\right)}{A\left(\dfrac{n+m+p}{2}\right)} \cdot \frac{2p+1}{n+m+p+1} P_p$$

$$+ \&c.$$

Hence
$$\int_{-1}^{1} P_m P_n P_p \, d\mu$$

$$= \frac{A\left(\dfrac{m+p-n}{2}\right) A\left(\dfrac{n+m-p}{2}\right) A\left(\dfrac{n+p-m}{2}\right)}{A\left(\dfrac{n+m+p}{2}\right)} \frac{2p+1}{n+m+p+1} \int_{-1}^{1} (P_p)^2 \, d\mu$$

$$= \frac{2}{n+m+p+1} \frac{A\left(\dfrac{m+p-n}{2}\right) A\left(\dfrac{n+m-p}{2}\right) A\left(\dfrac{n+p-m}{2}\right)}{A\left(\dfrac{n+m+p}{2}\right)};$$

or if
$$\frac{n+m+p}{2} = s,$$

$$\int_{-1}^{1} P_m P_n P_p \, d\mu = \frac{2}{2s+1} \frac{A(s-m) A(s-n) A(s-p)}{A(s)},$$

where as above

$$A(m) = \frac{1 \cdot 3 \cdot 5 \ldots (2m-1)}{1 \cdot 2 \cdot 3 \ldots m} = 2^m \cdot \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \ldots \left(m - \frac{1}{2}\right)}{1 \cdot 2 \cdot 3 \ldots m}.$$

It is clear that, in order that this integral may be finite, no one of the quantities m, n, and p must be greater than the sum of the other two, and that $m+n+p$ must be an even integer.

I learn from Mr Ferrers that, in the course of the year 1874, he likewise obtained the expression for the product of two Legendre's coefficients, by a method very similar to mine. In his work on "Spherical Harmonics," recently published, he gives, without proof, the above result for the value of the definite integral $\int_{-1}^{1} P_m P_n P_p \, d\mu$.

61.

SUR LES ÉTOILES FILANTES DE NOVEMBRE.

(LETTRE À M. DELAUNAY.)

[*Paris Academy of Sciences, Compt. Rend.* XLIV., 1867.]

Observatoire de Cambridge, 23 *Mars,* 1867.

JE me suis occupé des météores de Novembre et j'ai obtenu quelques résultats qui me paraissent importants. Si vous pensez qu'ils puissent intéresser l'Académie, je vous serai obligé de les lui communiquer à sa prochaine séance. Je les ai fait connaître verbalement à la séance de la Société philosophique de Cambridge de lundi dernier, mais ils n'ont pas encore été imprimés.

Adoptant la position suivante du point radiant:

$$Æ = 149° \, 12'$$
$$\text{Decl.} = \; 23° \, 1' \, \text{N}.$$

qui est la moyenne de ma propre détermination et de cinq autres, et tenant compte de l'action de la Terre sur les météores lorsqu'ils se sont approchés de nous, je trouve les éléments suivants de l'orbite:

Période	33·25 années (admise)
Moyenne distance	10·3402
Excentricité	0·9047
Distance périhélie	0·9855
Inclinaison	16° 46′
Longitude du nœud	51° 28′
Distance du périhélie au nœud	6° 51′
Mouvement rétrograde	

L'accord de ces éléments avec ceux de la comète de Tempel (I., 1866) est encore plus grand que celui que présentent les éléments calculés il y a quelque temps par M. Le Verrier.

Avec les éléments, j'ai calculé la variation séculaire du nœud de l'orbite des météores due à l'action des planètes Jupiter, Saturne et Uranus.

J'ai employé la méthode de Gauss donnée dans sa *Determinatio Attractionis etc.*, et j'ai trouvé que, dans une période totale des météores, c'est-à-dire en 33·25 années, le mouvement du nœud est

Par l'action de Jupiter, de.........20′
„ „ Saturne, de ...... $7'\frac{3}{4}$
„ „ Uranus, de ...... $1'\frac{1}{4}$

De sorte que le mouvement totale du nœud en 33·25 années serait de 29 minutes, ce qui s'accorde presque exactement avec la détermination du moyen mouvement du nœud d'après l'observation faite par le professeur Newton dans son Mémoire sur les pluies d'étoiles de Novembre, inséré dans les nos. 111 et 112 du *Journal Américain de Science et Arts.*

Cela me paraît mettre hors de doute l'exactitude de la période de 33·25 années.

62.

THE LUNAR INEQUALITIES DUE TO THE ELLIPTICITY OF THE EARTH.

[From *the Observatory*, No. 108 (1886).]

It is well known that M. Delaunay was unfortunately prevented by a premature death from completely carrying out his purpose of determining all the sensible inequalities of the Moon's motion by means of his very original and beautiful method of treating that subject. Happily the two magnificent volumes in which he determines the inequalities which are caused by the disturbing force of the Sun, on the supposition that the motion of the Earth about the Sun is purely elliptic, are complete in themselves. The small effects due to the action of the planets and the spheroidal figure of the Earth, as well as those which arise from the disturbances of the Earth's motion, remained to be determined.

Mr G. W. Hill, who is already well known for his skilful treatment of special portions of the lunar theory, has, in the paper now to be noticed, produced a valuable supplement to Delaunay's work by applying the same method to the determination of the lunar inequalities which are due to the ellipticity of the Earth. This paper forms part 2 of vol. III. of the valuable series of astronomical papers prepared for the use of the *American Ephemeris* and *Nautical Almanac.*

The author begins by developing the terms of the disturbing function which are introduced by the ellipticity of the Earth, by substituting for the Moon's coordinates their disturbed values as already given by Delaunay's work. Some idea of the length and complexity of this substitution may be formed when it is stated that the development so obtained contains one constant term accompanied by 121 periodic terms.

The next process is by a series of transformations of the variables involved gradually to remove these periodic terms from the disturbing function, so that it is at length reduced to the form of a constant term.

The number of such operations required to effect this reduction amounts to 103, although each operation is individually sufficiently simple.

By the essential principle of Delaunay's method the differential equations throughout these transformations always preserve their canonical form, and therefore when the disturbing function has been reduced to the above-mentioned simple form, the integrals are at once obtained.

In the next place the transformations indicated in the 103 operations above mentioned are also made in Delaunay's expressions for the three coordinates of the Moon, so that finally the values of these coordinates are found in terms of three arbitrary constants and three angles, each of which consists of a term proportional to the time joined to an arbitrary constant.

The coordinates thus expressed are the longitude, the latitude, and the reciprocal of the radius vector. As this last quantity is only intended to be employed in finding the Moon's parallax, it is given by Delaunay with much less precision than the other two coordinates, a circumstance which is to be regretted as an imperfection from a theoretical point of view.

The expressions thus found are purely analytical, that is the coefficients are expressed in series of powers and products of Delaunay's constants m, e, e', γ, each term also involving as a factor a constant quantity which depends on the figure of the Earth.

In order to make his work more complete, Mr Hill determines the numerical value of this last-mentioned factor by a very elaborate discussion of the results of numerous pendulum experiments.

Finally, by the substitution of the known values of the constants employed, the numerical expressions for the perturbations of the Moon's coordinates produced by the figure of the Earth are obtained.

It will be remarked that comparatively few of the coefficients so found amount to an appreciable quantity, by far the larger number being utterly insensible.

The quantity m denoting, as in Delaunay, the ratio of the mean motion of the Moon to that of the Sun, it is found that the analytical expressions of most of the coefficients involve negative powers of m. This circumstance, which never happens in the case of the perturbations due to the Sun's action, has given rise to a difficulty in some minds as to the admissibility of Mr Hill's results. Mr Stockwell, in particular, in an article in the twenty-ninth volume of the *American Journal of Science*, asserts that the value given to the coefficient of the principal equation of latitude leads to a manifest absurdity, and "justifies the suspicion that the entire solution is erroneous."

The difficulty thus noticed by Mr Stockwell, however, admits of an easy explanation. He applies Mr Hill's formulæ to a case in which they are not applicable, and for which they were not intended. The form of development in series adopted by Mr Hill is founded on the supposition that the perturbations due to the Earth's figure which he wishes to determine are very small compared with those due to the action of the Sun, and therefore he expressly neglects quantities which are proportional to the square of the first-named perturbations. Now, in the case of our Moon, which is that treated by Mr Hill, the above-mentioned supposition certainly holds good, and consequently his formulæ are sufficiently accurate.

If, however, the Sun's distance from the Earth were very much greater than it is, or if the Moon's distance were very much less than it actually is, then the perturbations arising from the Earth's figure might be much greater than those which arise from the Sun's action, and a different form of development would have to be adopted.

In this latter case it would be better to refer the motion of the Moon, not to the ecliptic, but to a fundamental plane passing through the line of intersection of the equator and ecliptic, and occupying a definite intermediate position between those two planes. If the perturbations due to the action of the Sun are much greater than those due to the Earth's figure, this fundamental plane nearly coincides with the ecliptic, whereas if the latter perturbations are much greater than the former, the fundamental plane nearly coincides with the equator. In Mr Hill's formula, the principal term in the expression for the latitude nearly represents the distance of the fundamental plane from the ecliptic corresponding to the actual longitude of the Moon at the time.

A simple analytical illustration of the change of form of the coefficient of this term of the latitude in different circumstances may be given.

If m have its usual meaning as before stated, and if c be a small positive constant depending on the ellipticity of the Earth, then the value of the coefficient in question is approximately proportional to—

$$\frac{c}{m^2+c}.$$

Now, if, as in the case of our Moon, c is very much smaller than m^2, so that we may neglect the square of c compared with that of m^2, the quantity just mentioned becomes approximately $=\frac{c}{m^2}$; whereas if m^2 is small compared with c, the same quantity becomes nearly $=1$, and the coefficient becomes nearly independent of the ellipticity of the Earth, as it should do, since in this case the coefficient of this term is approximately equal to the sine of the obliquity of the ecliptic.

Mr Stockwell's second objection, that Mr Hill has omitted to take into account the modification of the Sun's disturbing force which is caused by the alterations of the Moon's coordinates due to the ellipticity of the Earth, seems to arise from a misapprehension on his part of the spirit of Delaunay's method. These alterations of the Moon's coordinates are implicitly involved in the variables a, e, γ, l, g, h, throughout the series of operations by which Delaunay gradually removes from R the periodic terms arising from the action of the Sun.

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